Complete all of problems 1–4 and one of problems 5 or 6.

1. Let \( f : \mathbb{R} \to \mathbb{R} \) be the function defined by \( f(x) = x - \lfloor x \rfloor \) where \( \lfloor x \rfloor \) is the greatest integer \( m \leq x \). The quantity \( f(x) \) is often referred to as the fractional part of \( x \), because \( x \) is always equal to an integer plus its fractional part.

   a. Prove that \( f \) is continuous at \( x \) if and only if \( x \not\in \mathbb{Z} \).

      Suppose \( x \) is an integer. For each \( n \), choose \( y_n \) such that \( x - \frac{1}{n} < y_n < x \). We have \( \lim(y_n) = x \). Since \( \lfloor y_n \rfloor = x - 1 \) for all \( n \), we have \( f(y_n) = y_n - (x - 1) \) for all \( n \). Thus,
      \[
      \lim(f(y_n)) = \lim(y_n) - (x - 1) = x - (x - 1) = 1
      \]
      but \( f(\lim(y_n)) = f(x) = x - \lfloor x \rfloor = x - x = 0 \). Thus, \( f \) is not (sequentially) continuous at \( x \).

      Now suppose that \( x \) is not an integer. There exists \( \delta \) small enough so that \( (x - \delta, x + \delta) \) contains no integers, and \( \lfloor y \rfloor \) is therefore equal to a constant \( c \) on this range. So, \( f(y) = y - c \) on \( (x - \delta, x + \delta) \), so \( f \) is continuous on \( (x - \delta, x + \delta) \), and it is therefore continuous at \( x \) itself. □

   b. Let \( \alpha \) be an irrational number. Prove that the function defined by \( k \mapsto f(k\alpha) \) is an injection (that is, a “one-to-one” function) \( \mathbb{Z} \to [0, 1] \).

      Suppose that \( f(i\alpha) = f(j\alpha) \) for \( i, j \in \mathbb{Z} \) but that \( i \neq j \). Then \( i\alpha - \lfloor i\alpha \rfloor = j\alpha - \lfloor j\alpha \rfloor \), but then \( \alpha = \frac{\lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor}{i - j} \). This implies that \( \alpha \) is a ratio of integers, and therefore \( \alpha \in \mathbb{Q} \), which is a contradiction. □

   c. Prove that for every \( \epsilon > 0 \) we can find \( k, n \in \mathbb{Z} \) with \( k \neq 0 \) such that \( |k\alpha - n| < \epsilon \).

      For example, \( 5741\sqrt{2} - 8119 \approx 0.0000615839140719 \ldots \)

      Hint: \( [0, 1] \) is compact.

      Let \( \epsilon > 0 \) and let \( \mathcal{G} = \{ (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) : x \in [0, 1] \} \). \( \mathcal{G} \) is an open cover of \( [0, 1] \), so we can find a finite subcover of \( \mathcal{G} \). In short, we can cover \( [0, 1] \) with finitely many open intervals of length \( \epsilon \).

      Suppose we can cover \( [0, 1] \) with \( N \) open intervals \( U_1, \ldots, U_N \) of length \( \epsilon \). By (b), \( f(\alpha), f(2\alpha), \ldots, f((N + 1)\alpha) \) are all distinct points of \( [0, 1] \), and therefore, there exists \( m \) such that \( f(i\alpha), f(j\alpha) \in U_m \) for distinct \( i, j \) in \( \{1, \ldots, N + 1\} \). Thus \( |f(i\alpha) - f(j\alpha)| < \epsilon \), which simplifies to \( |(i - j)\alpha - (\lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor)| < \epsilon \), and this is the result, with \( k = i - j \) and \( n = \lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor \). □

Note: This is the first step to showing that \( \{ f(k\alpha) : k \geq 1 \} \) is dense in \( [0, 1] \). In fact, if \( [a, b] \) is any subinterval of \( [0, 1] \), then the proportion of \( k \in \{1, 2, \ldots, N\} \) such that \( f(k\alpha) \in [a, b] \) tends to \( b - a \) as \( N \) tends to infinity.

