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# ON THE BEHAVIOUR OF A LONG CASCADE OF LINEAR RESERVOIRS

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**Abstract**

This paper describes the limiting asymptotic behaviour of a long cascade of linear reservoirs fed by stationary inflows into the first reservoir. We show that the storage in the $n$th reservoir becomes asymptotically deterministic as $n \to \infty$, and establish a central limit theorem for the random fluctuations about the deterministic approximation. In addition, we prove a large deviations theorem that provides precise logarithmic asymptotics for the tail probabilities associated with the storage in the $n$th reservoir when $n$ is large.

*Keywords:* Storage theory; linear cascade; large deviations; strong approximation; central limit theorem

AMS 1991 Subject Classification: Primary 60K30; 60F05; 60F10
Secondary 60F15

## 1. Introduction

Consider an infinite cascade of reservoirs, in which the reservoirs are placed in 'series' and the output from reservoir $n$ flows (exclusively) into reservoir $n+1$. We assume that at time $t = 0$, each of the reservoirs in the infinite cascade is empty and that we then begin releasing inflow into the first reservoir according to a stochastic process $\Gamma = (\Gamma(t) : t \geq 0)$ in which $\Gamma(t)$ describes the cumulative inflow to reservoir 1 over $[0, t]$. We further require that the reservoirs obey linear release rules, so that the rate at which content is released from reservoir $n$ into reservoir $n+1$ is equal to $\alpha_n S_n(t)$, where $S_n(t)$ is the content stored in reservoir $n$ at time $t$. In this paper, our interest focuses on the asymptotic behaviour of the steady-state of the $n$th reservoir as $n \to +\infty$.

The linear release rule assumption is widely used within the hydrological community; see, for example Nash [9] for a single reservoir, Klemeš and Boruvka [7] and Klemeš *et al.* [8] for a cascade of reservoirs. A diffusion approximation for a more general network of reservoirs with power law release (a generalization of linear release) has been investigated by Glynn and Glynn [5]. Some properties of a cascade of nonlinear reservoirs (subject to certain restrictions) have also been studied recently by Glynn and Glynn [6]. The asymptotics established in this paper can be used to develop both approximations and insight into the temporal/spatial behaviour of long cascades of such reservoirs.

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Historically, there has been a tight connection between the theory of storage systems and that of queueing systems. This paper continues that tradition, by extending to the storage-theoretic setting several results that have been obtained for queueing systems in which a large number of stations are placed in series. An early example of this type was the paper by Vere-Jones [11] in which a long series of infinite-server queues was studied. An infinite-server queue is basically the queueing analogue to our linear reservoir, since the departure rate for such a queue is roughly proportional to the number of customers present. It was shown that if each queue is an infinite-server station with common processing time distribution $F$ (with processing times at individual stations independent), with an arrival process to the first queue that is renewal, then the steady-state queue-length distribution at the $n$th queue becomes Poisson in the limit. As indicated above, we shall establish analogous results for the linear cascade. In particular, we shall show that the steady-state storage at the $n$th reservoir becomes asymptotically constant, and that the corresponding release rate is therefore also asymptotically constant. However, we are also able to obtain much 'finer' results that show, for example, that when $\alpha_n \equiv \alpha$ the stochastic fluctuation of the $n$th reservoir about its steady-state mean, when renormalized by a factor of $n^{1/4}$, converges to a Gaussian r.v. (see Theorem 2). So, regardless of the probabilistic structure of the initial inflow $\Gamma$ to the cascade, the steady-state storage at the $n$th reservoir looks roughly Gaussian, with a standard deviation of order $n^{-1/4}$. In addition, Theorem 3 provides an accompanying large deviations result, that yields logarithmic asymptotics for the tail probabilities associated with the $n$th reservoir.

