Math 115 – Homework 7 Solutions

Kevin Yang

These are solutions written by the author, for the students. This write-up is meant to illustrate what an example solution may look like, and to be a helpful resource for students. Please inform the author if you find any mistakes or are confused by something – I’m quite likely to write something incorrect (especially even if I don’t mean it)!

1. Ross, Exercise 32.7

If \( f = g \) on \([a, b]\) except for finitely many points \( \{x_1, \ldots, x_n\} \) in \([a, b]\), then \( f - g = 0 \) except at exactly these points. If we can show \( f - g \) is integrable with vanishing integral, then because \( f \) is integrable, we deduce \( g \) is integrable with integral \( \int_a^b f(x) \, dx \). In particular, we may assume \( f = 0 \).

If \( g = 0 \) on \([a, b]\) except at the finitely many points \( \{x_1, \ldots, x_n\} \), then \( g \) is piecewise continuous and thus integrable by Theorem 33.8 (whose result says exactly the integrability of piecewise continuous functions). Without loss of generality, we may re-index the points and assume \( x_1 < \ldots < x_n \). To compute the integral of \( g \) on \([a, b]\), consider the partition

(1.1) \[
P_{\epsilon} = \{ [x_i - \epsilon, x_i + \epsilon] \cap [a, b] \}_{i=1}^n \bigcup \mathcal{P} \left( \left\{ [x_i + \epsilon, x_{i+1} - \epsilon] \cap [a, b] \right\}_{i=1}^{n-1} \right) \bigcup \mathcal{P} \left( \left\{ [a, x_1 - \epsilon] \cap [a, b], [x_n + \epsilon, b] \cap [a, b] \right\} \right).
\]

The \( \mathcal{P} \)-notation for the second and third sets of intervals indicate any arbitrary partitions of those intervals. The upper and lower Darboux sums with respect to the partition are

(1.2) \[
U(f, P_{\epsilon}) = \sum_{i=1}^n \max(0, f(x_i)) 2\epsilon, \quad L(f, P_{\epsilon}) = \sum_{i=1}^n \min(0, f(x_i)) 2\epsilon.
\]

If we let \( F = \max_{i=1}^n |f(x_i)| \), then we have

(1.3) \[
-2Fn\epsilon \leq L(f, P_{\epsilon}) \leq U(f, P_{\epsilon}) \leq 2Fn\epsilon.
\]

Thus, we may take \( \epsilon \to 0 \) and deduce the upper and lower Darboux sums vanish, and thus \( \int_a^b g(x) \, dx = 0 \), so we’re done.

2. Ross, Exercise 33.7

a. We directly compute

(2.1) \[
U(f^2, \mathcal{P}) - L(f^2, \mathcal{P}) = \sum_{i=1}^N \left( \max_{x \in [t_i, t_{i+1}]} f(x)^2 - \min_{x \in [t_i, t_{i+1}]} f(x)^2 \right) (t_{i+1} - t_i)
\]

(2.2) \[
= \sum_{i=1}^N \left[ \max_{x, y \in [t_i, t_{i+1}]} \left( f(x)^2 - f(y)^2 \right) \right] (t_{i+1} - t_i)
\]

(2.3) \[
= \sum_{i=1}^N \left[ \max_{x, y \in [t_i, t_{i+1}]} (f(x) - f(y)) (f(x) + f(y)) \right] (t_{i+1} - t_i)
\]

(2.4) \[
\approx \sum_{i=1}^N 2B \max_{x, y \in [t_i, t_{i+1}]} (f(x) - f(y)) (t_{i+1} - t_i)
\]

(2.5) \[
= 2B \left( \sum_{i=1}^N \left( \max_{x \in [t_i, t_{i+1}]} f(x) - \min_{x \in [t_i, t_{i+1}]} f(x) \right) (t_{i+1} - t_i) \right)
\]

