Math 115 – Homework 3 Solutions

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These are solutions written by the author, for the students. This write-up is meant to illustrate what an example solution may look like, and to be a helpful resource for students. Please inform the author if you find any mistakes or are confused by something – I’m quite likely to write something incorrect (especially even if I don’t mean it!)

1. Ross, Problem 9.5

We first show the sequence \( \{t_n\}_{n=1}^{\infty} \) satisfies the condition that \( t_n > 0 \) for all \( n > 0 \). To this end, we proceed by induction. For \( n = 1 \), we know \( t_1 = 1 > 0 \), which proves the base case. For the inductive step, suppose \( t_n > 0 \) for some \( n \geq 1 \). Then

\[ t_{n+1} = \frac{t_n}{2} + \frac{1}{t_n} > 0, \]

where the positivity follows from the fact that each term on the RHS is positive, as \( t_n > 0 \). This completes the inductive step.

Assuming \( t_n \to_{n \to \infty} t \), if we take a limit of the equation

\[ t_{n+1} = \frac{t_n^2 + 2}{2t_n} \]

as \( n \to \infty \), i.e. take the limit of both sides as \( n \to \infty \), we see

\[ t = \lim_{n \to \infty} t_{n+1} = \lim_{n \to \infty} \frac{t_n^2 + 2}{2t_n} = \frac{t^2 + 2}{2t}, \]

where here we’re using the fact that \( t_n > 0 \) for all \( n \geq 1 \) (so that we may interchange the limit and fraction). Multiplying by \( 2t \) and rearranging, we see

\[ 2t^2 - t^2 + 2 = 0, \]

which implies \( t = \sqrt{2} \) (which we know exists!). Here, the limit \( t \) is the positive square root of 2, as each term \( t_n \) in the sequence is positive.

2. Ross, Problem 9.12

a. Suppose

\[ L = \lim_{n \to \infty} \left| \frac{s_{n+1}}{s_n} \right| \]

exists and is strictly less than 1. In particular, for some \( \delta > 0 \), (e.g. \( \delta = \frac{1-L}{2} \)), we know \( 0 < L + \delta < 1 \). By definition of convergence, there exists some \( N \) such that for all \( n > N \), we know

\[ 0 < \left| \frac{s_{n+1}}{s_n} \right| < L + \delta < 1. \]

Rearranging, we see

\[ |s_{n+1}| < (L + \delta) |s_n| \]

for all \( n > N \). From this we deduce (via induction) for \( m > 0 \)

\[ |s_{N+m}| < (L + \delta)^m |s_N|, \]

where \( N \) is fixed (and depends only on \( \delta \)). Because \( N \) is fixed, we know

\[ \limsup_{m \to \infty} |s_{N+m}| = \limsup_{n \to \infty} |s_n| < \limsup_{m \to \infty} (L + \delta)^m |s_N| = |s_N| \limsup_{m \to \infty} (L + \delta)^m. \]
As \( m \to \infty \), because \( 0 < L + \delta < 1 \), the last \( \limsup \) converges to 0. This implies

\[
\lim_{m \to \infty} \sup |s_{N+m}| = 0
\]

which in turn implies \( |s_{N+m}| \to 0 \) as \( m \to \infty \), and thus \( |s_n| \to 0 \) as \( n \to \infty \).

b. For each \( n \geq 1 \), define

\[
t_n = \frac{1}{s_n}
\]

so that

\[
\frac{s_{n+1}}{s_n} = \frac{t_n}{t_{n+1}} = t_{n+1}.
\]

Because \( \frac{s_{n+1}}{s_n} \to_n \infty L > 1 \), we know \( \frac{t_{n+1}}{t_n} \to_n \infty \frac{1}{L} < 1 \). Thus, by part (a) of this problem, we deduce \( |t_n| \to 0 \) as \( n \to \infty \). This implies \( \frac{1}{|s_n|} \to 0 \) as \( n \to \infty \). Because \( |s_n| \geq 0 \), this can only happen if \( |s_n| \to \infty \) as \( n \to \infty \). To see this, it suffices to show

\[
\lim_{n \to \infty} \inf |s_n| = \infty,
\]

as this would imply \( \limsup_{n \to \infty} |s_n| = \infty \) as well, and thus \( \lim_{n \to \infty} |s_n| = \infty \). To prove this claim concerning the \( \liminf \), suppose it were not true for the sake of contradiction. By the definition of \( \liminf \), this implies that for some constant \( C > 0 \) and infinitely many \( n \geq 1 \), we have \( |s_n| \leq C \). But this implies \( \frac{1}{|s_n|} \geq \frac{1}{C} \) for infinitely many \( n \geq 1 \). This contradicts the fact that \( \frac{1}{|s_n|} \to 0 \) as \( n \to \infty \). Thus, we deduce \( |s_n| \to \infty \) as \( n \to \infty \).