## 2. Model formulation and basic properties

Let $S_n(t)$ be the (storage) content of the $n$th reservoir at time $t$, and let $R_n(t)$ be the total amount of fluid released from the $n$th reservoir over $[0, t]$. Suppose that $\Gamma(t)$ is the cumulative inflow into the first reservoir over $[0, t]$. Then, the cascade structure of the reservoir system implies that

$$S_{n+1}(t) = S_{n+1}(0) + R_n(t) - R_{n+1}(t) \tag{2.1}$$

for $n \geq 1$, with the 'boundary condition'

$$S_1(t) = S_1(0) + \Gamma(t) - R_1(t). \tag{2.2}$$

The linearity of the release rule means that

$$R_n(t) = \alpha_n \int_0^t S_n(u)\,\mathrm{d}u \tag{2.3}$$

for some positive constant $\alpha_n$ $(n \geq 1)$. From Equations (2.1) and (2.3), it follows that

$$S_{n+1}(t) = \mathrm{e}^{-\alpha_{n+1}t} S_{n+1}(0) + \int_{[0,t]} \mathrm{e}^{-\alpha_{n+1}(t-s)} R_n(\mathrm{d}s). \tag{2.4}$$

where the integral is to be interpreted in a Lebesgue–Stieltjes sense.

Set

$$\beta_n(t) = S_n(0)(1 - \exp(-\alpha_n t)) \quad \text{and} \quad F_n(\mathrm{d}u) = \alpha_n \exp(-\alpha_n u)\mathbf{1}(u \geq 0)\,\mathrm{d}u.$$

Substituting (2.4) into (2.1) yields a recursive relationship for the release processes as a function of $n$:

$$\begin{aligned} R_{n+1}(t) &= \beta_{n+1}(t) + R_n(t) - \int_{[0,t]} \mathrm{e}^{-\alpha_{n+1}(t-s)} R_n(\mathrm{d}s) \\ &= \beta_{n+1}(t) + R_n(t) - \int_{[0,t]} \left(1 - \alpha_{n+1} \int_0^{t-s} \mathrm{e}^{-\alpha_{n+1}u}\,\mathrm{d}u\right) R_n(\mathrm{d}s) \\ &= \beta_{n+1}(t) + \int_{[0,t]} \alpha_{n+1} \int_0^{t-s} \mathrm{e}^{-\alpha_{n+1}u}\,\mathrm{d}u\, R_n(\mathrm{d}s) \\ &= \beta_{n+1}(t) + \int_0^t \int_{[0,t-u]} R_n(\mathrm{d}s)\alpha_{n+1}\mathrm{e}^{-\alpha_{n+1}u}\,\mathrm{d}u \\ &= \beta_{n+1}(t) + \int_0^t R_n(t-u)\alpha_{n+1}\mathrm{e}^{-\alpha_{n+1}u}\,\mathrm{d}u \\ &= \beta_{n+1}(t) + (F_{n+1} \star R_n)(t), \end{aligned} \tag{2.5}$$

where $\star$ denotes the convolution operation. Put $G_{n,k} = F_n \star F_{n-1} \star F_{n-1} \star \ldots \star F_{k+1}$ and $G_n = G_{n,0}$.

Successively back-substituting in (2.5) and taking advantage of (2.2) leads one to the conclusion that

$$R_n = \beta_n + G_{n,n-1} \star \beta_{n-1} + \ldots + G_{n,1} \star \beta_1 + G_n \star \Gamma\,.$$

In particular, if all reservoirs start empty so that $S_n(0) = 0$ for $n \geq 1$, we find that $R_n = G_n \star \Gamma$. Consequently, under the empty reservoir initial condition, (2.1) implies that

$$S_n = H_n \star \Gamma \tag{2.6}$$

with $H_n = G_{n-1} - G_n$ (and $G_0(t) \stackrel{\triangle}{=} \mathbf{1}(t \geq 0)$). It should be noted that (2.6) proves that the storage at the $n$th reservoir, in the initial empty condition, is independent of the order of the first $n-1$ reservoirs; this is a storage process analogue to the fact that the departure process from the $n$th queue in a serial infinite-server system has a distribution that is independent of the order of the first $n$ servers. (This queueing-theoretic result relies on the observation that the departures epochs from the $n$th queue are the arrival epochs to the first queue randomly translated by an independent sequence of convolutions of the first $n$ processing times.)