The claim in the note above is called the (Weyl’s?) equidistribution theorem.
2. Let \((X, d)\) be a (nonempty) complete metric space and let \(\phi : X \to X\) be a function such that there is \(L \in [0, 1)\) satisfying \(d(\phi(p), \phi(q)) \leq L \cdot d(p, q)\).

This kind of function (a Lipschitz continuous function \(X \to X\) with Lipschitz constant \(L < 1\)) is called a contraction because it “contracts” the space \(X\). Points in \(X\) “get closer together” under repeated applications of \(\phi\).

a. Prove that \(\phi\) is uniformly continuous on \(X\).

(This of course does not require that \(L < 1\).)

If \(L = 0\) then it follows that \(d(\phi(p), \phi(q)) = 0\) for all \(p, q \in X\), so \(f(p) = f(q)\) for all \(p, q \in X\), and therefore that \(\phi\) is a constant function (hence continuous).

Otherwise, for any \(\epsilon > 0\) simply pick \(\delta = \frac{\epsilon}{L}\) to prove uniform continuity. \(\Box\)

b. Let \(x_0 \in X\) and define \(x_{n+1} = \phi(x_n)\) for all \(n \geq 0\). Prove that \((x_n)_n\) is Cauchy and therefore that \((x_n)_n\) converges to some \(x^* \in X\).

Hint: Start by computing an upper bound on \(d(x_{n+k}, x_n)\) for any \(n, k \geq 0\). Remember that there is a formula for \(1 + L + L^2 + \cdots\) which can be used to bound sums like \(1 + L + L^2 + \cdots + L^m\) when \(L \in [0, 1)\).

First of all, by the triangle inequality, we have
\[
d(x_{n+k}, x_n) \leq d(x_{n+k}, x_{n+k-1}) + \cdots + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)
\]
For each of those terms, observe that
\[
d(x_{j+1}, x_j) = d(\phi(x_j), \phi(x_{j-1})) \leq L \cdot d(x_j, x_{j-1})
\]
so by induction, \(d(x_{j+1}, x_j) \leq L^j d(x_1, x_0)\). Combining these two inequalities,
\[
d(x_{n+k}, x_n) \leq (L^{n+k-1} + \cdots + L^n) \cdot d(x_1, x_0) = L^n (1 + \cdots + L^{k-1}) \cdot d(x_1, x_0)
\]
Since \(0 \leq L < 1\), \(1 + \cdots + L^{k-1} \leq 1 + L + L^2 + \cdots = \frac{1}{1-L}\) and so we obtain
\[
d(x_{n+k}, x_n) \leq \frac{L^n}{1-L} d(x_1, x_0)
\]
To formally prove that this shows \((x_n)_n\) is Cauchy, let \(\epsilon > 0\). Choose \(N\) so that \(\frac{L^N}{1-L} d(x_1, x_0) < \epsilon\). If \(m, n \geq N\), we then have
\[
d(x_m, x_n) \leq \frac{L^{\min(m,n)}}{1-L} d(x_1, x_0) \leq \frac{L^N}{1-L} d(x_1, x_0) < \epsilon
\]
as desired. \(\Box\)

c. Prove that \(\phi(x^*) = x^*\), and, furthermore, that \(x^*\) is the unique point of \(X\) with this property: if \(y = \phi(y)\), then \(y = x^*\).

By the continuity of \(\phi\) proved in (a),
\[
\phi(x^*) = \phi \left( \lim_{n \to \infty} x_n \right) = \lim_{n \to \infty} (\phi(x_n)) = \lim_{n \to \infty} (x_{n+1}) = x^*
\]
Now, if \(\phi(y) = y\), then \(d(x^*, y) = d(\phi(x^*), \phi(y)) \leq L \cdot d(x^*, y)\), so since \(L < 1\), it follows that \(d(x^*, y) = 0\), and therefore \(x^* = y\) since \(d\) is a metric. \(\Box\)
Note: Part (c) means that $x^*$ does not depend on the choice of $x_0$, it depends only on $\phi$. Every sequence generated as in (b) will converge to the same point!

The above theorem is important in applications because it provides an algorithm for constructing solutions to differential equations and proving their uniqueness under nice conditions. I’ll provide a reference in the solutions after this exam is due.

The theorem proved in this problem is called Brouwer’s fixed point theorem or the contraction mapping theorem.