(2.6) \[
= 2B \left( U(f, \mathcal{P}) - L(f, \mathcal{P}) \right).
\]
The second equality follows from the fact that the difference between the max and min of a function on some interval is the max over the difference of the function over pairs of points in the interval. The inequality follows from placing an absolute value around each term inside the sum and realizing \( f(x) - f(y) \) may be assumed non-negative when considering the max over pairs of points \( x, y \in [t_i, t_{i+1}] \).

b. To show integrability, it suffices to show that for any \( \varepsilon > 0 \) there exists a partition \( \mathcal{P} \) such that \( U(f^2, \mathcal{P}) - L(f^2, \mathcal{P}) < \varepsilon \), given that this difference is always non-negative. Because \( f \) is integrable by assumption, given any \( \varepsilon > 0 \) fixed, choose a partition \( \mathcal{P} \) such that \( U(f, \mathcal{P}) - L(f, \mathcal{P}) < \frac{\varepsilon}{2B} \). By part (a), we then know
\[
U(f^2, \mathcal{P}) - L(f^2, \mathcal{P}) \leq 2B (U(f, \mathcal{P}) - L(f, \mathcal{P})) < \varepsilon,
\]
so we’re done.

3. Ross, Exercise 33.12

a. To see \( f \) is not piecewise continuous on any interval \([a, b]\), it suffices to show \( f \) is not continuous on any interval of the form \((a, b)\) with \( a < b \). Recall that the points of discontinuity of \( f \) are exactly the rational numbers, which are dense, and thus there exists a point of discontinuity in \((a, b)\). To see \( f \) is not piecewise monotone on \([a, b]\), it suffices to find rational numbers \( q, p \) and an irrational number \( \alpha \) all in \([a, b]\) such that \( q < \alpha < p \). This we can clearly do.

b. To show \( f \) is integrable on any \([a, b]\) with vanishing integral, it suffices to show
\[
\inf_{\mathcal{P}} U(f, \mathcal{P}) = 0.
\]
Fix any (small) \( \varepsilon > 0 \), and fix \( M > 0 \) an integer such that \( \frac{1}{M} < \frac{\varepsilon}{2|K|} \). Let \( S_M \) denote the finitely many rational numbers in \([a, b]\) with denominators (in least/reduced form) less than \( M \). Consider a partition \( \mathcal{P} \) consisting of an interval of radius at most \( \frac{\varepsilon}{2|S_M|} \) on either side around each rational number/element in \( S_M \), and then partitioning the rest of the interval in any arbitrary manner. Because \( f \) is bounded by 1 everywhere and bounded above by \( \frac{1}{M} \) outside these intervals around points of \( S_M \), and because \( f \) is non-negative, we see
\[
U(f, \mathcal{P}) \leq \sum_{x \in S_M} \frac{\varepsilon}{2|S_M|} + \frac{(b-a)}{M} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

4. Supplemental Problem 4

a. Given any increasing sequences \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) in \([a, b]\) with the same limit \( x \), because \( f \) is monotone increasing bounded function on \([a, b]\), the sequences \( \{f(x_n)\}_{n=1}^{\infty} \) and \( \{f(y_n)\}_{n=1}^{\infty} \) are both bounded monotone increasing sequences, and thus both converge to the respective suprema. It now suffices to show they have the same suprema. This follows from the observation that for any \( x_N \), there exists an \( M \) such that \( x_N \leq y_M \leq x \), and similarly for any \( y_K \), there exists an \( L \) such that \( y_K \leq x_L \leq x \).