A reasonable assumption for our inflow process $\Gamma$ is the requirement that $\Gamma$ have stationary increments. Specifically, we will require that:

(A1) $(\Gamma(t) : -\infty < t < \infty)$ is a non-decreasing process possessing stationary increments, with $\Gamma(0) = 0$ and $\mathrm{E}\,\Gamma(1) < \infty$.

Under (A1), (2.6) establishes that

$$\begin{aligned} S_n(t) &= \int_{[0,t]} \Gamma(t-s) H_n(\mathrm{d}s) \\ &\stackrel{\mathcal{D}}{=} \int_{[0,t]} \Gamma(-s) H_n(\mathrm{d}s) - \Gamma(-t) H_n(t)\,, \end{aligned}$$

where $\stackrel{\mathcal{D}}{=}$ denotes 'equality in distribution'. Since $H_n(t) \to 0$ exponentially fast and $\Gamma(t)/t = O(1)$ a.s. (as is evident from the finiteness of $\mathrm{E}\,\Gamma(1)$ and the ergodic theorem), it follows that $\Gamma(-t)H_n(t) \Rightarrow 0$ as $t \to \infty$, yielding

$$S_n(t) \Rightarrow \int_{[0,\infty)} \Gamma(-s)H_n(\mathrm{d}s) \tag{2.7}$$

as $t \to \infty$. The limit r.v. appearing in (2.7) can be expected to correspond to a stationary regime for the $n$th storage process. More precisely, we can prove the following:

**Theorem 1.** *Assume (A1) and let* $S_n^*(t) = \int_{[0,\infty)} \Gamma(t-s)H_n(\mathrm{d}s)$ *for* $n \geq 1$ *and* $t \geq 0$*, Then, for each* $n \geq 1$*,*

$$((S_1^*(t), \dots, S_n^*(t)) : t \geq 0)$$

*is a (strictly) stationary process satisfying* $S_n^*(t) = S_n^*(0) + R_{n-1}^*(t) - R_n^*(t)$ *for* $n \geq 1$ *and* $t \geq 0$*, with* $R_0^*(t) = \Gamma(t)$ *and* $R_n^*(t) = \alpha_n \int_0^t S_n^*(u)\,\mathrm{d}u$*.*

*Proof.* Recall that $H_n(\infty) = 0$. Consequently, the stationary increments structure of $\Gamma$ implies that for each $t \geq 0$,

$$\begin{aligned} S_n^*(t+\cdot) &= \int_{[0,\infty)} [\Gamma(t+\cdot-s) - \Gamma(t)]H_n(\mathrm{d}s) \\ &= \int_{[0,\infty)} [\Gamma(\cdot-s) - \Gamma(t)]H_n(\mathrm{d}s) = S_n^*(\cdot), \end{aligned}$$

proving the stationarity of $S_n^*$; the stationarity of the joint $n$-dimensional process may be similarly handled. To establish that the joint process $(S_1^*, \dots, S_n^*)$ has the appropriate dynamics, the key step is to show that

$$S_{n+1}^*(0) + \alpha_n \int_0^t S_n^*(u)\,\mathrm{d}u - \alpha_{n+1} \int_0^t S_{n+1}^*(u)\,\mathrm{d}u \tag{2.8}$$

defines the r.v. $S_{n+1}^*(t)$. This computation is straightforward and is therefore omitted.

In the next section, we study behaviour of the stationary regime as $n$ tends to infinity.

## 3. A central limit theorem for the stationary regime

In order to obtain asymptotic approximations for the behaviour of the $n$th reservoir, we need to impose some additional regularity hypotheses on the inflow process $\Gamma$.

