Math 115 – Homework 2 Solutions

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These are solutions written by the author, for the students. This write-up is meant to illustrate what an example solution may look like, and to be a helpful resources for students. Please inform the author if you find any mistakes or are confused by something – I’m quite likely to write something incorrect (especially even if I don’t mean it)!

1. Ross: Exercise 4.7

a. For any nonempty set $S$, the inequality $\inf S \leq \sup S$ holds, since for any $x \in S$, we have $\inf S \leq x \leq \sup S$.

We now prove the inequality $\inf T \leq \inf S$. Because $S \subseteq T$, we see $\inf T$ is a lower bound for $S$, since it is a lower bound for $T$. Precisely, for any $x \in S$, we see $x \in T$, which implies $\inf T \leq x$. Because $\inf S$ is the greatest lower bound for $S$, we see $\inf T \leq \inf S$.

It remains to show $\sup S \leq \sup T$. Again because $S \subseteq T$, for any $x \in S$, we see $x \in T$, which implies $x \leq \sup T$. Because $\sup S$ is the least upper bound for $S$, this implies $\sup S \leq \sup T$.

b. We first note $S \subseteq S \cup T$ and $T \subseteq S \cup T$. By part (a), this implies

\begin{equation}
\sup S \leq \sup(S \cup T), \quad \sup T \leq \sup(S \cup T).
\end{equation}

In particular, we deduce $\max(\sup S, \sup T) \leq \sup(S \cup T)$. It remains to prove the reverse inequality. To this end, it suffices to show that $\max(\sup S, \sup T)$ is an upper bound for $S \cup T$. For any $x \in S \cup T$, suppose $x \in S$. Then we know

\begin{equation}
x \leq \sup S \leq \max(\sup S, \sup T).
\end{equation}

Similarly, if $x \in T$, we know

\begin{equation}
x \leq \sup T \leq \max(\sup S, \sup T).
\end{equation}
We begin by proving (1), i.e.

\[ -\sup(-S) \leq s, \quad \forall s \in S. \tag{2.1} \]

To this end, suppose \( s \in S \), so that \( -s \in -S \). Then

\[ -s \leq \sup(-S). \tag{2.2} \]

Applying negative signs (i.e. multiplying both sides by \(-1\)), we reverse the inequality and deduce (1).

We now prove (2), i.e. if \( t \leq s \) for all \( s \in S \), then \( t \leq -s_0 \), where \( s_0 = \sup(-S) \). To this end, suppose \( t \leq s \) for all \( s \in S \). Then \( -t \geq -s \) for all \( s \in S \). In particular, \(-t\) is an upper bound for \(-S\), so \( s_0 \leq -t \). As above, \( t \leq -s_0 \).
To show there exist infinitely many rational numbers between \( a \) and \( b \), it suffices to show that for any \( N \geq 1 \), there exist rational numbers \( r_1, \ldots, r_N \) such that \( a < r_1 < r_2 < \ldots < r_N < b \) for all \( N \geq 1 \). To this end, we proceed by induction. By density of \( \mathbb{Q} \), we may find some rational number \( r \) such that \( a < r < b \). Letting \( r_1 = r \), this proves the base case.

Suppose now that there exist rational numbers \( r_1, \ldots, r_N \) such that \( a < r_1 < \ldots < r_N < b \). By density of \( \mathbb{Q} \), we may find a rational number \( r \) such that \( r_N < r < b \). Letting \( r_{N+1} = r \), we see \( a < r_1 < \ldots < r_N < r_{N+1} < b \), which completes the inductive step.

As in Exercise 4.11, we may find rational numbers $r < s$ such that $a < r < s < b$. We now claim that for some positive integer $N$, we have

\[ r < r + \frac{\sqrt{2}}{N} < s. \]  

The first inequality is a consequence of $\sqrt{2} > 0$. To prove existence of such a positive integer $N > 0$, note $s - r > 0$ by assumption. By the Archimedean principle, we may find a positive integer $N$ such that

\[ \sqrt{2} < (s - r)N. \]  

Dividing by $N > 0$ on both sides, we see

\[ \frac{\sqrt{2}}{N} < s - r, \]

which implies

\[ r + \frac{\sqrt{2}}{N} < s. \]

In particular, because of our choices of $r$ and $s$, we see

\[ a < r < r + \frac{\sqrt{2}}{N} < s < b. \]

It now suffices to show $r + \frac{\sqrt{2}}{N}$ is irrational. To this end, suppose for the sake of contradiction that it is equal to some rational numbers $x \in \mathbb{Q}$, i.e.

\[ r + \frac{\sqrt{2}}{N} = x. \]

Rearranging, i.e. subtracting $r$ from both sides and multiplying by $N$ on both sides, we see

\[ \sqrt{2} = N(x - r). \]

Because the sum and product of rational numbers is rational, we see the right-hand side (RHS) is rational. But this implies $\sqrt{2}$ is rational, a contradiction.
We first show

\[(5.1) \quad \sup(A + B) \leq \sup A + \sup B.\]

For any \(x \in A + B\), we may find \(a \in A\) and \(b \in B\) such that \(x = a + b\) (by definition). Because \(a \leq \sup A\) and \(b \leq \sup B\), we see

\[(5.2) \quad x = a + b \leq \sup A + \sup B.\]

Thus, \(\sup A + \sup B\) is an upper bound for \(A + B\), which implies \(\sup(A + B) \leq \sup A + \sup B\).