3. Ross, Problem 10.6

a. To prove \( \{s_n\}_{n \geq 1} \) is a Cauchy sequence, we fix an arbitrary \( \varepsilon > 0 \). We now choose \( N \) such that \( 2^{-N} < \varepsilon \). For any \( n, m > N \), assuming \( m > n \) without loss of generality, by the triangle inequality we have

\[
|s_n - s_m| \leq \sum_{j=n}^{m-1} |s_{j+1} - s_j| < \sum_{j=n}^{m-1} 2^{-j} < \sum_{j=n}^{\infty} 2^{-j}.
\]

This last sum may be controlled by a geometric series. In particular, in general for any \( r < 1 \), we have

\[
\sum_{j=n}^{\infty} r^j = \frac{r^n}{1 - r},
\]

so

\[
\sum_{j=n}^{\infty} 2^{-j} = 2^{-n} \frac{1}{1 - \frac{1}{2}} = 2^{n+1} \leq 2^{-N}
\]

where the last bound follows from the assumption that \( n > N \). Thus, we see

\[
|s_n - s_m| < 2^{-N} < \varepsilon
\]

for any \( n, m > N \). This shows the sequence \( \{s_n\}_{n \geq 1} \) is Cauchy.
b. It is NOT true in general. For example, define the sequence

\[ s_n = \sum_{j=1}^{n} \frac{1}{j}, \quad n \geq 1. \tag{3.5} \]

Then, we know

\[ |s_{n+1} - s_n| = \frac{1}{n} < \frac{1}{n+1}, \tag{3.6} \]

which is the required condition. But this sequence is not Cauchy as it does not converge to any real number, since

\[ \lim_{n \to \infty} s_n = \sum_{j=1}^{\infty} \frac{1}{j} = \infty. \tag{3.7} \]

4. Ross, Problem 10.10

a. By the formula

\[ s_{n+1} = \frac{1}{3} (s_n + 1) \tag{4.1} \]

and the condition \( s_1 = 1 \), we immediately compute

\[ s_2 = \frac{1}{3} (s_1 + 1) = \frac{2}{3}, \tag{4.2} \]
\[ s_3 = \frac{1}{3} (s_2 + 1) = \frac{5}{9}, \tag{4.3} \]
\[ s_4 = \frac{1}{3} (s_3 + 1) = \frac{14}{27}. \tag{4.4} \]

b. To prove \( s_n > \frac{1}{2} \) for all \( n \), we proceed by induction. The base case \( n = 1 \) is clear (and in the above we even have the cases \( n = 2, 3, 4 \)). For the inductive step, suppose \( s_n > \frac{1}{2} \) for some \( n > 0 \). We then have

\[ s_{n+1} = \frac{1}{3} (s_n + 1) > \frac{1}{3} \left( \frac{1}{2} + 1 \right) = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}, \tag{4.5} \]

which completes the inductive step.

c. To show \( \{s_n\}_{n \geq 1} \) is decreasing, by definition we have

\[ s_{n+1} = \frac{1}{3} s_n + \frac{1}{3} \tag{4.6} \]

for any \( n \geq 1 \). It now suffices to show that \( \frac{1}{3} < \frac{2}{3} s_n \). This is equivalent to \( s_n > \frac{1}{2} \), which follows from part (b).

d. We now have shown \( \{s_n\}_{n=1}^{\infty} \) is decreasing and is bounded below by \( \frac{1}{2} \). This implies the sequence converges to some limit \( s \geq 0 \). Taking a limit as \( n \to \infty \) of both sides of the equation

\[ s_{n+1} = \frac{1}{3} (s_n + 1) \tag{4.7} \]

we see

\[ s = \frac{1}{3} (s + 1). \tag{4.8} \]

Rearranging, we see \( \frac{2}{3} s = \frac{1}{3} \), so \( s = \frac{1}{2} \).
5. Ross, Problem 11.10

a. We claim the set of subsequential limits is

\[ \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \cup \{0\}. \]

To prove this claim, we first note that the sequence hits \( \frac{1}{n} \) for any positive integer \( n \) infinitely many times. For any \( n \geq 1 \), taking the subsequence that is just \( \frac{1}{n} \) for each term, we see \( \frac{1}{n} \) is a subsequential limit.

To see that 0 is a subsequential limit, by the same token as the above paragraph we have the following subsequence \( \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \), whose limit is 0.

It now remains to show any other real number cannot be a subsequential limit. Because each term in the sequence is bounded above by 1 and below by 0, we note any real number outside the closed interval \([0, 1]\) cannot be a subsequential limit.