(A2) For some $\delta \in (0, \frac{1}{2})$ and positive constants $\lambda$, $\sigma$, there exists a probability space supporting the inflow process $\Gamma$ and a standard Brownian motion $(B(t) : -\infty < t < \infty)$ with $B(0) = 0$ such that

$$\Gamma(t) = \lambda t + \sigma B(t) + O(|t|^\delta) \quad \text{a.s. as } |t| \to \infty.$$

We can and will assume that the probability space used in the remainder of this paper is that space guaranteed by (A2). The 'strong approximation' assumption (A2) is satisfied by many weakly dependent stochastic processes, including martingales and additive functionals Markov processes and mixing processes; see Philipp and Stout [10] for details. In terms of practical interpretation, (A2) can be viewed as one means of mathematically asserting that $\Gamma$ satisfies a central limit theorem (CLT). Typically, the constants $\lambda$ and $\sigma^2$ can be identified from the corresponding mean and variance of $\Gamma(\cdot)$ as follows:

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \,\mathrm{E}\,\Gamma(t) \quad \text{and} \quad \sigma^2 = \lim_{t\to\infty} \frac{1}{t} \,\mathrm{var}\,\Gamma(t).$$

For example, relation (3) would hold for Lévy input processes $\Gamma$ under suitable moment conditions on the increments. The parameter $\delta$ appearing in (A2) reflects both the degree of autocorrelation present in $\Gamma$ and the non-Gaussianity of its increments.

To establish a CLT for $S_n^*(\cdot)$, it seems clear that we need to consider the behaviour of the convolution density $h_n$ as $n \to \infty$. In particular, it turns that we shall need a local CLT-type result for the 'signed density' $h_n$; such an approximation is most directly obtained if we assume:

(A3) Assume that there exist two constants $c_1$ and $c_2$ such that

$$0 < c_1 \triangleq \inf\{\alpha_n : n \geq 1\} \leq \sup\{\alpha_n : n \geq 1\} \triangleq c_2 < \infty.$$

Set $m_n = \sum_{i=1}^n 1/\alpha_i$, $v_n = \sum_{i=1}^n 1/\alpha_i^2$, $s_n = v_n^{1/2}$, and let $\varphi(\cdot)$ be the density of an $N(0,1)$ r.v.

**Proposition 1.** *Under (A3),*

$$\alpha_{n+1} v_n h_{n+1}(m_n + s_n x) + x\varphi(x) = o(1)$$

*uniformly in $x$ as $n \to \infty$.*

*Proof.* We show that

$$|\alpha_{n+1} v_n h_{n+1}(m_n + s_n x) - \alpha_{n+1} s_n(\varphi(x) - \varphi(x - 1/(\alpha_{n+1} s_n)))| \tag{3.1}$$

converges to 0 uniformly in $x$ as $n \to \infty$. This will establish the result as it is easily shown (using the boundedness of $\varphi'$) that

$$\alpha_{n+1} s_n(\varphi(x) - \varphi(x - 1/(\alpha_{n+1} s_n))) = -x\varphi(x) + o(1)$$

uniformly in $x$ as $n \to \infty$.

To establish (3) we apply the local CLT to the probability density $s_n g_n(m_n + s_n\cdot)$; this latter density is the density of $(\chi_n - \mathrm{E}\,\chi_n)/\sqrt{\mathrm{var}\,\chi_n}$, where $\chi_n$ is sum of $n$ independent r.v.'s, in which the $j$th summand is exponential with parameter $\alpha_j$. The relevant local CLT can be found on p. 194 of [1].

We are now prepared to state our main result of this section, namely a CLT (in the sense of convergence to a stationary Gaussian process) for the stationary regime $(S_n^*(t) : t \geq 0)$ when $n$ is large.