The application to differential equations is the Picard–Lindelöf theorem. In that setting, $X$ is a complete subspace (TBD) of $\mathcal{C}^\infty([-\delta, \delta])$ (continuous functions on $[-\delta, \delta]$ that are smooth in the interior). Suppose we are trying to solve $y' = F(t, y)$ with initial condition $y(0) = y_0$. Then, $y(t)$ is a solution iff $\phi(y) = y$ where $\phi : X \to X$ is given by

$$\phi(f)(t) = y_0 + \int_0^t F(s, y(s)) \, ds.$$

Under certain conditions, one can choose $\delta$ and $X$ so that $\phi$ is a contraction of $X$ (with respect to the $\mathcal{C}^\infty$-metric, for example) guaranteeing the existence of a local solution (i.e., one defined on $[-\delta, \delta]$) to the diff eq $y' = F(t, y)$ with $y(0) = y_0$.

The first part of the above Note in fact shows that you can find a solution by starting with any element of $X$ and applying $\phi$ to it over and over again. If, for example, $F$ is a polynomial, this could for example result in a power series that satisfies the differential equation.

3. a. Suppose that $g : [0, 1] \to \mathbb{R}$ is continuous, $g(x) \geq 0$ for all $x \in [0, 1]$, and $g(p) > 0$ for some $p \in [0, 1]$. Prove that $\int_0^1 g(x) \, dx > 0$.

Let $y = g(p)$, with $y > 0$. Since $g$ is continuous, there exists a $\delta > 0$ such that $g(x) \geq \frac{1}{2}y$ on $[p-\delta, p+\delta] \cap [0, 1]$. Let $s$ be the step function $\frac{1}{2}y \cdot 1_{[p-\delta, p+\delta]}$. Then, $s \leq g$ on $[0, 1]$, and $\int_0^1 s$ is positive because it is positive on an interval of positive length, namely $[p-\delta, p+\delta] \cap [0, 1]$, and zero everywhere else. It follows that $\int_0^1 g$ is also positive, by monotonicity of the integral. □

b. Prove that if $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^1 f(x)x^n \, dx = 0$ for $n = 0, 1, 2, \ldots$, then $f(x) = 0$ for all $x \in [0, 1]$.

Note: Unlike $g$ in part (a), $f$ is allowed to take negative values.

Hint: Use Weierstrass’ approximation theorem to evaluate $\int_0^1 f^2$.

From the condition on $\int_0^1 f(x)x^n \, dx$, it follows by induction and linearity of the integral that $\int_0^1 f(x) \cdot P(x) \, dx = 0$ for any polynomial $P(x)$. In particular, for any polynomial,

$$\int_0^1 f^2 = \int_0^1 (f + (f - P)) = \int_0^1 fP + \int_0^1 (f - P) = \int_0^1 (f - P)$$

Let $\epsilon > 0$. By Weierstrass’ theorem we can pick $P$ such that $\|f - P\|_\infty < \epsilon$. Now

$$\left| \int_0^1 (f - P) \right| \leq \int_0^1 |f - P| \leq \epsilon$$
so we obtain $-\epsilon \leq \int_0^1 f^2 \leq \epsilon$. Since this is true for any $\epsilon > 0$, it follows that \( \int_0^1 f^2 = 0 \). By part (a) of this problem, it follows that \( f(x)^2 = 0 \) for all \( x \in [0, 1] \), and finally, that \( f = 0 \) on \([0, 1]\]. □

4. Let \( \mathcal{R}^1([0,1]) \) be the space of all (bounded) Riemann–Darboux integrable functions \([0,1] \rightarrow \mathbb{R} \). If \( f \in \mathcal{R}^1([0,1]) \), then \( |f| \in \mathcal{R}^1([0,1]) \) and we define the \( L^1 \)-norm of \( f \):

\[
\|f\|_1 = \int_0^1 |f(x)| \, dx.
\]

a. Prove that \( d(f,g) = \|f-g\|_1 \) is a metric (the \( L^1 \)-metric) on \( \mathcal{C}([0,1]) \) (the space of continuous functions \([0,1] \rightarrow \mathbb{R} \)). It is not quite a metric on \( \mathcal{R}^1([0,1]) \)—you don’t have to prove this but it’s still worth giving a moment of thought.