b. Suppose \( f \) is not continuous on \( I \), so that there exists a point \( x \in I \) and two sequences \( \{\alpha_n\}_{n=1}^{\infty} \) and \( \{\beta_n\}_{n=1}^{\infty} \) converging to \( x \) such that \( \lim_{n \to \infty} f(\alpha_n) \neq \lim_{n \to \infty} f(\beta_n) \). Call the first limit \( \alpha \) and the latter limit \( \beta \). Because these limits exist and the original sequences \( \{\alpha_n\}_{n=1}^{\infty} \) and \( \{\beta_n\}_{n=1}^{\infty} \) are convergent, we may assume these two sequences are monotone. By part (a), we may assume \( \{\alpha_n\}_{n=1}^{\infty} \) must be monotone increasing and \( \{\beta_n\}_{n=1}^{\infty} \) must be monotone decreasing. Because \( f \) is monotone increasing, this implies
\[
\lim_{n \to \infty} f(\alpha_n) =: \alpha < \beta := \lim_{n \to \infty} f(\beta_n).
\]
Consider any \( \gamma \in (\alpha, \beta) \) such that \( \gamma \neq f(x) \). Because \( J \) is an interval containing \( \alpha \) and \( \beta \), it must contain this midpoint \( \gamma \). However, this point is not in the range of \( f \). To see this, for any \( y \in I \), if \( y < x \), then for some \( N > 1 \) we know \( \alpha_N \geq y \), and thus \( \gamma > \alpha \geq f(\alpha_n) \geq f(y) \). Similarly, if \( y > x \), we see \( \gamma < \beta \leq f(\beta_N) \leq f(y) \) for some \( N > 1 \). Lastly, \( f(x) \neq \gamma \) by assumption, so we’re done.
We first show the Cantor function $\phi$ is surjective onto $[0, 1]$. To see this, first note the range of $\phi$ is contained in $[0, 1]$. For any $x \in [0, 1]$, we write $x$ in binary expansion as $x = 0.a_1a_2a_3\ldots$, and consider the element in $[0, 1]$ with the ternary expansion $y = 0.b_1b_2b_3\ldots$ with $b_i = 2a_i$ for each $i \geq 1$. Because $b_i \in \{0, 2\}$, we know $y$ is in the Cantor set and moreover $\phi(y) = x$. Thus, for any $x \in [0, 1]$, there exists $y$ such that $\phi(y) = x$, which shows $\phi$ is surjective onto $[0, 1]$.

By Supplemental Problem 4, it suffices to show that $\phi$ is increasing. To this end, suppose we have two ternary expansions $(d_n)_{n=1}^\infty$ and $(\tilde{d}_n)_{n=1}^\infty$ with $d_n, \tilde{d}_n \in \{0, 2\}$, and suppose the first ternary expansion is larger than the second ternary expansion. In other words, there exists some $N > 1$ such that $d_n = \tilde{d}_n$ for all $n \leq N$ and $d_{N+1} > \tilde{d}_{N+1}$. Thus, $d_{N+1} = 2$ and $\tilde{d}_{N+1} = 0$ because we’re working in the Cantor set. Thus, we know

$$\phi((\tilde{d}_n)_{n=1}^\infty) = \sum_{n=1}^{\infty} \frac{\tilde{d}_n}{2} 2^{-n} = \sum_{n=1}^{N} \frac{d_n}{2} 2^{-n} + \sum_{n=N+2}^{\infty} \frac{\tilde{d}_n}{2} 2^{-n} \leq \sum_{n=1}^{N} \frac{d_n}{2} 2^{-n} + 2^{-N} \leq \phi((d_n)_{n=1}^\infty).$$

(5.1)

To show $\phi$ is increasing on the entire interval $[0, 1]$, for any $x < y$, if $x, y$ are in the Cantor set, the above argument shows $\phi(x) \leq \phi(y)$. If $x$ is in the Cantor set and $y$ is not, then the closest point in the Cantor set to the left of $y$ must be at least $x$, and thus $\phi(x) \leq \phi(y)$ by the same token. If $x$ is not in the Cantor set and $y$ is, then the closest point in the Cantor set to the left of $x$ must be less than $y$, and so $\phi(x) \leq \phi(y)$ again. Lastly, if neither $x$ nor $y$ is in the Cantor set, then the closest point in the Cantor set to the left of $x$ must be to the left of $y$, and thus $\phi(x) \leq \phi(y)$, so we’re done.