Exponentiation

The purpose of Math 115 is to make calculus rigorous, so we ought to spend some time constructing the functions familiar from a calculus course.

Though the field axioms set up the rules of addition and multiplication (and subtraction and division), the operation of exponentiation is not intrinsically defined. That is, given a real number \( x > 0 \) and a real number \( p \), how do we define \( x^p \)?

**Integer exponentiation**

The answer is clear if \( p = m \) for an integer \( m \): If \( m \) is positive, then

\[ x^m = \underbrace{x \cdot \cdots \cdot x}_m \]

and we extend to negative numbers using \( x^{-m} = \frac{1}{x^m} \). Of course, we also set \( x^0 = 1 \) for any \( x > 0 \) (it is often convenient to set \( 0^0 = 1 \), as well). It is easy to prove the following properties of integer exponentiation: For all real \( x, y > 0 \) and all integers \( m, n \),

- (E1) \( x^m \cdot x^n = x^{m+n} \)
- (E2) \( (x^m)^n = x^{m \cdot n} \)
- (E3) \( (x \cdot y)^m = x^m \cdot y^m \)

Also, if \( x > 1 \) and \( m < n \), we have \( x^m < x^n \), and if \( 1 < x < y \) and \( m > 0 \), then \( x^m < y^m \).

**Rational exponentiation**

Now that exponentiation is defined for integer exponents, we will next define it for *rational* exponents. It is enough to define for every \( x > 0 \) and every positive integer \( m \), a number \( x^{1/m} \) such that \( (x^{1/m})^m = x \). That is, to define rational exponentiation, we need to explain how \( m \)th roots are defined!

Obviously, this cannot be done if we restrict ourselves to the rational numbers (since e.g., \( \sqrt{2} = 2^{1/2} \notin \mathbb{Q} \)), so we will need to use the completeness axiom in our construction. We will use Theorem 10.2 (bounded monotone sequences converge), which relies on the completeness axiom. In addition, we will also need the binomial theorem.

**The binomial theorem**

The binomial theorem explains how to expand a two-variable expression like \((x + y)^m\):

\[
(x + y)^m = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k
\]
where \( \binom{m}{k} \) is the number of ways to choose \( k \) (distinct) elements from a set of \( m \) (distinct) elements. A formula for \( \binom{m}{k} \) is

\[
\binom{m}{k} = \frac{m!}{k!(m-k)!}
\]

and these binomial coefficients are the entries in the famous Pascal’s triangle. It is important that \( \binom{m}{k} \) is a positive integer for all \( k \in \{0, \ldots, m\} \) (this can be proven using the counting interpretation of \( \binom{m}{k} \) or using Pascal’s triangle—I won’t get into the details here).

Applying the binomial theorem with \( x = y = 1 \), we have

\[
2^m = \sum_{k=0}^{m} \binom{m}{k}
\]

so in particular, for all \( k \in \{0, \ldots, m\} \) we have the “trivial” bounds on binomial coefficients

\[
1 \leq \binom{m}{k} \leq 2^m \quad \text{(whenever } 0 \leq k \leq m)\]

We’ll need these bounds in our proof.

**Note:** Ross has a bit on the binomial theorem in Section 1, specifically Exercise 1.12, in which the reader (you?) proves the binomial theorem by induction.

**The idea of the proof**

Say we want to find \( (2/5)^{1/3} = \sqrt[3]{\frac{2}{5}} \). That is, we are looking for a number \( \beta \) such that \( \beta^3 = \frac{2}{5} \). How do we find such a number? One way is to make a sequence of increasingly precise rational approximations... but, again, how?