**Theorem 2.** *Fix $d \geq 1$. Under (A1)–(A3),*

$$s_n^{1/2}(\alpha_n(S_n^*(s_n t) - \lambda/\alpha_n), \ldots, \alpha_{n+d}(S_{n+d}^*(s_n t) - \lambda/\alpha_{n+d})) \\ \Rightarrow \left(\sigma \int_{-\infty}^{\infty} B(t-y) y\varphi(y)\,\mathrm{d}y, \ldots, \sigma \int_{-\infty}^{\infty} B(t-y) y\varphi(y)\,\mathrm{d}y\right)$$

*in $C_{\mathbf{R}^d}[0,\infty)$.*

*Proof.* Set $\hat{\Gamma}(t) = \Gamma(t) - \lambda t$, and observe that for $\epsilon > 0$, (A2) guarantees that

$$
\begin{aligned}
&s_n^{1/2}(\alpha_{n+k}(S^*_{n+k}(s_n t) - \lambda/\alpha_{n+k})) \\
&\quad = \left(\frac{s_n}{s_{n+k}}\right)^{1/2} s_{n+k}^{-1/2} \int_{-(m_{n+k})/(s_{n+k})}^{\infty} \hat{\Gamma}(s_n t - m_{n+k} - s_{n+k} y) \\
&\qquad\qquad \times \alpha_{n+k} v_{n+k} h_{n+k}(m_{n+k} + s_{n+k} y)\, \mathrm{d}y \\
&\quad \overset{\mathcal{D}}{=} \left(\frac{s_n}{s_{n+k}}\right)^{1/2} s_{n+k}^{-1/2} \int_{-(m_{n+k})/(s_{n+k})}^{\infty} \hat{\Gamma}(s_n t - s_n^{1+\epsilon} - s_{n+k} y) \\
&\qquad\qquad \times \alpha_{n+k} v_{n+k} h_{n+k}(m_{n+k} + s_{n+k} y)\, \mathrm{d}y.
\end{aligned}
$$

By (A2), $\hat{\Gamma}(t) = O(|t|)$ a.s. as $t \to -\infty$. Since $\alpha_n v_n h_n(m_n + s_n y) \to 0$ faster than any power of $y$ uniformly in $n$, it is evident that

$$
\int_{|y|>n^\epsilon} \hat{\Gamma}(s_n t - s_n^{1+\epsilon} - s_{n+k} y)\alpha_{n+k} v_{n+k} h_{n+k}(m_{n+k} + s_{n+k} y)\, \mathrm{d}y \to 0 \qquad \text{a.s.}
$$

uniformly in compact $t$-sets. Furthermore, by (A2) and Proposition 1,

$$
\begin{aligned}
&s_{n+k}^{-1/2} \int_{|y|\le n^\epsilon} \hat{\Gamma}(s_n t - s_n^{1+\epsilon} - s_{n+k} y)\alpha_{n+k} v_{n+k} h_{n+k}(m_{n+k} + s_{n+k} y)\, \mathrm{d}y \\
&\qquad = -s_{n+k}^{-1/2} \int_{|y|\le n^\epsilon} (\sigma B(s_n t - s_n^{1+\epsilon} - s_{n+k} y) \\
&\qquad\qquad + O(|s_n t - s_n^{1+\epsilon} - s_{n+k} y|^\delta))(y\varphi(y) + o(1))\, \mathrm{d}y \\
&\qquad = -s_{n+k}^{-1/2} \int_{|y|\le n^\epsilon} \sigma B(s_n t - s_n^{1+\epsilon} - s_{n+k} y) y\varphi(y)\, \mathrm{d}y + o(1) \qquad \text{a.s.}
\end{aligned}
$$

uniformly in compact $t$-sets. Because $\varphi(\cdot)$ decays faster than any power in its tails, it follows that the above quantity equals

$$
\begin{aligned}
&- s_{n+k}^{-1/2} \int_{-\infty}^{\infty} \sigma B(s_n t - s_n^{1+\epsilon} - s_{n+k} y) y\varphi(y)\, \mathrm{d}y + o(1) \qquad \text{a.s.} \\
&\quad = -s_{n+k}^{-1/2} \int_{-\infty}^{\infty} \sigma[B(s_n t - s_n^{1+\epsilon} - s_{n+k} y) - B(-s_n^{1+\epsilon})] y\varphi(y)\, \mathrm{d}y + o(1) \qquad \text{a.s.}
\end{aligned}
$$

uniformly in compact $t$-sets. But the above integral has the same distribution, viewed as a process, as

$$
\sigma \int_{-\infty}^{\infty} B\left(\frac{s_n}{s_{n+k}} t - y\right) y\varphi(y)\, \mathrm{d}y.
$$

