Supplemental Problems
Problem 8: Functions, injections, surjections, and bijections

Recall that a function \( f : X \rightarrow Y \) is a rule that assigns for every \( x \in X \) an element \( f(x) \in Y \). \( X \) is called the domain of \( f \), \( X = \text{dom}(f) \), and \( Y \) is called the codomain. The codomain is different from the image (which you may also known as the range):

\[
\text{im}(f) = f(X) = \{ y \in Y : \exists x \in X \ f(x) = y \}
\]

The image must be a subset of the codomain, but it need not be equal to it.

- \( f \) is called injective (or one-to-one) if

\[
\forall x_1, x_2 \in X : f(x_1) = f(x_2) \Rightarrow x_1 = x_2
\]

In short, \( f \) is injective if two different inputs always map to different outputs.

- \( f \) is called surjective (or onto) if the image and codomain are equal. That is, if

\[
\forall y \in Y, \exists x \in X : f(x) = y
\]

- \( f \) is called bijective if it is both injective and surjective.

Bijective functions are important because they are exactly the functions that have an inverse: If \( f : X \rightarrow Y \) is a bijection, there exists a (unique) function \( f^{-1} : Y \rightarrow X \) such that

\[
\forall x \in X, \forall y \in Y : f(x) = y \Leftrightarrow f^{-1}(y) = x
\]

Problems below:

a. Prove that a function between two finite sets \textbf{of the same cardinality} is injective iff it is surjective.
b. Let \([n] = \{ k \in \mathbb{Z} : 1 \leq k \leq n \} = \{1, \ldots, n\}\). How many functions are there \([m] \rightarrow [n]\)?

c. Suppose \([m] \leq [n]\). How many injections are there \([m] \rightarrow [n]\)? How about if \([m] > [n]\)?

d. Suppose \([m] \geq [n]\). How many surjections are there \([m] \rightarrow [n]\)? How about if \([m] < [n]\)?

e. Using a computer, plot the following data:

\[
\left( n, \log \left( \frac{\text{number of bijections} [n] \rightarrow [n]}{\text{number of functions} [n] \rightarrow [n]} \right) \right)
\]

for \(n = 1, 2, \ldots, 9, 10\). What do your data suggest? We’ll prove this suggested relationship in a later homework, maybe.

**Problem 9: Cardinality**

The **cardinality** of a set is the number of elements in that set. The cardinality of \(X\) is denoted \(|X|\). For a finite set, it is obvious how to compute cardinality: You count the elements: \(|\emptyset| = 0\), \(|\{1, 2, 3\}| = 3\), and so on.

What about infinite sets? As we saw briefly in class (using Cantor’s diagonalization argument), it is not enough to say that a set is infinite, because there are different sizes of infinite cardinals. Here’s how we compare sizes of sets in general:

- We say that \(|X| \leq |Y|\) if there exists an **injective** function \(X \rightarrow Y\).
- We say that \(|X| \geq |Y|\) if there exists a **surjective** function \(X \rightarrow Y\), and
- We say that \(|X| = |Y|\) if there exists a **bijective** function \(X \rightarrow Y\).

The cardinality of the natural numbers, which happens to be the smallest infinite cardinal, is denoted \(\aleph_0\) (aleph-null). This is also called **countable infinity**. A set \(X\) is called **countable** if \(|X| \leq \aleph_0\), and it is called **countably infinite** if \(|X| = \aleph_0\). A set \(X\) is called **uncountable** or **uncountably infinite** if \(|X| > \aleph_0\).

a. Though he does not state it in these terms, Ross proves that the set of rational numbers is countably infinite. Where is this in the textbook? Explain.