Let’s pick a reasonably large denominator, say 1024, and look at all the integers \( a \geq 0 \) such that \( \left( \frac{a}{1024} \right)^3 \leq \frac{2}{5} = 0.4 \):

\[
\left( \frac{0}{1024} \right)^3 = 0 \leq 0.4 \\
\left( \frac{1}{1024} \right)^3 \approx 0.000000001 \leq 0.4 \\
\vdots \\
\left( \frac{750}{1024} \right)^3 \approx 0.3929017112 \leq 0.4 \\
\left( \frac{751}{1024} \right)^3 \approx 0.3944754144 \leq 0.4 \\
\left( \frac{752}{1024} \right)^3 \approx 0.3960533142 \leq 0.4 \\
\left( \frac{753}{1024} \right)^3 \approx 0.3976354161 \leq 0.4 \\
\left( \frac{754}{1024} \right)^3 \approx 0.3992217258 \leq 0.4 \\
\left( \frac{755}{1024} \right)^3 \approx 0.4008122487 > 0.4
\]

so \( a = 754 \) is the largest integer satisfying \( \left( \frac{a}{1024} \right)^3 \leq 0.4 \), and so \( \frac{754}{1024} \) should be a “pretty good” approximation of \( \sqrt[3]{\frac{2}{5}} \). Indeed, in decimals,

\[
\sqrt[3]{\frac{2}{5}} \approx 0.7368062997 \quad \text{and} \quad \frac{754}{1024} \approx 0.736328125
\]
This suggests the following strategy for finding $\alpha^{1/m}$. For a sequence $(d_n)_n$ of increasing denominators (we will pick $d_n = 2^n$ for convenience), let $a_n$ be the largest integer such that $(\frac{a_n}{d_n})^m \leq \alpha$ and set $s_n = \frac{a_n}{d_n}$. If we play our cards right, the sequence $(s_n)_n$ will converge (to some number $\beta$), and $(s_n^m)_n$ will converge to $\alpha$, after which it will follow that $\beta^m = \alpha$.

For example, the first eleven terms of our sequence $(s_n)_n$ for $\sqrt[3]{\frac{2}{5}}$ will be

$$0 < \frac{1}{2} = \frac{2}{4} < \frac{5}{8} < \frac{11}{16} < \frac{23}{32} < \frac{47}{64} = \frac{94}{128} = \frac{188}{256} < \frac{377}{512} = \frac{754}{1024}, \ldots$$

Where each numerator is chosen to be the largest $a$ satisfying $(\frac{a}{2^n})^3 \leq \frac{2}{5}$. Note that the binary expansion of $\sqrt[3]{\frac{2}{5}}$ begins $0.1011110011 \ldots$, and the =s and <s in the sequence above correspond precisely to the 0s and 1s (after the .) in the binary expansion. (!)

**Proof that $m$th roots exist**

Here’s the precise statement we are going to prove:

**Theorem.** For all positive real numbers $\alpha$ and all positive integers $m$, there exists a unique positive real number $\beta$ such that $\beta^m = \alpha$.

**Proof.** First of all, once existence is established, uniqueness is easy: Suppose that $\beta^m = \gamma^m = \alpha$ and then show that both $\beta < \gamma$ and $\beta > \gamma$ lead to a contradiction.

Second of all, it is enough to prove the claim assuming that $0 < \alpha \leq 1$ (once we’ve done so, if we’re given an $\alpha > 1$, we have proven that there is $\beta'$ with $(\beta')^m = \frac{1}{\alpha}$, so $\beta = \frac{1}{\beta'}$ will satisfy $\beta^m = \alpha$).

Let $n \geq 0$. We define

$$a_n = \max \left\{ a \in \mathbb{Z} : \left( \frac{a}{2^n} \right)^m \leq \alpha \right\}$$

Such an $a_n$ exists because the set above is bounded above (by $2^n$, since $\alpha \leq 1$) and nonempty (since 0 is an element of the set). We also define $s_n = \frac{a_n}{2^n}$.

The idea is that $s_n^m = \left( \frac{a_n}{2^n} \right)^m \leq \alpha$ but as $n$ increases, the two sides of the inequality gets closer and closer together. We will therefore show two things: that $(s_n)_n$ converges and that $\lim (s_n^m)_n = \alpha$. Then one can use Theorem 9.3 (and an easy inductive argument) to show that $(\lim s_n)_n = \alpha$.

