Failure to abide by the instructions below will be considered a violation of the Stanford Honor Code (if applicable to you) and the expectations of academic integrity as described in the Stanford Summer Session Program Handbook:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone or any other electronics during the exam for any reason.
- You may not look at anyone else's solutions.
- You may not communicate with anyone other than Dr. Schaeffer during the exam.
- You may not sit directly adjacent to any other student.

I understand and agree to the above.

Signature: ______________________________________________________

Instructions: Complete Problems 1–17. A list of theorems (possibly abbreviated or with omissions) is on the next page.

GOOD LUCK!
List of selected results from class/book

- WO: Well-ordering property
- BWO: Bounded well-ordering property
- NT, A1–A4, M1–M4, DL: Field axioms
- O1–O5: Order axioms
- Theorem 3.1: Consequences of field axioms
- Theorem 3.2: Consequences of order axioms
- Theorem 3.5: Properties of absolute value
- Theorem 3.6: Triangle inequality
- —: Uniqueness of identities and inverses in a field
- —: \((\forall \epsilon > 0 : x \leq y + \epsilon) \Rightarrow x \leq y\)
- C: Completeness axiom
- Theorem 4.6: Archimedean property
- ROL: Really Obvious Lemma
- Theorem 4.7: Density of \(\mathbb{Q}\) in \(\mathbb{R}\)
- Exercise 8.5: Squeeze theorem
- Exercise 8.9: Bounds on the terms of a convergent sequence bound the limit
- Theorem 9.1: Convergent sequences are bounded
- Theorems 9.2–9.5, 9.9, 9.10: Sequence limit theorems
- Theorem 9.7: Basic limits
- Theorem 10.2: Bounded monotone sequences converge
- Theorem 10.7: Relationship between \(\lim, \limsup, \liminf\)
- Theorem 10.11: Relationship between Cauchy sequences and convergent sequences
- Section 11: Subsequences and subsequential limits
- Section 14: Series convergence tests, geometric sum and series formulas
- Section 15: \(p\)-series, integral test, alternating series test
- Theorems 17.3 and 17.4: Behavior of continuous functions under operations
- Theorems 18.1 and 18.2: Extreme and intermediate value theorems
- Section 19: Uniform continuity
- Section 20: Limits
- Theorem 28.2: Continuous functions are differentiable
- Theorem 28.3: Linearity of the derivative, product and quotient rules
- Theorem 28.4: Chain rule
- Theorem 29.1: Critical points theorem
- Theorem 29.2: Rolle’s theorem
- Theorem 29.4: Functions with derivative equal to zero
- Theorem 32.2 and 32.3: Darboux sum theorems
- Theorem 32.4: Lower integral \(\leq\) upper integral
- Theorem 32.5: Cauchy criterion for integrability
- Theorem 32.9: Equivalence of Darboux and Riemann integrals
- Theorems 33.1 and 33.2: Monotonic or continuous functions are integrable
- Theorems 33.3–5: Basic properties of the integral
- Theorem 33.6: Breaking apart integrals
- —: Integral endpoint conventions
- Theorem 33.9: IVT/MVT for integrals
- Theorems 34.1 and 34.3: Fundamental theorem of calculus

Generally, you are allowed to use any result we’ve covered (up to and excluding the one you are currently proving), unless otherwise mentioned.
1. a. Let \((s_n)_{n=0}^\infty\) be a sequence.

Write down a logical statement that translates to “the sequence \((s_n)_n\) converges.”

b. Write down the negation of the statement you wrote in (a), in logical notation, without \(\neg\).

2. a. Which of the statements below is/are true if we fill each _____ with rational number(s)?

   Circle the correct answer(s).

   i. If \(a\) and \(b\) are _____ such that \(a < b\) there exists \(r \in \mathbb{Q}\) such that \(a < r < b\).

   ii. If \((s_n)_n\) is an increasing sequence of _____ that is bounded above, it converges to a _____.

   iii. If \(\alpha\) is a _____ and \(m\) is a positive integer, there is a _____ \(\beta\) such that \(\beta^m = \alpha\).

   iv. If \(a\) and \(b\) are two distinct positive _____ there is a positive integer \(n\) such that \(na > b\).

   v. Every sequence \((s_n)_n\) of _____ has a monotone subsequence.

   vi. If \(a\) is a _____ then there exists an integer \(m\) such that \(m \leq a < m + 1\).

   vii. Every Cauchy sequence \((s_n)_n\) of _____ converges to \(a(n)\) _____.

   viii. None of the above.

b. Is statement (vii.) above TRUE or FALSE if we fill both _____s in with integer(s)?
3. For each of the following statements, say whether the statement is TRUE or FALSE. If you wrote TRUE, no further justification is necessary. If you wrote FALSE, write down a counterexample (a sequence or sequences for which the statement is false):

a. If \((s_n)_n\) converges and \((s_n \cdot t_n)_n\) converges, then \((t_n)_n\) must also converge.

b. If \((s_{2n})_n\) and \((s_{2n+1})_n\) both converge to \(s\), then \((s_n)_n\) also converges to \(s\).

c. If \(|s_n|)_n\) converges, then \((s_n)_n\) converges.
d. If \((s_n)_n\) converges and \(s_n \neq 0\) for all \(n\), then \(\left(\frac{s_n}{s_n}\right)_n\) converges.