It now remains to show

\[(5.3) \quad \sup A + \sup B \leq \sup(A + B).\]

By the result in Problem 4, we may instead show

\[(5.4) \quad \sup A + \sup B \leq \sup(A + B) + \frac{1}{n}\]

for all positive integers \(n\). To this end, for any positive \(\varepsilon > 0\), we may find \(a \in A\) and \(b \in B\) such that

\[(5.5) \quad \sup A < a + \varepsilon, \quad \sup B < b + \varepsilon.\]

Indeed, if this were not true, then \(a < \sup A\) and \(b < \sup B\) would be smaller upper bounds for \(A\) and \(B\), respectively. Thus, adding these two inequalities, we see

\[(5.6) \quad \sup A + \sup B < a + b + 2\varepsilon.\]

Because \(a + b \leq \sup(A + B)\), we see

\[(5.7) \quad \sup A + \sup B < \sup(A + B) + 2\varepsilon.\]

Here, \(\varepsilon > 0\) was arbitrary. Thus, we can set \(\varepsilon = 1/(2n)\) for all positive integers \(n\) to see

\[(5.8) \quad \sup A + \sup B < \sup(A + B) + \frac{1}{n}\]

for all positive integers \(n\), which is exactly what we wanted to show.
We note \( a \) is an upper bound on the set \( \{ r \in \mathbb{Q} : r < a \} \), so that

\[
\text{(6.1)} \quad \sup \{ r \in \mathbb{Q} : r < a \} \leq a.
\]

To prove the reverse inequality, suppose for the sake of contradiction that the inequality is strict. By density of \( \mathbb{Q} \), we may find \( r_0 \) such that

\[
\text{(6.2)} \quad \sup \{ r \in \mathbb{Q} : r < a \} < r_0 < a.
\]

But \( r_0 < a \), which implies \( r_0 \leq \sup \{ r \in \mathbb{Q} : r < a \} \). But this contradicts the inequality above.
a. Given the inequalities

\[ a_n \leq s_n \leq b_n \]  

for all \( n \geq 1 \), subtract \( s \) from all three terms above to deduce

\[ a_n - s \leq s_n - s \leq b_n - s. \]  

We aim to show that for any \( \varepsilon > 0 \), there exists some \( N \) such that \( |s_n - s| < \varepsilon \) for all \( n \geq N \). Because \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = s \), for any \( \varepsilon > 0 \), there exists \( N \) such that \( |a_n - s| < \varepsilon \) and \( |b_n - s| < \varepsilon \) for all \( n \geq N \). Thus, we see

\[ -\varepsilon < a_n - s \leq s_n - s \leq b_n - s < \varepsilon, \quad \forall n \geq N. \]  

In particular, we see \( |s_n - s| < \varepsilon \) for all \( n \geq N \), so we’re done.

b. If \( |s_n| \leq t_n \) for all \( n \geq 1 \), we see

\[ -t_n \leq s_n \leq t_n \]  

for all \( n \geq 1 \). Because \( t_n \to 0 \) as \( n \to \infty \), we see \( -t_n \to 0 \) as \( n \to \infty \) as well. By part (a), we deduce \( s_n \to 0 \) as \( n \to \infty \).
8. Ross: Exercise 8.9

a. Suppose \( \lim_{n \to \infty} s_n = s < a \) for the sake of contradiction. By assumption, there exists some \( N_0 \) such that for all \( n \geq N_0 \), we see \( s_n \geq a \). Let \( \varepsilon = \frac{a - s}{2} > 0 \). Because \( s_n \to s \) as \( n \to \infty \), there exists some \( N_1 \) such that for all \( n \geq N_1 \) we know \( |s - s_n| < \varepsilon \). For \( n > \max(N_0, N_1) \), we rewrite

\[
(8.1) \quad s_n - a = s - a + (s - s_n).
\]

But we know \( |s_n - s| < |s - a| \) by our choice of \( n \) and \( \varepsilon \). Moreover, \( s - a < 0 \). These last two statements imply \( s_n - a < 0 \), a contradiction.

b. If \( s_n \leq b \) for all but finitely many \( n \geq 1 \), then \( -s_n \geq -b \) for all but finitely many \( n \geq 1 \). If \( s_n \to s \) as \( n \to \infty \), then \( -s_n \to -s \) as \( n \to \infty \). By part (a), we deduce

\[
(8.2) \quad -s = \lim_{n \to \infty} (-s_n) \geq -b.
\]

Multiplying by \(-1\) on both sides, we deduce \( s \leq b \), so we’re done.

