Math 115 - Summer 2017, Schaeffer
Lecture 3

Axioms of \( \mathbb{R} \)

(NT) \( 0 \neq 1 \)

(A1) Addition is associative

(A2) Addition is commutative

(A3) 0 is the identity element for +.

(A4) Every element has an additive inverse.

\( \forall x \in \mathbb{R} \exists y \in \mathbb{R} \; x + y = 0 \)

(M1) Multiplication is associative

(M2) Multiplication is commutative

(M3) 1 is the identity for .

(M4) Every nonzero element has a multiplicative inverse.

\( \forall x \in \mathbb{R} \; x \neq 0 \Rightarrow \exists y \in \mathbb{R} \; xy = 1 \)

(DL) Multiplication distributes over addition.

(O1) Dichotomy of \( \leq \): \( \forall x, y \in \mathbb{R} \; x \leq y \vee y \leq x \)

(O2) Antisymmetry of \( \leq \): \( \forall x, y \in \mathbb{R} \)

\( (x \leq y) \wedge (y \leq x) \Rightarrow x = y \)

(O3) Transitivity of \( \leq \): \( \forall x, y, z \in \mathbb{R} \)

\( (x \leq y) \wedge (y \leq z) \Rightarrow x \leq z \)

(04) Additive translation

\( \forall x, y, z \in \mathbb{R} \; x \leq y \Rightarrow x + z \leq y + z \)

(05) Nonzero scaling

\( \forall x, y, z \in \mathbb{R} \; x \leq y \wedge z > 0 \Rightarrow xz \leq yz \)

(C) Completeness (tomorrow)

Axioms valid in the \( \# \) systems

\( \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \)

- NT - DL are the "field axioms"
  A field is an algebraic structure
  where the arithmetic you learned
  in grade school works approx.
  like you expect. (or does it?)

- 01-03 are the axioms of a
  "total order." Relxing 01
  yields a "partial order"

- 04 + 05 are added to show that
  + and * cooperate w/ \( \leq \).

- NT-05: axioms of a totally ordered field.

Notes:

- NT - DL are the "field axioms"
  A field is an algebraic structure
  where the arithmetic you learned
  in grade school works approx.
  like you expect. (or does it?)
Proving basic arithmetic from the axioms

**Theorem 3.1**

(i) Addition is cancellative; (\( \Rightarrow -a \) is unique)

(ii) \( \forall a \in \mathbb{R} \quad a \cdot 0 = 0 \)

(iii) \( \forall a, b \in \mathbb{R} \quad (-a) \cdot b = -ab \)

(iv) \( \forall a, b \in \mathbb{R} \quad (-a) \cdot (-b) = ab \) \( (\Rightarrow a \) is unique \)

(v) \( \forall a, b, c \in \mathbb{R} \quad c \neq 0 \land (ac = bc) \Rightarrow a = b \)

(vi) \( \forall a, b \in \mathbb{R} \quad ab = 0 \Rightarrow a = 0 \lor b = 0 \)

**Theorem 3.2**

(i) \( \forall a, b \in \mathbb{R} \quad a \leq b \Rightarrow -b \leq -a \)

(ii) \( \forall a, b, c \in \mathbb{R} \quad a \leq b \land c \leq 0 \Rightarrow bc \leq ac \)

(iii) \( \forall a, b \in \mathbb{R} \quad a \geq 0 \land b \geq 0 \Rightarrow ab \geq 0 \)

(iv) \( \forall a \in \mathbb{R} \quad a^2 \geq 0 \)

(v) \( \forall a \in \mathbb{R} \quad 0 < 1 \)

(vi) \( \forall a \in \mathbb{R} \quad a > 0 \Rightarrow a^{-1} > 0 \)

(vii) \( \forall a, b \in \mathbb{R} \quad 0 < a < b \Rightarrow 0 < b^{-1} < a^{-1} \)

**Remarks**

You may be familiar w/ \( \mathbb{C} \), the field of complex #s:

\[ \mathbb{C} = \{ a + b\sqrt{-1} : a, b \in \mathbb{R} \} \]

Since \( \sqrt{-1} \in \mathbb{C} \) and \( (\sqrt{-1})^2 = -1 \), there is no way to order \( \mathbb{C} \) (and have \( 01\rightarrow05 \) valid)!

However, \( \mathbb{C} \) does have other nice properties, which we won't get into.

**Absolute value and distance**

\( \forall a \in \mathbb{R} \) define \( |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} \)

**Theorem 3.5** \( \forall a, b, c \in \mathbb{R} \)

(i) \( |a| \geq 0 \)

(ii) \( |ab| = |a||b| \)

(iii) \( |a+b| \leq |a| + |b| \)

\( d(a, b) = |a - b| \). \( \text{Thm 3.6} \quad d(a, c) \leq d(a, b) + d(b, c) \)

**Remark:** All of these valid for \( \mathbb{Q} \), as well!