Final Practice Questions (Post-Midterm Material)

IMPORTANT: The final will cover all of the material in this course. It will be twice as long as the midterm, with roughly half of the exam covering the first part of the course, and half covering the second part. The questions below only cover the second half or “post-midterm” material. (Refer to the practice midterm and actual midterm for practice with the “pre-midterm” material).

1. Let $\alpha$ be the number whose binary expansion is 0.011101110111…2 (the 0111 repeats forever). Find integers $p, q \geq 1$ such that $\alpha = p/q$.

The point is to write the binary expansion as a geometric series, and then apply the geometric series formula $\sum_{n=0}^{\infty} r^{-n} = \frac{1}{1-r}$ valid when $|r| < 1$:

$$0.011101110111…2 = \left(\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right) + \left(\frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7}\right) + \left(\frac{1}{2^{10}} + \frac{1}{2^{11}} + \frac{1}{2^{12}}\right) \cdots$$

$$= \frac{7}{16} + \frac{7}{16} + \frac{7}{16} \cdots$$

$$= \frac{7}{16} \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n$$

$$= \frac{7}{16} \cdot \frac{1}{1 - \frac{1}{16}} = \frac{7}{16} \cdot \frac{16}{15} = \frac{7}{15}$$

2. Determine whether the series in Ross, Exercise 14.1.a–e converge or diverge, and justify your answer (stating which test you used to determine each is enough).

a. Converges by ratio test.
b. Converges by ratio test.
c. Converges by ratio test.
d. Diverges by ratio test.
e. Converges by comparison with $\sum \frac{1}{n^2}$ (using $0 \leq \cos^2 n \leq 1$).

3. Let $f : X \rightarrow \mathbb{R}$ be a function (where $X$ is a subset of $\mathbb{R}$). Write out the statements “$f$ is continuous on $X$” and “$f$ is uniformly continuous on $X$” in logical notation, and then briefly explain the practical difference between these statements.

$f$ is continuous on $X$ if

$$\forall x \in X, \forall \epsilon > 0, \exists \delta > 0, \forall y \in X : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

$f$ is uniformly continuous on $X$ if

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in X : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$
Formally, the distinction is the order of the quantifiers: $\forall \epsilon \forall x \exists \delta \forall y$ changes to $\forall \epsilon \exists \delta \forall x \forall y$. Practically speaking, this translates to the following difference: If $f$ is uniformly continuous, then for every $\epsilon$, the same $\delta$ must work at every point in $X$.

4. Let $P(x)$ be a polynomial. Suppose that $P'(x)$ has exactly one real root (there is a unique $a \in \mathbb{R}$ such that $P'(a) = 0$). Which of the following must be true?

i. $P$ has at least one real root.
ii. $P$ has at most one real root.
iii. $P$ has even degree.
iv. None of the above.

The answer is “None of the above.” Here are counterexamples to the other three statements: (i.) $x^2 + 1$ has derivative $2x$, which has exactly one real root $x = 0$, but $x^2 + 1$ has no real roots. (ii.) $x^2 - 1$ also has derivative $2x$, but it has two real roots. (iii.) $x^3$ has odd degree, but $3x^2$ has only one real root.

5. Let $X, Y \subseteq \mathbb{R}$ and let $f : X \to Y$ be a function with an inverse $f^{-1} : Y \to X$:

$$[\forall x \in X : f^{-1}(f(x)) = x] \land [\forall y \in Y : f(f^{-1}(y)) = y]$$

Suppose that $f$ is differentiable on $X$ and that $f^{-1}$ is be differentiable on $Y$.
Derive a formula for $(f^{-1})'(y)$ in terms of $f, f', f^{-1}, y$.
We apply the chain rule to $f(f^{-1})(y) = y$:

$$f'(f^{-1}(y))(f^{-1})'(y) = 1$$

so $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$.

6. For each integer $n \geq 1$, let $f_n : [-1, 1] \to [0, 1]$ be the function defined by $f_n(x) = 1 - x^{2n}$.
Let $f : [-1, 1] \to [0, 1]$ be the function defined by $f(x) = \lim_{n \to \infty} f_n(x)$ (this limit always exists, and is in $[0, 1]$).

a. Which of the following statements is/are true?

   i. Every function $f_n$ is continuous.
   ii. The function $f$ is continuous.
   iii. Every function $f_n$ is integrable.
   iv. The function $f$ is integrable.
   v. None of the above.

