Final Practice Questions (Post-Midterm Material)

IMPORTANT: The final will cover all of the material in this course. It will be twice as long as the midterm, with roughly half of the exam covering the first part of the course, and half covering the second part. The questions below only cover the second half or “post-midterm” material. (Refer to the practice midterm and actual midterm for practice with the “pre-midterm” material).

1. Let $\alpha$ be the number whose binary expansion is $0.011101110111\ldots_2$ (the 0111 repeats forever). Find integers $p, q \geq 1$ such that $\alpha = p/q$.

2. Determine whether the series in Ross, Exercise 14.1.a–e converge or diverge, and justify your answer (stating which test you used to determine each is enough).

3. Let $f : X \to \mathbb{R}$ be a function (where $X$ is a subset of $\mathbb{R}$). Write out the statements “$f$ is continuous on $X$” and “$f$ is uniformly continuous on $X$” in logical notation, and then briefly explain the practical difference between these statements.

4. Let $P(x)$ be a polynomial. Suppose that $P'(x)$ has exactly one real root (there is a unique $a \in \mathbb{R}$ such that $P'(a) = 0$). Which of the following must be true?
   
   i. $P$ has at least one real root.
   ii. $P$ has at most one real root.
   iii. $P$ has even degree.
   iv. None of the above.

5. Let $X, Y \subseteq \mathbb{R}$ and let $f : X \to Y$ be a function with an inverse $f^{-1} : Y \to X$:

   \[ [\forall x \in X : f^{-1}(f(x)) = x] \land [\forall y \in Y : f(f^{-1}(y)) = y] \]

   Suppose that $f$ is differentiable on $X$ and that $f^{-1}$ is be differentiable on $Y$.
   Derive a formula for $(f^{-1})'(y)$ in terms of $f, f', f^{-1}, y$.

6. For each integer $n \geq 1$, let $f_n : [-1, 1] \to [0, 1]$ be the function defined by $f_n(x) = 1 - x^{2n}$. Let $f : [-1, 1] \to [0, 1]$ be the function defined by $f(x) = \lim_{n \to \infty} f_n(x)$ (this limit always exists, and is in $[0, 1]$).

   a. Which of the following statements is/are true?
      
      i. Every function $f_n$ is continuous.
      ii. The function $f$ is continuous.
      iii. Every function $f_n$ is integrable.
      iv. The function $f$ is integrable.
      v. None of the above.
b. If you circled (iv.) above, what is the value of \( \int_{-1}^{1} f \)?

7. Let \( g: [a, b] \rightarrow \mathbb{R} \) be a bounded function. If \( g \) and/or its derivative have certain properties, the fundamental theorem of calculus guarantees \( \int_{a}^{b} g' = g(b) - g(a) \). What are the required properties?

8. Let \( Z = \{ a + b\sqrt{2} : a, b \in \mathbb{Z} \} \) and let \( \tau = -1 + \sqrt{2} \).

   a. Prove that for all integers \( n \geq 1 \) we have \( \tau^n \in \mathbb{Z} \).
   
   b. We have \( \lim(\tau^n) = 0 \). Why? (Just give a reason, no proof is necessary.)
   
   c. Prove that \( Z \) is dense in \( \mathbb{R} \): That is, if \( x, y \in \mathbb{R} \) with \( x < y \), then there exists \( z \in \mathbb{Z} \) such that \( x < z < y \).
   
   d. Let \( f: [0, 1] \rightarrow \mathbb{R} \) be the following function:

\[
 f(x) = \begin{cases} 
 1 & \text{if } x \in Z, \text{ and} \\
 0 & \text{otherwise.}
\end{cases}
\]

   Prove that \( f \) is not integrable.

   Hint: Given that \( \sqrt{2} \) is irrational, under what conditions is \( a + b\sqrt{2} \) rational? \( (a, b \in \mathbb{Z}) \)

9. Let \( f: [a, b] \rightarrow \mathbb{R} \) be a continuous function such that \( f \) is differentiable on \( (a, b) \). Prove that if \( f \) achieves its maximum at \( x \in [a, b] \), then \( x = a \), \( x = b \), or \( f'(x) = 0 \).

10. Prove that if \( f: [a, b] \rightarrow \mathbb{R} \) is a continuous function, then \( f \) is integrable.

   Note: Above, “integrable” always means “Riemann/Darboux integrable.”