We verify the three axioms of a metric. Let \( f, g \in \mathcal{C}([0,1]) \).

- i. \( d(f,g) = 0 \) iff \( f = g \).
  
  If \( f = g \) on \([0,1]\), then \( |f-g| = 0 \) and so \( d(f,g) = \int_0^1 |f-g| = \int_0^1 0 = 0 \).

  On the other hand, suppose that \( d(f,g) = \int_0^1 |f-g| = 0 \). |\( f-g \) is a continuous function on \([0,1]\) whose integral is zero, so by 3a, \( |f-g| = 0 \) on \([0,1]\). It follows that \( f = g \) on \([0,1]\).

- ii. \( d(f,g) = d(g,f) \)
  
  Immediate from \( |f-g| = |g-f| \) (with the usual absolute value).

- iii. \( d(f,g) \leq d(f,h) + d(h,g) \) for any \( h \in \mathcal{C}([0,1]) \).

  We have \( |f-g| = |(f-h) + (h-g)| \leq |f-h| + |h-g| \) (by the regular triangle inequality in \( \mathbb{R} \)). It follows by linearity and monotonicity of the integral that

\[
d(f,g) = \int_0^1 |f-g| \leq \int_0^1 |f-h| + \int_0^1 |h-g| = d(f,h) + d(h,g). \quad □
\]

b. Let \( g = 1_{[\frac{1}{2},1]} \). That is, \( g(x) = 1 \) if \( x \in [\frac{1}{2}, 1] \) and \( g(x) = 0 \) otherwise. Construct a sequence \( (g_n) \) in \( \mathcal{C}([0,1]) \) (of continuous functions \( g_n \)) so that

\[
\lim_{n \to \infty} \int_0^1 |g_n(x) - g(x)| \, dx = 0.
\]

There are several choices for the sequence \( g_n \). Here’s one: Let \( n \geq 2 \) and let \( g_n \) be the graph is the union of the following line segments: \( (0,0) \rightarrow (\frac{1}{2} - \frac{1}{n},0) \rightarrow (\frac{1}{2} + \frac{1}{n},1) \rightarrow (1,1) \). We have \( g_n \geq g = 0 \) on \([0,\frac{1}{2}]\) and \( g_n \leq g = 1 \) on \([\frac{1}{2},1]\); so one can check using geometry that

\[
\int_0^1 |g_n - g| = \int_0^{1/2} (g_n - 0) + \int_{1/2}^1 (1 - g_n) = \frac{1}{2} \cdot \frac{1}{n} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2n}
\]

that is, the combined areas of two triangles with a base of length \( \frac{1}{n} \) and a height of \( \frac{1}{2} \). This clearly tends to 0 as \( n \to \infty \). □
c. Use (b) to prove that $\mathcal{C}([0,1])$ is not complete with respect to the $L^1$-metric.

Let $\epsilon > 0$ and choose $N$ large enough that $\frac{1}{N} < \epsilon$. Then for $m,n \geq N$ we have

$$\int_0^1 |g_m - g_n| \leq \int_0^1 (|g_m - g| + |g_n - g|) = \int_0^1 |g_m - g| + \int_0^1 |g_n - g| = \frac{1}{2m} + \frac{1}{2n} \leq \frac{1}{N} < \epsilon.$$ 

So $(g_n)_n$ is Cauchy with respect to the $L^1$ norm...

- **Sufficient for full credit:** However, the pointwise limit of $(g_n)_n$ is $g \notin \mathcal{C}([0,1])$, so $(g_n)_n$ is an example of a Cauchy sequence in this metric space that is not convergent.

- **Pedantic but completely rigorous (extra step of proving that the pointwise limit would be the limit in $L^1$):** Suppose for contradiction that $(g_n)_n \to h \in \mathcal{C}([0,1])$ with respect to the $L^1$-metric. Then since $\int_0^1 |g - h| \leq \int_0^1 |g_n - g| + \int_0^1 |g_n - h|$, by (b) and convergence $(g_n)_n \to h$ in $L^1$, we find that $\int_0^1 |g - h| = 0$. Thus, for every $\delta \in (0,\frac{1}{2})$ we would have $\int_0^{1/2-\delta} |g - h| = \int_{1/2}^1 |g - h| = 0$, but by 3a yet again, this would mean that $g = h$ on $\bigcup_{0<\delta<1/2}[0,\frac{1}{2}-\delta] \cup \frac{1}{2},1] = [0,1]$. This is not possible, since $h$ is continuous but $g$ is not.