But this latter integral converges to

$$
\sigma \int_{-\infty}^{\infty} B(t - y) y\varphi(y)\, \mathrm{d}y
$$

uniformly in $t$, verifying the theorem.

This CLT shows that for $n$ large, we have the following approximation for the process $S_n^*(\cdot)$:

$$S_n^*(t) \overset{\mathcal{D}}{\approx} \frac{\lambda}{\alpha_n} + \frac{\sigma}{s_n^{1/2}\alpha_n} \int_{-\infty}^{\infty} B(t/s_n - y) y\varphi(y)\,\mathrm{d}y,$$

where $\overset{\mathcal{D}}{\approx}$ denotes 'has approximately the same distribution as'. One implication of this CLT is that $S_n^*(t)$, for large $n$, becomes more and more deterministic. Since one expects additional reservoirs to successively smooth the inflow, this is not surprising. However, Theorem 2 identifies the precise rate of convergence to the deterministic limit as $s_n^{-1/2}$ (which is of order $n^{-1/4}$). Furthermore, the random fluctuations of $S_n^*(\cdot)$ occur on a time scale of order $n^{1/2}$. In addition, $S_n^*(\cdot)$ behaves asymptotically like a stationary Gaussian process having continuous paths. Our final result of this section computes the variance of the limiting Gaussian process.

**Proposition 2.**

$$\int_{-\infty}^{\infty} B(t-y) y\varphi(y)\,\mathrm{d}y \overset{\mathcal{D}}{=} \frac{1}{\sqrt{2}\pi^{1/4}} N(0,1).$$

*Proof.* Clearly,

$$\begin{aligned}\int_{-\infty}^{\infty} B(t-y) y\varphi(y)\,\mathrm{d}y &= \int_{-\infty}^{\infty} [B(t-y) - B(t)] y\varphi(y)\,\mathrm{d}y \\ &\overset{\mathcal{D}}{=} \int_{-\infty}^{0} B(-y) y\varphi(y)\,\mathrm{d}y + \int_{0}^{\infty} B(y) y\varphi(y)\,\mathrm{d}y \\ &\overset{\mathcal{D}}{=} Z_1 + Z_2,\end{aligned}$$

where $Z_1$, $Z_2$ are i.i.d. r.v.'s, with $Z_1 = \int_0^\infty B(t) t\varphi(t)\,\mathrm{d}t$. Since $Z_1$ is an integral of a Gaussian process, it is straightforward to verify that it is normally distributed, having mean zero and

$$\begin{aligned}\operatorname{var} Z_1 &= 2\int_0^{+\infty}\int_s^{\infty} s\varphi(s) t\varphi(t) \operatorname{cov}(B(s), B(t))\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_0^{+\infty}\int_s^{\infty} s^2\varphi(s) t\varphi(t)\,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{2}{\sqrt{2\pi}}\int_0^{\infty} s^2\varphi(s)\exp(-s^2/2)\,\mathrm{d}s \\ &= \frac{1}{\pi}\int_0^{\infty} s^2\exp(-s^2)\,\mathrm{d}s \\ &= \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} s^2\exp(-s^2)\,\mathrm{d}s \\ &= \frac{1}{2\sqrt{\pi}} \mathrm{E}\,(N(0,1)/\sqrt{2})^2 \\ &= \frac{1}{4\sqrt{\pi}}.\end{aligned}$$

Consequently, $Z_1 + Z_2 \overset{\mathcal{D}}{=} N(0, 1/(2\sqrt{\pi}))$.