**Claim 1:** $(s_n)_n$ converges. Since $0 \leq a_n \leq 2^n$ for all $n$, we have $0 \leq s_n \leq 1$ for all $n$. Thus, $(s_n)_n$ is bounded (above as well as below). Next, note that for any $n$,

$$\left( \frac{2a_n}{2^{n+1}} \right)^m = \left( \frac{a_n}{2^n} \right)^m \leq \alpha$$

so $2a_n \in \{ a \in \mathbb{Z} : (a/2^{n+1})^m \leq \alpha \}$ and therefore $a_{n+1} \geq 2a_n$ (since $a_{n+1}$ is the maximum of that set). Dividing through by $2^{n+1}$ yields $s_{n+1} \geq s_n$, so $(s_n)_n$ is increasing. By Theorem 10.2, $(s_n)_n$ converges; we will call its limit $\beta$. 

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Claim 2: There exists \( N \) such that \( a_n \geq 1 \) for all \( n > N \). As we saw above, \( a_{n+1} \geq 2a_n \) for all \( n \), so it is enough to show that \((a_n)_n\) is not the zero sequence \((0)_n\). Otherwise, we would have, by the definition of \( a_n \)

\[
0 = \left( \frac{0}{2^n} \right)^m \leq \alpha < \left( \frac{1}{2^n} \right)^m = \frac{1}{2^{nm}}
\]

for all \( n \), and this clearly implies that \( \alpha = 0 \), against our hypothesis that \( 0 < \alpha \leq 1 \).

Claim 3: \((s_n^m)_n\) converges to \( \alpha \). Let \( \epsilon > 0 \). Since \((\frac{m \cdot 2^m}{2^n})_n\) converges to 0, we can choose \( N \) large enough so that \( n > N \) guarantees

\[
0 < \frac{m \cdot 2^m}{2^n} < \epsilon
\]

whenever \( n > N \). By Claim 2, we may also choose this \( N \) so that \( a_n \geq 1 \) whenever \( n > N \).

Let \( n > N \). By the definitions of \( s_n \) and of \( a_n \), we have

\[
s_n^m = \left( \frac{a_n}{2^n} \right)^m \leq \alpha < \left( \frac{a_n + 1}{2^n} \right)^m
\]

We want to show that the gap between the expressions around \( \alpha \) is smaller than \( \epsilon \). By the binomial theorem,

\[
(a_n + 1)^m = \sum_{k=0}^{m} \binom{m}{k} a_n^k = a_n^m + \sum_{k=0}^{m-1} \binom{m}{k} a_n^k
\]

(we used \( \binom{m}{m} = 1 \) above—this is easy to check using the formula or counting interpretation of the binomial coefficients). Dividing the above by \( 2^{nm} \), and rearranging,

\[
0 \leq |s_n^m - \alpha| < \frac{1}{2^{nm}} \sum_{k=0}^{m-1} \binom{m}{k} a_n^k
\]

Next, we apply the following inequalities: \( 1 \leq \binom{m}{k} \leq 2^m \) (proven earlier), and, if \( 0 \leq k \leq m - 1 \), then \( 1 \leq a_n^k \leq a_n^{m-1} \leq (2^n)^{m-1} \) (remember that we have \( 1 \leq a_n \leq 2^n \) for all \( n > N \)). Thus,

\[
\sum_{k=0}^{m-1} \binom{m}{k} a_n^k \leq m \cdot 2^m \cdot (2^n)^{m-1} = \frac{m \cdot 2^m \cdot 2^{nm}}{2^n},
\]

and so

\[
0 \leq |s_n^m - \alpha| < \frac{1}{2^{nm}} \cdot \frac{m \cdot 2^m \cdot 2^{nm}}{2^n} = \frac{m \cdot 2^m}{2^n} < \epsilon,
\]

as desired.

Finally,

\[
\alpha = \lim(s_n^m)_n = (\lim(s_n)_n)^m = \beta^m,
\]

(where the second equality follows from Theorem 9.3 plus a quick induction proof) and this completes the argument. \( \Box \)