In contrast, $\mathcal{C}([0,1])$ is complete with respect to the $L^\infty$-metric—this is a restatement of “uniform limits of continuous functions are continuous,” which can also be interpreted as “$\mathcal{C}([0,1])$ is a closed set in $\mathcal{B}([0,1])$” since closed subspaces of complete metric spaces are complete.

**Complete one of problems 5 or 6.**

5. Recall the construction of the Cantor set:

$$C_0 = [0,1], \quad C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1], \quad C_2 = [0,\frac{1}{3}] \cup [\frac{2}{3},\frac{1}{2}] \cup [\frac{2}{3},\frac{2}{3}] \cup [\frac{2}{3},1], \ldots, \quad C_\infty = \bigcap_{n=0}^{\infty} C_n.$$ 

Some basic facts: $C_n$ is a union of $2^n$ many intervals each of length $3^{-n}$, so $\text{length}(C_n) = (2/3)^n$. The sequence $(C_n)_n$ is decreasing in the sense that $C_0 \supset C_1 \supset C_2 \supset \cdots$.

Let $f_n = \left(\frac{3}{2}\right)^n \cdot 1_{C_n}$ (where as usual $1_{C_n}(x) = 1$ if $x \in C_n$ and $1_{C_n}(x) = 0$ if $x \notin C_n$) and observe that since $C_n$ is a finite union of closed intervals, $f_n$ is a step function. Let

$$F_n(x) = \int_0^x f_n(t) \, dt$$

for each $x \in [0,1]$. In what follows, we will consider the sequence of functions $(F_n)_n$.

a. Prove that the sequence $(f_n)_n$ (with lowercase $f$) does not converge (pointwise) to any function $[0,1] \to \mathbb{R}$.

For pointwise convergence we would have to have $\lim f_n(0) = f(0)$. However, $\lim f_n(0) = (3/2)^n$ (since $0 \in C_\infty$), and this diverges to $+\infty$. So $f$ cannot be defined. □
b. Prove that each $F_n$ is a continuous, increasing function $[0, 1] \to [0, 1]$.

Functions defined as integrals of integrable functions are necessarily (Lipschitz) continuous. Furthermore, $f_n \geq 0$, so $F_n$ is increasing. Formally, if $x \leq y$, then
\[
F_n(x) = \int_0^x f_n(t) \, dt = \int_0^y f_n(t) \, dt - \int_x^y f_n(t) \, dt \leq \int_0^y f_n(t) \, dt = F_n(y)
\]

It is a fact that $|F_{n+1}(x) - F_n(x)|$ is maximized at $x = \frac{1}{3^n+1}$.

c. Using the above fact, carefully compute
\[
\|F_{n+1} - F_n\|_\infty = \sup_{x \in [0,1]} |F_{n+1}(x) - F_n(x)|.
\]

We just need to compute $F_{n+1}(\frac{1}{3^n+1}) - F_n(\frac{1}{3^n+1})$. Note that $[0, \frac{1}{3^n+1}] \subseteq C_{n+1} \subseteq C_n$ (it is the first interval included in $C_{n+1}$), so $F_{n+1}$ and $F_n$ take the constant values $(3/2)^{n+1}$ and $(3/2)^n$ on the whole interval of integration. Thus,
\[
F_{n+1}(\frac{1}{3}) - F_n(\frac{1}{3^n+1}) = \int_0^{1/3} \left[(3/2)^{n+1} - (3/2)^n\right] \, dt = \frac{(3/2)^{n+1} - (3/2)^n}{3^{n+1}} = \frac{1}{6 \cdot 2^n}
\]

d. With a bit more work (which we will skip, since it is very similar to 2b), it follows from the (correct) calculation in (c) that $(F_n)_n$ is a Cauchy sequence with respect to the $L^\infty$-metric. Since $B([0,1])$ is complete with respect to this metric (it’s in the notes!), $(F_n)_n$ converges to some function $F : [0,1] \to [0,1]$ *uniformly*.