## 4. Tail behaviour of the stationary regime

The CLT of the previous section describes the random fluctuations of order $n^{-1/4}$ about the mean of the $n$th reservoir. However, in a hydrological context, one is often interested in calculating the likelihood of flooding, in which event one may wish to accurately compute and approximate the tail probabilities corresponding to $S_n^*(t)$.

A key to this analysis is the application of large deviations ideas. In particular, our main limit theorem of this section will depend upon the Gärtner–Ellis theorem (see, for example, Bucklew [3]), the hypotheses of which involve the moment generating functions of the random variables under consideration. As a result, we start by computing $\mathrm{E}\exp(\theta S_n^*(t))$.

We shall assume here that

(A4) $\Gamma$ is a compound Poisson process, so that $\Gamma(t) = \sum_{i=0}^{N(t)} X_i$, with $X_0 = 0$ and $(X_1, X_{-1}, X_2, X_{-2}, \dots)$ i.i.d. and independent of the Poisson process $N = (N(t) : -\infty < t < \infty)$, where $N(0) = 0$. Furthermore, $N$ has rate $\gamma$ and

$$m_X(\theta) \stackrel{\triangle}{=} \mathrm{E}\exp(\theta X_i) < \infty \qquad \text{for} \quad 0 \le \theta < \theta_0.$$

We start by noting that

$$\begin{aligned} S_n^*(t) &= \int_{[0,\infty)} \Gamma(t-s) H_n(\mathrm{d}s) \\ &\stackrel{\mathcal{D}}{=} \sum_{i=1}^{\infty} X_i H_n(T_i), \end{aligned}$$

where $T_1, T_2, \dots$ are the transition epochs of the Poisson process $(N(t) : t \ge 0)$. Because $\Gamma$ is non-decreasing, $X_i \ge 0$ so that $\sum_{i=1}^{\infty} X_i H_n(T_i)$ exists almost surely. A standard computation then reveals that

$$\log \mathrm{E}\exp\Big(\theta S_n^*(t)\Big) = \gamma \int_0^\infty (m_X(\theta H_n(s)) - 1)\,\mathrm{d}s\,. \tag{4.1}$$

We are now ready to state our large deviations theorem for the tail probabilities of $S_n^*(t)$ when $n$ is large. Set $\psi(\theta) = \gamma \int_{-\infty}^{\infty} (m_X(\theta\varphi(s)) - 1)\,\mathrm{d}s$.

**Theorem 3.** *Assume (A1), (A2), and (A4) and suppose $x > \gamma \mathrm{E} X$. If there exists a root $\theta_x^* < \theta_0\sqrt{2\pi}$ to the equation*

$$\gamma \int_{-\infty}^{\infty} m_X'(\theta\varphi(s))\varphi(s)\,\mathrm{d}s = x, \tag{4.2}$$

*then*

$$\frac{1}{s_n} \log \mathrm{P}(\alpha_n S_n^*(t) > x) \to -\theta_x^* x + \psi(\theta_x^*). \tag{4.3}$$

*as $n \to \infty$.*

*Proof.* This result follows from the Gärtner–Ellis theorem, provided that we prove that

$$\frac{1}{s_n}\psi_n(\alpha_n s_n \theta) \to \psi(\theta) \tag{4.4}$$

as $n \to \infty$, and that $\psi'(\theta)$ is given by the left-hand side of (4.2). Relation (4.1) yields

$$\begin{aligned}\frac{1}{s_n}\psi_n(\alpha_n s_n \theta) &= \frac{\gamma}{s_n}\int_0^\infty (m_X(\theta\alpha_n s_n H_n(u)) - 1)\,\mathrm{d}u \\ &= \gamma \int_{-m_n/s_n}^\infty \left(m_X\left(\theta\alpha_n v_n \int_{-\infty}^y h_n(m_n + s_n z)\,\mathrm{d}z\right) - 1\right)\mathrm{d}y.\end{aligned}$$