For any \( \alpha \in [0, 1] \) not of the form \( \frac{1}{n} \) for any \( n \geq 1 \), by the Archimedean principle, we may find \( N \) such that \( \frac{1}{N+1} < \alpha < \frac{1}{N} \).

Thus, for any \( n \geq 1 \), we see

\[ |\alpha - \frac{1}{n}| \geq \min\left( |\alpha - \frac{1}{N}|, |\alpha - \frac{1}{N+1}| \right) \]

which is strictly positive. Thus, choosing \( \varepsilon \) to be half the minimum on the RHS, there exists no \( n \geq 1 \) such that \( |\alpha - \frac{1}{n}| < \varepsilon \), and thus \( \alpha \) cannot be a subsequential limit.

6. Supplemental Problem 6

a. To show \( S \) is closed, take \( x \) to be any adherent point of \( S \). I.e., there exists a sequence \( \{t_n\}_{n=1}^{\infty} \) of adherent points of \( S \) such that \( t_n \to_{n\to\infty} x \). Because each \( t_n \) is an adherent point of \( S \), for each \( n \geq 1 \) there exists a sequence \( \{s_{k_n}^{(n)}\}_{k=1}^{\infty} \) of elements of \( S \) such that \( s_{k_n}^{(n)} \to_{k\to\infty} t_n \). Thus, for each \( n \geq 1 \), there exists \( k_n \) such that

\[ |s_{k_n}^{(n)} - t_n| < \frac{1}{n}. \]

Consider the sequence \( \{s_{k_n}^{(n)}\}_{n=1}^{\infty} \). For any \( \varepsilon > 0 \), choose \( N \) such that

\[ |x - t_n| < \frac{\varepsilon}{2}, \quad \frac{1}{n} < \frac{\varepsilon}{2} \]

for all \( n > N \). Thus, by the triangle inequality, for all such \( n \), we see

\[ |x - s_{k_n}^{(n)}| \leq |x - t_n| + |t_n - s_{k_n}^{(n)}| < \frac{\varepsilon}{2} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

This shows \( s_{k_n}^{(n)} \to_{n\to\infty} x \). Because each \( s_{k_n}^{(n)} \in S \), this shows \( x \) is also an adherent point of \( S \). Thus, \( x \in S \), which shows \( S \) is closed.

b. Because \( \mathbb{R} \) is complete and \( \mathbb{Q} \subset \mathbb{R} \), any Cauchy sequence of rational numbers is a Cauchy sequence of real numbers, and thus has a limit in \( \mathbb{R} \) by completeness of \( \mathbb{R} \). This shows \( \overline{\mathbb{Q}} \subset \mathbb{R} \), so it suffices to show \( \overline{\mathbb{Q}} \supset \mathbb{R} \) in order to show \( \overline{\mathbb{Q}} = \mathbb{R} \).

Take \( x \in \mathbb{R} \). By the density of \( \mathbb{Q} \), for any \( n \geq 1 \), we may find a rational number \( q_n \in \mathbb{Q} \) such that

\[ |x - q_n| < \frac{1}{n}. \]

Thus, \( q_n \to x \) as \( n \to \infty \), which shows \( x \) is an adherent point of \( \mathbb{Q} \). This shows \( \overline{\mathbb{Q}} \supset \mathbb{R} \), so we’re done.
7. Supplemental Problem 7

a. Suppose $U$ is open, and for the sake of contradiction suppose $U^C$ is not closed. In particular, there exists a sequence of points $\{x_n\}_{n=1}^\infty$ in $U^C$ that converges to a point $x \in U$. But $U$ is open, so for some $\varepsilon > 0$,

$$ (x - \varepsilon, x + \varepsilon) \subset U. $$

Because $x_n \to_{n \to \infty} x$, we deduce, for $n$ sufficiently large, we have $|x_n - x| < \varepsilon$, and thus $x_n \in (x - \varepsilon, x + \varepsilon) \subset U$. But now $x_n \in U \cap U^C$, a contradiction.

Suppose now that $U^C$ is closed, and that $U$ is not open again for the sake of contradiction. Thus, for some $x \in U$, for all $n > 0$ we may find a point $y_n \in U^C \cap (x - \frac{1}{n}, x + \frac{1}{n})$. Because $y_n \in (x - \frac{1}{n}, x + \frac{1}{n})$ for all $n \geq 1$, we see $y_n \to_{n \to \infty} x$, where $y_n \in U^C$ for all $n \geq 1$. Because $U^C$ is closed, we know its limit $x$ is also in $U^C$. But this contradicts $x \in U$.