Explain briefly why we can conclude that $F$ is continuous.

Uniform limits of continuous functions are continuous. □

e. Let $I$ be an open interval that is disjoint from the Cantor set. Prove that there exists $N$ large enough such that $n \geq N$ implies that $F_n$ is differentiable on $I$, with $F'_n = 0$ on $I$.

Since $C_\infty \cap I = \emptyset$, There is $N$ large enough such that $C_n \cap I = \emptyset$ for all $n \geq N$.

Now, suppose that $n \geq N$ and that $x, y \in I$ with $x \leq y$ (wlog). Since $f_n = 0$ on $[x,y]$, we have
\[
F_n(x) = \int_0^x f_n(t) \, dt = \int_0^y f_n(t) \, dt - \int_x^y f_n(t) \, dt = \int_0^y f_n(t) \, dt - 0 = F_n(y)
\]

so $F_n$ is constant on $I$, hence differentiable on $I$ with derivative zero. □

It follows that $F$ is differentiable with $F' = 0$ on $[0,1] \setminus C_\infty$, that is, $F$ is “locally constant almost everywhere” on the interval $[0,1]$. Yet, $F$ is not actually constant.

Fun fact: Computing $\{ x \in [0,1] : F$ is differentiable at $x \}$ is an open problem.

$F$ is the *Cantor function* that I mentioned on the last day of class. It can be interpreted also in terms of probability: Suppose that we take a random walk on the real line where on the $n$th step (starting at $n = 1$) we can either move to the right $2 \cdot 3^{-n}$ units or
stay put and let $X$ be the random variable determined by where our random walk ends (after $n \to \infty$)—note that $X \in C_\infty$. Then $F(x) = \text{Prob}(X \leq x)$. That is, $F$ is the cumulative distribution function for $X$, but there is no probability density function for this random variable because $F$ is not differentiable everywhere on $(0, 1)$.

Even so, there is an associated measure $\mu$ on $[0, 1]$, the Cantor measure which assigns to any $E \subseteq [0, 1]$ the size of $E \cap C_\infty$ “relative to $C_\infty$.” For example, $\mu([0, 1/3]) = F(1/3) - F(0) = 1/2$, since $[0, 1/3] \cap C_\infty$ is “the left half” of the Cantor set. This measure is also atomless: $\mu(\{p\}) = 0$ for any $p \in [0, 1]$, even if $p \in C_\infty$. This measure is an important example in measure theory of what is called a singular measure.

6. Let $n \geq 1$ and let $I^n$ denote the $n$-dimensional unit hypercube, so $$I^n = [0, 1]^n = [0, 1] \times \cdots \times [0, 1].$$

For any $\delta > 0$ let $\delta \cdot I^n = [0, \delta]^n$, so $\delta \cdot I^n$ is the hypercube with sides of length $\delta$.

For any $\delta > 0$ and any bounded set $S \subseteq \mathbb{R}^n$, let $C(S, \delta)$ be the smallest integer $N \geq 0$ such that $S$ can be covered by $N$ translated copies of $\delta \cdot I^n$. That is, $$C(S, \delta) = \min \left\{ N : \exists \mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^n \text{ such that } S \subseteq \bigcup_{i=1}^{N} (\mathbf{x}_i + \delta \cdot I^n) \right\}$$

In the case of $S \subseteq \mathbb{R}^1$ for instance, $C(S, \delta)$ is the least number of closed intervals of length $\delta$ required to cover $S$. Finally, let $$\varrho(S) = \lim_{\delta \to 0^+} \frac{\log C(S, \delta)}{\log(1/\delta)} \quad \text{where log = log}_b \text{ for your favorite } b > 1,$$

provided this limit exists (the character $\varrho$ is LaTeX \texttt{\textbackslash varrho} and the choice of $b > 1$ makes no difference whatsoever). If the limit exists we say that $S$ is renderable.

I believe (but am not totally sure) that $\varrho(S)$ is equal to the Hausdorff dimension of $S$ as usually defined, provided it exists.