c. By part (a), we deduce \( s \geq a \), and by part (b), we deduce \( s \leq b \). Thus, \( s \in [a, b] \), so we’re done.
9. Problem 9

a. If $y + x = x$ for all $x \in \mathbb{R}$, then in particular $y + 0 = 0$. But $y + 0 = y$ because 0 is an additive identity. Thus, $y = 0$, so any additive identity is equal to 0.

b. If $x + y = 0$, then $x + y + z = 0 + z = z$. But by commutativity we know $x + y + z = x + z + y = 0 + y = y$. Thus, we see $z = y$.

c. Suppose $xy = x$ for all $x \in \mathbb{R}$. Because 1 is a multiplicative identity, it remains to show $y = 1$. In particular, we know $1 \cdot y = 1$. But know $1 \cdot y = y$ since 1 is a multiplicative identity, so $y = 1$.

d. Because $x^{-1}x = 1$, if we suppose $yx = 1$ then it suffices to show $y = x^{-1}$. We know $yx x^{-1} = y \cdot 1 = y$, where we use commutativity of multiplication. By the same token, we know $y x x^{-1} = 1 \cdot x^{-1} = x^{-1}$, so $y = x^{-1}$. 
10. Problem 10

a. We note that \( a_n \) must satisfy \( \left( \frac{a_n}{2^n} \right)^2 \leq \frac{1}{2} \leq 1 \). In particular, \( 0 \leq \frac{a_n}{2^n} \leq 1 \). Thus \( s_n = \frac{a_n}{2^n} \) is bounded independent of \( n \).

b. We want to show \( s_n \leq s_{n+1} \) for all \( n \geq 1 \). Rewriting \( s_n = \frac{a_n}{2^n} \) and \( s_{n+1} = \frac{a_{n+1}}{2^{n+1}} \), this inequality is equivalent to

\[
\frac{a_n}{2^n} \leq \frac{a_{n+1}}{2^{n+1}},
\]

which, after multiplying by \( 2^{n+1} \), is equivalent to the inequality

\[
2a_n \leq a_{n+1}.
\]

To show this last inequality, it suffices to show that \( 2a_n \) satisfies the condition

\[
\left( \frac{2a_n}{2^{n+1}} \right)^2 \leq \frac{1}{2},
\]

since \( a_{n+1} \) was defined to be the maximum of such integers. This last inequality is equivalent to

\[
\left( \frac{a_n}{2^n} \right)^2 \leq \frac{1}{2},
\]

which is true as we defined \( a_n \) to satisfy such an inequality.

c. We note that for all \( n \geq 1 \), we have

\[
0 \leq \frac{1}{2} - \frac{2 \cdot 2^n + 1}{4^n}.
\]

Moreover, we know

\[
\frac{2 \cdot 2^n + 1}{4^n} = 2 \left( \frac{1}{2} \right)^n + \frac{1}{4^n}.
\]

We know \( \frac{1}{2^n} \) vanishes as \( n \to \infty \) and \( \frac{1}{4^n} \) vanishes as \( n \to \infty \). Thus, the RHS vanishes as \( n \to \infty \), i.e.

\[
\lim_{n \to \infty} \left( 2 \left( \frac{1}{2} \right)^n + \frac{1}{4^n} \right) = \lim_{n \to \infty} 2 \left( \frac{1}{2} \right)^n + \lim_{n \to \infty} \frac{1}{4^n} = 2 \lim_{n \to \infty} \left( \frac{1}{2} \right)^n + \lim_{n \to \infty} \frac{1}{4^n} = 0.
\]

Thus, because 0 is constant in \( n \), by the squeeze theorem we deduce

\[
\lim_{n \to \infty} \frac{2 \cdot 2^n + 1}{4^n} = 0.
\]

d. The string of inequalities tells us

\[
s_n^2 \leq \frac{1}{2} \leq s_n^2 + \frac{2a_n + 1}{4^n}.
\]

Subtracting \( s_n^2 \) from all three, we see

\[
0 \leq \frac{1}{2} - s_n^2 \leq \frac{2a_n + 1}{4^n} \leq \frac{2 \cdot 2^n + 1}{4^n},
\]

where the last inequality follows from noting \( a_n \leq 2^n \) as in part (a). By part (c), the far RHS vanishes as \( n \to \infty \). Thus, by the squeeze theorem again, we see

\[
\lim_{n \to \infty} \left( s_n^2 - \frac{1}{2} \right) = 0.
\]

e. Because \( \frac{1}{\sigma} \neq 0 \), it suffices to show \( \sigma^2 = \frac{1}{2} \). But

\[
\sigma^2 = \left( \lim_{n \to \infty} s_n \right)^2 = \lim_{n \to \infty} s_n^2 = \frac{1}{2},
\]

so \( \frac{1}{\sigma} \) is a positive real number (as it is a limit of \( s_n \) and each \( s_n \) is positive) whose square is 2. This implies \( \sqrt{2} = \frac{1}{\sigma} \in \mathbb{R} \) exists.