Problem Set 4: Q&A

Homework Questions

1. *How would the enumeration of the set \( \mathbb{Q} \) look? Does it matter?*

   There are many, many, MANY, enumerations of \( \mathbb{Q} \). You can’t pick any particular one. Your argument should work for *any* enumeration of \( \mathbb{Q} \). A particularly cool one to consider is the *Calkin–Wilf* sequence (which has its own wikipedia article).

   *Can we construct a sequence?*

   You should explain how, given an arbitrary enumeration of \( \mathbb{Q} \), one can construct an unbounded, monotonically increasing subsequence.

   Your solution should start out “Let \((r_n)_n\) be an enumeration of \( \mathbb{Q} \),” and from that sequence \((r_n)_n\), you want to derive some subsequence that tends to +\( \infty \).

   Here’s a hint: Every positive integer will appear as a term in the sequence, but be careful: they may not appear in order.

   *Can we use the fact the \((r_n)_n\), the enumeration of rational numbers, is unbounded above? If so, this question is direct result of thm 11.2 (ii). Or we need to prove that \((r_n)_n\) is unbounded above before using thm 11.2(ii)?*

   You would need to prove that \((r_n)_n\) is unbounded above. This may be somewhat less obvious than you think. To show it is unbounded, you will essentially need to show that there is an unbounded subsequence, but then you have solved the problem! Maybe... just do the problem! :)

   *Can we use example 3 on page 70?*

   You can, but it probably won’t help you too much, because this is only one of many possible enumerations of \( \mathbb{Q} \).

   *Can we use example 7 or example 11 on page 73?*

   No, come on! Ross claims the content of Problem 1 in this example, without proof, but in Example 7 says that the proof is left to you. Simply stating “Ross said so in Example 7 [or 11], [without proof!]” will net you a zero on this problem and a dirty look from yours truly. :)

7. *When showing \( \sum a_n \) is a geometric series, can we assume that \( n \) starts at 0?*

   It doesn’t have to. Though you’re correct that “the geometric series” usually refers to the one that starts at 0, more generally we say an infinite series is geometric if there exists a nonzero \( r \in \mathbb{R} \) such that the \((n + 1)\)th term is always \( r \) times the \( n \)th term. That is, any series that looks like

   \[
   \sum_{n=k}^{\infty} a_n r^n
   \]

   can be called a geometric series.

   *Should we prove the inferred recursive formula \( a_{n+1} = Ca_n \) by induction, or can we use as is?*

   No need to prove this as it is fairly obvious.
9. Problem 9 is in our hearts. Problem 10a was the friends we made along the way. (These questions do not exist—Problem 9 will be free points, and Problem 10a will not be graded, since it doesn’t exist.)

10. There may be some technical difficulty with this problem, so just do part (b). (The reordered series will still converge to $\frac{1}{2} \log(2) \neq \log(2)$; it’s just less obvious than I think it is to bound the reordered series.)