In what follows, you may use any log/calculus rules you need, and you may assume that all sets taken as input to $\varrho$ are renderable. Under this assumption,

$$\varrho(S) = \lim_{n \to \infty} \frac{\log C(S, \delta_n)}{\log(1/\delta_n)}$$

for any choice of sequence $(\delta_n)_n$ of positive numbers such that $(\delta_n)_n \to 0^+$.

a. Explain why $C(S, \delta)$ will be finite for any $\delta > 0$ and any bounded $S \subseteq \mathbb{R}^n$. Since $S$ is bounded and a subset of $\mathbb{R}^n$, $\overline{S}$ is compact (by Heine–Borel). Let $\mathcal{G}$ be a cover of $\overline{S}$ by copies of $(\delta \cdot I^n)^\circ$. By compactness, it has a finite subcover $\mathcal{G}'$, and this finite subcover covers $S$ as well. Replacing each element of the subcover with its closure (a copy of $\delta \cdot I^n$) shows us that $S$ can be covered by finitely many copies of $\delta \cdot I^n$. 

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b. Remembering that you can assume all sets input to \( \varrho \) are renderable, show that

(i) \( \varrho([0,1]) = 1 \) in \( \mathbb{R}^1 \), (ii) \( \varrho([0,1] \times [0,1]) = 2 \) in \( \mathbb{R}^2 \), and

(iii) \( \varrho(\{(t,t) : 0 \leq t \leq 1\}) = 1 \) in \( \mathbb{R}^2 \).

Your arguments may appeal informally to geometric intuition (e.g., about length and area). Your arguments do not need to be totally rigorous, but they should be convincing—evaluate \( \varrho(S) \) directly via an explicit (and wise) choice of \( (\delta_n)_n \).

i. Let \( \delta = \frac{1}{n} \). We require \( n \) copies of \( \delta_n \cdot I \) to cover \( [0,1] \): You do not need more (since \([0, \frac{1}{n}] \cup \cdots \cup [\frac{n-1}{n},1] \) has \( n \) terms) and fewer copies would have insufficient length. Thus \( C([0,1],1/n) = n \) and so \( \varrho(S) = \lim_{n \to \infty} \frac{\log(n)}{\log(n)} = 1. \)

ii. Let \( \delta_n = \frac{1}{n} \). We need \( n^2 \) copies of \( \delta_n \cdot I^2 \) to cover \( I^2 \) by a similar argument to (i). Thus \( \varrho(S) = \lim_{n \to \infty} \frac{\log(n^2)}{\log(n)} = \lim_{n \to \infty} 2 \frac{\log(n)}{\log(n)} = 2. \)

iii. The set \( S \) is the closed line segment from \((0,0)\) to \((1,1)\). Let \( \delta_n = \frac{\sqrt{2}}{n} \). We have \( C(S,\delta_n) = n \): We need at least \( n \) copies because the set \( S \) contains points at distance \( \sqrt{2} \), and the diameter of \( \delta_n \cdot I^2 \) (the max distance between two points in \( \delta_n \cdot I^2 \)) is \( \sqrt{2}/n \). Furthermore, there is an obvious cover of \( S \) by \( n \) squares of sidelength \( \sqrt{2}/n \). Thus, \( \varrho(S) = \lim_{n \to \infty} \frac{\log(n)}{\log(\sqrt{2}n)} = \lim_{n \to \infty} \frac{\log(n)}{\log(n)+\log(\sqrt{2})} = 1 \).

Note that this dovetails with our naïve notions of dimension: The unit interval has dimension 1, the unit square has dimension 2, and any line segment in \( \mathbb{R}^2 \) still has dimension 1 (even if it is part of an ambient space having dimension 2).

c. Prove that \( 0 < \varrho(C_\infty) < 1 \) where \( C_\infty \) is the Cantor set. (As stated above, you may assume without proof that \( C_\infty \) is renderable, and previous comments about logs, calculus, rigor, and intuition apply to this problem as well.)

