Problem Set 4: Q&A

General Questions

• When proving a function is continuous, is there a general intuition for which definition of continuity we should use?

   Not really, but I think in this assignment, the sequential definition is typically easiest to use (I find it the easiest to use personally, but everyone is different).

• If a theorem holds for any function where for all \( x_0 \) in the domain, \( \lim (x_n) = x_0 \) implies \( \lim (f(x_n)) = \lim (f(x_0)) \), then can we just state the theorem as “Theorem X holds for any function continuous on its domain”? i.e. do we always need to state the first condition in order to justify “function continuous on its domain” explicitly? This is specifically for Theorem 17.4.

   You don’t have to state this explicitly, since the definition of continuity on a set is “continuous at every point in the set.”

Homework Questions

1. **Do we need to prove \( f(x) = x \) is continuous on \( \mathbb{R} \) for Problem 1a?**

   No need to prove this—we proved it in class and Ross does it in an example in Section 17 somewhere. Here’s the proof that it is \( \epsilon-\delta \) continuous: Take \( \delta = \epsilon \). End of proof! :)

2. **For 17.12a, when we take a sequence of rational numbers \( (x_n) \) approaching an irrational number \( \alpha \), do we need to prove the limit \( (x_n) = \alpha \) or we can just state it “take a rational sequence approaching some irrational number”?**

   I believe we may have proved that every irrational number is the limit of a sequence whose terms are all rational numbers, but if you want to use this without proof, you’ll have to find where we did it (in an assigned homework exercise, or somewhere in the book).

   In any case, here’s a quick proof: For all \( n \geq 1 \), there is a rational number \( r_n \) such that \( x < r_n < x + \frac{1}{n} \) (why? justify this). The sequence \( (r_n) \) converges to \( x \) by The Squeeze (which you did prove in an earlier problem set).

3. **When we take limit a rational sequence to approach a irrational number \( \alpha \), do we need to prove that the limit exists?**

   I’m not sure I understand. If you don’t prove the limit exists, how do you know the sequence approaches the proposed irrational number limit in the first place?

4. **Can we assume that \( \text{dom}(f) \) is the natural domain?**

   Ross defines the function for all real numbers, so the domain of \( f \) is \( \mathbb{R} \).
Can we use the fact that the irrationals are also dense in \( \mathbb{R} \), and thus for every real number, we can find a sequence of irrational numbers that converge to it, even though we haven’t explicitly proven it in class?

If we did it in class, you can cite it. If we didn’t, and it seems nontrivial (since you asked this question, I suspect you think it is nontrivial!), you should at least give some quick reasoning.

Here’s a hint: Since the product of two rational numbers is always rational, the product of an irrational number and a nonzero rational number must be irrational (why?). Thus, for example, \( \left( \frac{1}{\sqrt{n}} \right)_{n \geq 1} \) is a sequence of irrational numbers that approaches zero—that is, there are arbitrarily small positive irrational numbers. How do you use this to prove density of \( \mathbb{R} - \mathbb{Q} \)? If you have a rational number \( r \) such that \( x < r < y \), then, let \( \iota \) be a positive irrational number smaller than \( y - r \). We have \( x < r < r + \iota < r + (y - r) = y \), and \( r + \iota \) is irrational (why?).

8. Maybe I’m not reading carefully enough, but where is \( [m] \) defined?

\( [n] \) is defined to be the set of all integers \( k \) such that \( 1 \leq k \leq n \) (this is notated \([n] = \{1, \ldots, n\}\) in the assignment), so \([m] \) is the set of all integers \( k \) such that \( 1 \leq k \leq m \). I should have been a bit more careful and said that \( n, m \) are both positive integers here, but I felt it should have been clear by now from context. Guess not? :/

What is 8b. talking about? what does it mean to say HOW MANY FUNCTIONS are there \([m] \to [n] \)?

8b is talking about functions. A function is anything that assigns outputs to inputs. This problem is asking you to count all possible functions with a domain of size \( m \) and a codomain of size \( n \).

It might help to start with a small example, such as counting functions \([2] \to [3] \). To define a function \([2] \to [3] \) you have to assign two values: \( f(1) \) and \( f(2) \). How many choices are there for \( f(1) \)? How many for \( f(2) \)? Once you figure this out, generalize your argument to count functions \([m] \to [n] \).

If you still don’t understand what a function is in this context, read the first few bits of the Wikipedia article on “Functions (mathematics)” which has some helpful diagrams.

Maybe it doesn’t matter and we only need to work with elements of \([n] \)?

It does matter because \( m \) and \( n \) can be different numbers. Don’t just count functions \( \{1, \ldots, n\} \to \{1, \ldots, n\} \), for example.

For counting injections and/or surjections, may we use the inclusion-exclusion principle without its general proof?

Sure, though I am not sure you’ll need it.