**Problem Set 7 (due Tuesday, August 15th)**

**Textbook problems**
1. 33.7
2. 33.8
3. 33.12
4. 34.10

**Problem 5: A very flat function**

a. Let \( g : (0, \infty) \to \mathbb{R} \) be defined by \( g(x) = e^{-1/x} \). Prove that for all integers \( n \geq 0 \) the function \( g^{(n)} \), the \( n \)th derivative of \( g \), has the form

\[
g^{(n)}(x) = \frac{P_n(x)e^{-1/x}}{x^{2n}}
\]

where \( P_n \) is a polynomial.

*Use induction. The 0th derivative of a function is just the function itself. I believe it is slightly easier in this case to prove \( n = 0 \) separately and then make \( n = 1 \) the base case for induction. You may use any calculus facts about derivatives you need, including, e.g., that \((e^x)’ = e^x\).*

b. Let

\[
f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}
\]

Using (without proof) the fact that \( \lim_{x \to 0^+} \frac{e^{-1/x}}{x^m} = 0 \) for all integers \( m \geq 0 \), prove that \( f^{(n)}(0) \) exists and is equal to 0 for all \( n \geq 0 \).

If you remember your calculus this might be slightly disturbing! \( f(x) \) is *not* itself the zero function, but its Taylor series expansion will be equal to 0 everywhere. That is, the Taylor expansion of \( f \) does *not* satisfactorily model \( f \) at 0. What gives?

The reason this happens is that as \( n \to \infty \), \( g^{(n)} \) has a critical point closer and closer to \( x = 0 \) at which \( g^{(n)} \) is very, very large. So, despite \( g^{(n)} \) being continuous at \( x = 0 \), it *very steeply increases* to some enormous value just to right left of \( x = 0 \). Since the Taylor series for \( f \) can only work with the values of the derivatives at \( x = 0 \), it “cannot detect” this behavior. (Time permitting, I’ll say more about this in class.)