b. For any $x \in (a, b)$, define

$$ \varepsilon = \min \left( \frac{|x - a|, |x - b|}{2} \right). $$

Because $a < x < b$, we know $\varepsilon > 0$. We now claim that $(x - \varepsilon, x + \varepsilon) \subset (a, b)$. In particular, it suffices to show $x - \varepsilon > a$ and $x + \varepsilon < b$. By definition of $\varepsilon$, we know

$$ \varepsilon < \min \left( \frac{|x - a|, |x - b|} \right) \leq |x - a|, $$

so

$$ x - \varepsilon \geq x - |x - a| = x - (x - a) = a. $$

Here, we note $x > a$ by assumption, so $|x - a| = x - a$. Similarly, we see

$$ x + \varepsilon < x + |x - b| = x + b - x = b, $$

where we used the assumption $x < b$ to note $|x - b| = b - x$.

c. Take $x \in U \cap V$. Because $U$ and $V$ are open, we may find $\varepsilon > 0$ and $\delta > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset U$ and $(x - \delta, x + \delta) \subset V$.

Take $\eta = \min(\varepsilon, \delta)$, which is positive as $\varepsilon, \delta > 0$. Then we know

$$ (x - \delta, x + \delta) \subset (x - \varepsilon, x + \varepsilon) \subset U $$

and

$$ (x - \delta, x + \delta) \subset (x - \varepsilon, x + \varepsilon) \subset V.$$  

Thus, $(x - \delta, x + \delta) \subset U \cap V$, so that $x \in U \cup V$ is an interior point. Thus $U \cap V$ is closed as $x$ was an arbitrary point in $U \cap V$.

d. Given any arbitrary point $x \in \bigcup_{i \in I} U_i$, we know $x \in U_{i_0}$ for some $i_0 \in I$. Because $U_{i_0}$ is open, for some $\varepsilon > 0$ we know $(x - \varepsilon, x + \varepsilon) \subset U_{i_0} \subset \bigcup_{i \in I} U_i$. Thus, $x$ is an interior point of the union, which shows the union is open as $x$ was an arbitrary point of the union.

e. Consider the sequence of open sets

$$ U_n = \left( -\frac{1}{n}, \frac{1}{n} \right) $$

for all $n \geq 1$. We now claim

$$ \bigcup_{n=1}^\infty U_n = \{0\}, $$

which is not open (as any neighborhood of 0 will contain a number that is not 0). To prove this identity, note $0 \in \left( -\frac{1}{n}, \frac{1}{n} \right)$ for all $n \geq 1$. On the other hand, for any $x \neq 0$, we know $|x| > 0$. Thus, for some $N$ we know $|x| > \frac{1}{N}$. This implies $x \notin U_N$, and thus not in the intersection.
8. Supplemental Problem 8

a. For any positive integer $N > 0$, because $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ are Cauchy, there exists $M > 0$ such that for all $n, m > M$ we have

\begin{align}
|s_n - s_m| &< \frac{1}{2N}, \quad |t_n - t_m| < \frac{1}{2N}.
\end{align}

Thus, for such $n, m > M$, we see by the triangle inequality

\begin{align}
|s_n + t_n - (s_m + t_m)| &\leq |s_n - s_m| + |t_n - t_m| < \frac{1}{N}.
\end{align}

This implies $\{s_n + t_n\}_{n=1}^\infty$ is Cauchy.

b. With the same set-up as in part (a), for any $n, m > M$, we see

\begin{align}
|s_n t_n - s_m t_m| &= |s_n t_n - s_m t_n + s_m t_n - s_m t_m| \\
&\leq |s_n - s_m| |t_n| + |t_n - t_m| |s_m|.
\end{align}

We now note that $\{t_n\}$ and $\{s_n\}$ are Cauchy and thus uniformly bounded by $A$ and $B$ respectively. Thus, we see

\begin{align}
|s_n t_n - s_m t_m| &\leq \frac{A}{2N} + \frac{B}{2N} = \frac{A + B}{2N}
\end{align}

for any $m, n > M$. This shows that $\{s_n t_n\}_{n=1}^\infty$ is Cauchy.

c. Retaining the same set-up as in parts (a) and (b), we may write

\begin{align}
\left| \frac{1}{s_n} - \frac{1}{s_m} \right| &= \left| \frac{s_m - s_n}{s_n s_m} \right|.
\end{align}

Because $s_n, s_m \geq \mu > 0$, we see $\frac{1}{s_n s_m} \leq \mu^{-2}$. Thus, for $m, n > M$, we see

\begin{align}
\left| \frac{1}{s_n} - \frac{1}{s_m} \right| &\leq |s_n - s_m| \mu^{-2} < \frac{1}{\mu^2 N},
\end{align}

which shows $\{\frac{1}{s_n}\}_{n=1}^\infty$ is Cauchy.