   We have

   $$f(x) = \begin{cases} 
   0 & \text{if } x = \pm 1 \\
   1 & \text{otherwise.}
   \end{cases}$$

   (i.) is true, but (ii.) is false. (iii.) and (iv.) are both true.

b. If you circled (iv.) above, what is the value of $\int_{-1}^{1} f$?

   $\int_{-1}^{1} f = 2$. 
7. Let \( g : [a, b] \to \mathbb{R} \) be a bounded function. If \( g \) and/or its derivative have certain properties, the fundamental theorem of calculus guarantees \( \int_a^b g' = g(b) - g(a) \). What are the required properties?

We must have \( g \) continuous on \([a, b]\) and differentiable on \((a, b)\), and we must have \( g' \) integrable on \([a, b]\).

8. Let \( Z = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\} \) and let \( \tau = -1 + \sqrt{2} \).

a. Prove that for all integers \( n \geq 1 \) we have \( \tau^n \in Z \).

This is an easy induction: Clearly \( \tau^1 \in Z \). Now, suppose \( \tau^n \in Z \) with \( \tau^n = a + b\sqrt{2} \).

\[
\tau^{n+1} = \tau \cdot \tau^n = (-1 + \sqrt{2})(a + b\sqrt{2}) = -a + a\sqrt{2} - b\sqrt{2} + 2b = (-a + 2b) + (a - b)\sqrt{2}
\]

so since \(-a + 2b\) and \(a - b\) are both integers, \( \tau^{n+1} \in Z \) as well.

b. We have \( \lim(\tau^n) = 0 \). Why? (Just give a reason, no proof is necessary.)

Because \(-1 + \sqrt{2} < 1\).

c. Prove that \( Z \) is dense in \( \mathbb{R} \): That is, if \( x, y \in \mathbb{R} \) with \( x < y \), then there exists \( z \in Z \) such that \( x < z < y \).

Let \( x, y \in \mathbb{R} \) with \( x < y \). Since \( \lim(\tau^n) = 0 \), there is \( n \) such that \( \tau^n < y - x \). That is, \( \frac{y - x}{\tau^n} > 1 \). By the ROL, there exists an integer \( N \) such that \( \frac{x}{\tau^n} < N < \frac{y}{\tau^n} \), and so \( x < N\tau^n < y \). Since \( \tau^n \in Z \) we clearly have \( N\tau^n \in Z \) (since \( N(a + b\sqrt{2}) = (Na) + (Nb)\sqrt{2} \)).

d. Let \( f : [0, 1] \to \mathbb{R} \) be the following function:

\[
f(x) = \begin{cases} 
1 & \text{if } x \in Z, \\
0 & \text{otherwise.}
\end{cases}
\]

Prove that \( f \) is not integrable.

**Hint:** Given that \( \sqrt{2} \) is irrational, under what conditions is \( a + b\sqrt{2} \) rational? \((a, b \in \mathbb{Z})\)

The only rational numbers in \( Z \) are integers: If \( b \neq 0 \), then \( b\sqrt{2} \) is irrational, so \( a + b\sqrt{2} \) is irrational (rational plus irrational is irrational; this follows from the fact that \( \mathbb{Q} \) is closed under addition).

Let \( I = [a, b] \) be any subinterval of \([0, 1]\) with \( a < b \). Since \( \mathbb{Q} \) and \( Z \) are both dense, there exist \( r \in \mathbb{Q} - \{0, 1\} \) and \( z \in Z \) such that \( r, z \in [a, b] \); note that \( r \notin Z \) by the blurb above. But then \( \inf_{x \in I} f(x) = 0 \) and \( \sup_{x \in I} f(x) = 1 \). Thus, \( \int_0^1 f = 0 \) but \( \int_0^1 f = 1 \).

9. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function such that \( f \) is differentiable on \((a, b)\). Prove that if \( f \) achieves its maximum at \( x \in [a, b] \), then \( x = a, x = b, \) or \( f'(x) = 0 \).

10. Prove that if \( f : [a, b] \to \mathbb{R} \) is a continuous function, then \( f \) is integrable.

These are in Ross.

**Note:** Above, “integrable” always means “Riemann/ Darboux integrable.”