4. a. Let \(S \subseteq \mathbb{R}\). Explain what it means for the set \(S\) to be closed.

b. State the Bolzano–Weierstrass theorem.
5. Consider the following diagram, called the Calkin–Wilf tree, a rooted binary tree whose nodes are labeled with positive rational numbers:

In the above diagram, we have $\frac{1}{1}$ at the root of the tree (the topmost vertex); if a node is labeled $\frac{a}{b}$, then its children (i.e., the two nodes below it in the diagram) are $\frac{a}{a+b}$ on the left and $\frac{a+b}{b}$ on the right.

Let $(q_n)_n$ be the sequence of positive rational numbers obtained by reading the rows of the Calkin–Wilf tree from left to right, starting with $q_1 = \frac{1}{1} = 1$:

$$(q_n)_{n=1}^{\infty} = \left(\frac{1}{1}, 1, \frac{1}{2}, 1, \frac{2}{1}, 1, \frac{3}{2}, 2, \frac{2}{3}, 3, 1, 4, \frac{1}{2}, 5, \frac{3}{1}, 2, 6, \frac{4}{3}, 3, 7, 1, \frac{4}{2}, \ldots \right)$$

It is a fun and intriguing fact that every positive rational number will appear exactly once as a term of the sequence $(q_n)_n$ (and in fact, it will always appear as a fraction in lowest terms).

a. Using the information above, what is the set of subsequential limits of $(q_n)_n$ equal to?
b. Write down a subsequence of \((q_n)_n\) that converges to 1.

c. Let \(n_k = 2^k - 1\). What is \(\lim_{k \to \infty} (q_{n_k})\) equal to?
d. (OPTIONAL!!! FOR EXTRA CREDIT.)

Consider the subsequence \((q'_n)_n\) of \((q_n)_n\) obtained as follows: Let \(q'_1 = \frac{1}{1}\). If \(n \geq 1\) is odd, let \(q'_{n+1}\) be the left child of \(q'_n\). If \(n \geq 2\) is even, let \(q'_{n+1}\) be the right child of \(q'_n\). That is, \((q'_n)_n\) is obtained by starting at \(\frac{1}{1}\) and then following the path that alternates left and right:

\[ \frac{1}{1} \rightarrow \frac{1}{2} \rightarrow \frac{1}{3} \rightarrow \frac{3}{5} \rightarrow \ldots \]

\((q'_n)_n\) has exactly two subsequential limits. What are they?

*A complete, formal proof is not required, but show your work.*
6. **Prove** that $a \cdot 0 = 0$ for all real numbers $a$.

   You may only cite the axioms of the real numbers, plus Theorem 3.1.i: If $a + c = b + c$, then $a = b$. 
7. Let \((s_n)_n\) and \((t_n)_n\) be convergent sequences. \textbf{Prove} that \((s_n + t_n)_n\) also converges, and

\[
\lim(s_n + t_n)_n = \lim(s_n)_n + \lim(t_n)_n
\]

You may only cite results in Sections 1–8 of the textbook.
8. A polynomial of degree \( n \) is a function of the form

\[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

where \( a_n \neq 0 \). A polynomial of degree 0 is therefore a nonzero constant function (the zero constant function is a polynomial of degree \(-\infty\), by convention—this is not important to the problem).

A real root of the polynomial \( P \) is a value \( r \in \mathbb{R} \) such that \( P(r) = 0 \).

Prove that if \( P(x) \) is a polynomial of degree \( n \), then \( P(x) \) has at most \( n \) distinct real roots. You may use any theorem we proved in this course.
9. a. Let \((a_n)_{n=0}^{\infty}\) be the sequence defined by \(a_0 = 1\), then

\[
a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } n \text{ is even,} \\ \frac{2a_n}{3} & \text{if } n \text{ is odd.} \end{cases}
\]

What are \(\lim \inf_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|\) and \(\lim \sup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|\) equal to?

b. Which of the following is/are true about the infinite series \(\sum_{n=0}^{\infty} a_n\)?

\(\text{Circle all true statements.}\)

i. It converges absolutely.

ii. It converges conditionally.

iii. It diverges.

iv. The limit of the partial sums lies in the interval \([2, 3]\).

v. The ratio test gives no information, when applied to this series.

vi. The root test gives no information, when applied to this series.

vii. None of the above.

10. For which values of \(r \in \mathbb{R}\) will the series \(\sum_{n=0}^{\infty} \frac{r^n}{n + 1}\) converge (either absolutely or conditionally)?
11. We gave three different definitions of what it means for a function $f : X \to \mathbb{R}$ to be continuous at a point $x \in X$. Write down two of them:

a. $f$ is continuous at $x$ if...

b. $f$ is continuous at $x$ if...