We claim that \( C(C_\infty,3^{-n}) = 2^n \). By construction, \( C_\infty \subseteq C_n \), and \( C_n \) is itself a union of \( 2^n \) copies of \( 3^{-n} \cdot I \). This proves that \( C(C_\infty,3^{-n}) \leq 2^n \). On the other hand, fewer copies will fail to cover \( \partial C_n \) because it consists of \( 2 \cdot 2^n \) points at distance \( \geq 3^{-n} \) from each other. Finally,

\[
\varrho(S) = \lim_{n \to \infty} \frac{\log(C(C_\infty,3^{-n}))}{\log(3^n)} = \lim_{n \to \infty} \frac{\log(2^n)}{\log(3^n)} = \frac{\log(2)}{\log(3)} = \log_3(2)
\]

and we have \( 0 < \log_3(2) < 1 \) (by basic log rules—you don’t need a calculator for this). \( \Box \)

Surprise! Fractals may have non-integer dimension. :D This was famously observed by Mandelbrot, who asserted that coastlines (e.g., of Britain) cannot always be adequately measured as a length because of how fractally they get.

d. Recall that a function \( f : X \to Y \) between metric spaces is called Lipschitz continuous if there is a real number \( L \geq 0 \) such that \( d_Y(f(p), f(q)) \leq L \cdot d_X(p, q) \) for all \( p, q \in X \). A Lipschitz continuous function is always (uniformly) continuous on its domain (by the same argument as in 2a).

Suppose that \( f : [0,1] \to \mathbb{R}^2 \) is Lipschitz continuous and that \( f([0,1]) \) is renderable. Prove that \( \varrho(f([0,1])) \leq 1 \).
Fix $n \geq 1$ and let $I_k = [\frac{k}{n}, \frac{k+1}{n}]$. Note that $\max_{x \in I_k} d(\frac{k}{n}, x) = \frac{1}{n}$, so by Lipschitz continuity,

$$\max_{y \in f(I_k)} d \left( f \left( \frac{k}{n} \right), y \right) \leq \frac{L}{n}$$

so in particular, $f(I_k) \subseteq \mathcal{B}_{L/n}(f(k/n))$. The ball $\mathcal{B}_{L/n}(f(k/n))$ is contained in the copy of $(\frac{2L}{n}) \cdot I^2$ centered at $f(k/n)$ (in terms of Euclidean geometry, the ball is “inscribed” in the square).

So, since $f(I) = \bigcup_{k=0}^{n-1} f(I_k)$, we have $C(f(I), 2L/n) \leq n$ (≤ because we have only shown a cover of size $n$ exists but not excluded the possibility that we need all these sets—for example if $f$ is constant, $f(I)$ is a point, and $C(f(I), \delta) = 1$ for any $\delta > 0$). We therefore choose $\delta_n = 2L/n$ and compute

$$\varrho(f(I)) = \lim_{n \to \infty} \frac{\log C(f(I), 2L/n)}{\log(n/2L)} \leq \lim_{n \to \infty} \frac{\log n}{\log n + \log(2L)} = 1$$

as claimed. □

If $f : I^n \to \mathbb{R}^m$ is Lipschitz continuous, then must $\varrho(f(I^n))$ be an integer ≤ $\min\{m, n\}$? That is, is the image of $I^n$ under a Lipschitz continuous map always a non-fractal set of dimension ≤ $n$? I feel like this should be true, but I can’t find a reference.

It’s worth noting that if $f$ is Lipschitz continuous, then $f(I)$ cannot be the Cantor set (because $I$ is connected but $C_\infty$ is not), or the Takagi/blancmange curve, or the boundary of the Koch snowflake (because Lipschitz continuous functions are almost everywhere differentiable and these curves are not anywhere differentiable).

e. Show by counterexample that the conclusion of (d) is false if we only assume that $f : [0, 1] \to \mathbb{R}^2$ is continuous. Hint: You may cite an example from the homework.

Problem 1 on HW7 was about a continuous surjection $\Phi : [0, 1] \to [0, 1]^2$ (one must choose the $f$ in that problem to satisfy $0 \leq f \leq 1$, but that is not a problem). By (b) we have $\varrho([0, 1]) = 1$ but $\varrho(f([0, 1])) = 2$, so the conclusion of (d) fails if we relax the assumption of Lipschitz continuity. □

So the function defined in that problem is apparently not Lipschitz continuous! This is essentially because any function involving unbounded derivatives (where defined) will fail to be Lipschitz continuous. The prototypical example is that $f(x) = \sqrt{x}$ is not Lipschitz continuous on $[0, 1]$ because of its unbounded slope at $x = 0$. 

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