*Hint: A function \(s : \mathbb{N} \rightarrow \mathbb{R}\) is also called a sequence.*

If \(X\) is a set, let \(\mathcal{P}(X)\) denote its power set: The set of all subsets of \(X\). For example,

\[
\mathcal{P}(\emptyset) = \{\emptyset\}, \quad \mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}
\]

It’s important to remember that \(\{1, 1\} = \{1\}\) and \(\{1, 2\} = \{2, 1\}\) because sets don’t care about repeats or order, only membership. Also, while \(|\emptyset| = 0\), \(|\{\emptyset\}| = 1\).

b. Prove that if \(X\) is a **finite** set, then \(|\mathcal{P}(X)| = 2^{|X|}\).

*Hint: Here’s a clever way to do this—though maybe not the easiest. Every subset \(S \subseteq X\) comes with an “indicator function” \(i_S : X \rightarrow \{0, 1\}\) defined by \(i_S(x) = 1\) if \(x \in S\) and \(i_S(x) = 0\) if \(x \notin S\). That is, the number of subsets of \(X\) has the same cardinality as the number of functions \(X \rightarrow \{0, 1\}\). Now, use a previous supplemental problem.*
c. Prove that for any set $X$, $|X| < |\mathcal{P}(X)|$. Here’s how: Suppose for contradiction that $|X| \geq |\mathcal{P}(X)|$. Then there is a surjection $f : X \to \mathcal{P}(X)$. Now, define

$$Y = \{ x \in X : x \notin f(x) \}$$

But this set $Y$ causes some nasty trouble! Why? Since $Y \in \mathcal{P}(X)$ and $f$ is surjective, there is $y \in X$ such that $f(y) = Y$. Finally, consider the question of whether $y \in Y$ or $y \notin Y$.

**Problem 10: The difference set of the Cantor set is $[-1, 1]$**

Recall the Cantor set, constructed in class and in Ross in §13 (which you don’t have to read). In class I mentioned that the Cantor set can be characterized as all numbers in $[0, 1]$ that have a base 3 expansion not containing any 1s digits (for example, $1 = 0.222\ldots$ in base 3). That is,

$$C = \left\{ \sum_{n=1}^{\infty} d_n3^{-n} : d_1, d_2, d_3, \ldots \in \{0, 2\} \right\}.$$  

The Cantor set is closed. You may use these facts without proving them.

Another fact I mentioned in class is that 

$$[-1, 1] = \{ x - y : x, y \in C \}$$

That is, every element of the interval $[-1, 1]$ can be expressed as $x - y$ for some $x, y \in C$. You are going to prove this theorem, in steps!

a. For every integer $m \geq 0$ Let

$$I_m = \left[ \frac{3^m+1}{2}, \frac{3^{m+1}-1}{2} \right] \cap \mathbb{Z}$$

Prove that every positive integer is contained in exactly one $I_m$ and that $3^m \in I_m$.

b. Prove the following statement:

Every positive integer $n$ can be expressed in the form

$$n = \sum_{k=0}^{m} \sigma_k3^k$$

where $m$ satisfies $n \in I_m$, $\sigma_k \in \{-1, 0, 1\}$ for $k = 0, \ldots, m$, and $\sigma_m = 1$.

For example, $17 = 27 - 9 - 1 = 3^3 - 3^2 - 3^0$.

*Hint: Use strong induction. Treat $n = 3^m$ and $n \neq 3^m$ separately.*

c. Prove that if $a, b \in \mathbb{R}$ with $a < b$, then there exists an even integer $d$ and an integer $m \geq 0$ such that $a \leq \frac{d}{3^m} < b$. *Hint: Use the Really Obvious Lemma.*

d. Prove that if $S$ is closed and bounded then $\{ x - y : x, y \in S \}$ is also closed and bounded. *Hint: Bolz.–Wei.*

e. Let $C$ denote the Cantor set. Prove that $\{ x - y : x, y \in C \}$ contains every number in $[-1, 1]$ of the form $\frac{d}{3^m}$ for $d$ even and $m \geq 0$.

f. Finally, prove that $\{ x - y : x, y \in C \} = [-1, 1]$. 