12. Let $f : [a, b] \to \mathbb{R}$ be continuous. Which of the following must be true?

Circle all true statements.

i. $f$ is bounded on $[a, b]$.
ii. $f$ achieves a maximum value and a minimum value on $[a, b]$.
iii. $f$ is uniformly continuous on $[a, b]$.
iv. $f$ is differentiable at some point in $(a, b)$.
v. $f$ is integrable on $[a, b]$.
vi. None of the above.
13. Which of the following functions is/are uniformly continuous?

*Circle the correct answer(s).*

i. \( x \) on \((-\infty, \infty)\).

ii. \( x^2 \) on \((-\infty, \infty)\).

iii. \( \sqrt{x} \) on \((0, \infty)\).

iv. \( \sin(\frac{1}{x}) \) on \((0, \pi)\).

v. \( \frac{1}{x^2} \) on \((0, \infty)\).

vi. \( \frac{1}{x^2} \) on \((0, \infty)\) where \(a > 0\).

vii. \( \frac{1}{1+x^2} \) on \((0, \infty)\).

viii. None of the above.

14. For this problem let \( f \) be “everyone’s favorite function”:

\[
f(x) = \begin{cases} 
0 & \text{if } x \text{ is irrational, and} \\
1/q & \text{if } x \text{ is rational and } x = p/q \text{ is in lowest terms, } q \geq 1.
\end{cases}
\]

Note that \( f(k) = 1 \) for any integer, since \( k/1 \) is a fraction in lowest terms.

a. Which of the following statements are true about \( f \)? *Circle all true statements.*

i. \( f \) is continuous at every rational number.

ii. \( f \) is continuous at every irrational number.

iii. \( f \) is piecewise monotonic.

iv. \( f \) is piecewise continuous.

v. For any \( a, b \in \mathbb{R} \) with \( a < b \), \( f \) is integrable on \([a, b]\).

vi. None of the above.

b. For all \( x \in \mathbb{R} \), let \( F(x) = \lim_{n \to \infty} f(x)^{1/n} \) (this limit always exists). What function is \( F \)?

*It suffices to write down a formula for computing \( F(x) \).*
15. Let $f : (a, b) \to \mathbb{R}$ be unbounded above and unbounded below, but differentiable (and therefore continuous) on $(a, b)$ with $f'(x) > 0$ for all $x \in (a, b)$.

The function $\tan(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ is an example of such a function.

a. Prove that $f$ is injective.

$f$ is injective if for all $x_1, x_2 \in \text{dom}(f)$, we have $f(x_1) = f(x_2)$ only when $x_1 = x_2$.

b. Prove that $f$ is surjective.

$f$ is surjective if for all $y \in \text{codom}(f)$, there is $x \in \text{dom}(f)$ such that $f(x) = y$.

c. Is it possible for $f$ to be uniformly continuous? Briefly explain (but do not give a full proof).
16. a. The fundamental theorem of calculus, part I (FTC, I), says that if a function $g : [a, b] \to \mathbb{R}$ meets certain conditions, then

$$\int_a^b g' = g(b) - g(a)$$

What are the required conditions?

b. In the proof of FTC, I, we have for any partition $P = \{t_0 < t_1 < \cdots < t_{k-1} < t_k\}$,

$$g(b) - g(a) = \sum_{k=1}^{n} [g(t_k) - g(t_{k-1})] = \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1})$$

where $x_k \in (t_{k-1}, t_k)$ for $k = 1, \ldots, n$. What allows us to find these points $x_k$?
c. Finish the proof of the FTC, I.
17. Let $L : (0, \infty) \to \mathbb{R}$ be the function defined as follows:

$$L(x) = \int_1^x \frac{1}{t} \, dt$$

a. $L$ is continuous and differentiable on $(0, \infty)$. Without going into the details, what theorem plays a major role in proving that $L$ has these properties?

b. Fix $y \in (0, \infty)$ and let $g(x) = L(xy) - L(x) - L(y)$.

Note that $g(1) = 0$. Show that $g(x) = 0$ for all $x \in (0, \infty)$.

Hint: Compute $g'(x)$. Since $y$ is fixed, you may treat it as a constant. You may NOT use logarithm rules you know from calculus—we are proving those rules.

\* From (b) we may conclude that $L(xy) = L(x) + L(y)$ for all $x, y \in (0, \infty)$ and (by induction) that $L(x^n) = nL(x)$ for any $x \in (0, \infty)$ and any integer $n \geq 0$. 

c. Prove that \( L(1/x) = -L(x) \)

d. You may know the limit formula \( \lim_{n \to \infty} (1 + \frac{1}{n})^n = e \) (yes, that \( e \)). Prove that \( L(e) = 1 \).

*Hint: We have \( \frac{1}{n+1} \leq L(n+1) - L(n) \leq \frac{1}{n} \) for \( n \geq 1 \). Briefly explain why, if you use this fact.*
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