Now, for all $k \geq 0$,

$$\alpha_n v_n \int_{|z|>y} h_n(m_n + s_n z)\,\mathrm{d}z = O(y^{-k}) \tag{4.5}$$

uniformly in $n$, so it follows from Proposition 1 that

$$\alpha_n v_n \int_{-\infty}^y h_n(m_n + s_n z)\,\mathrm{d}z \to -\int_{-\infty}^y x\varphi(x)\,\mathrm{d}x = \varphi(y) \tag{4.6}$$

uniformly in $y$. Since $\alpha_n v_n \int_{-\infty}^y h_n(m_n + s_n z)\,\mathrm{d}z = O(1)$ uniformly in $y$ and $n$, it is evident that

$$\left|m_X\left(\theta\alpha_n v_n \int_{-\infty}^y h_n(m_n + s_n z)\,\mathrm{d}z\right) - 1\right| \leq k\alpha_n v_n \int_{-\infty}^y h_n(m_n + s_n z)\,\mathrm{d}z \tag{4.7}$$

for some constant $k$ depending on the derivative of $m'_X(\cdot)$. Now,

$$\begin{aligned}&\int_{-m_n/s_n}^\infty \left(m_X\left(\theta\alpha_n v_n \int_{-\infty}^y h_n(m_n + s_n z)\,\mathrm{d}z\right) - 1\right)\mathrm{d}y \\ &\quad= \int_{-\beta}^{\beta}\left(m_X\left(\theta\alpha_n v_n \int_{-\infty}^y h_n(m_n + s_n z)\,\mathrm{d}z\right) - 1\right)\mathrm{d}y \\ &\qquad+ \int_{|y|>\beta, y \geq -m_n/s_n}\left(m_X\left(\theta\alpha_n v_n \int_{-\infty}^y h_n(m_n + s_n z)\,\mathrm{d}z\right) - 1\right)\mathrm{d}y.\end{aligned}$$

The uniform convergence of the integrand (see (4.6)) in the first integral establishes that it converges to $\int_{-\beta}^{\beta}(m_X(\theta\varphi(y)) - 1)\,\mathrm{d}y$. On the other hand, by choosing $\beta$ sufficiently large and exploiting (4.5) and (4.7), we note that we can make the second integral above arbitrarily small uniformly in $n$. This gives us the desired relation (4.4). The formula for $\psi'(\theta)$ is immediate, upon recognizing that

$$h^{-1}|m_X((\theta + h)\varphi(s)) - m_X(\theta\varphi(s))| \leq k\varphi(s)$$

uniformly in $s$, providing the domination necessary to justify interchanging the integral and derivative.

Theorem 3 suggests the tail approximation

$$\mathrm{P}(S_n^*(t) > y) \approx \exp(s_n(-\theta^*_{\alpha_n y}\cdot(\alpha_n y) + \psi(\theta^*_{\alpha_n y}))) \tag{4.8}$$

where $\theta^*_{\alpha_n y}$ solves $\psi'(\theta^*_{\alpha_n y}) = \alpha_n y$. Of course, the large deviations approximation (4.8) can, in practice, be rather poor, as the logarithm appearing in Theorem 3 can hide a variety of sins.

In particular, any dependence of the probability that is of the order of $\exp(O(\sqrt{n}))$ cannot be described by Theorem 3. This makes the development of efficient numerical schemes for computing $\mathrm{P}(S_n^*(t) > y)$ of interest.

Specifically, we shall describe here a Monte Carlo-based scheme for computing this tail probability. The idea will be to use importance sampling (see, for example, Bucklew [3]) to simulate $\Gamma$ under a change of measure that makes the event $\{S_n^*(t) > y\}$ more likely.

We start by defining

$$\mathrm{P}_{\theta,n}(\cdot) = \mathrm{E}\exp\biggl(\theta\int_0^\infty H_n(u)\,\Gamma(\mathrm{d}u) - \psi_n(\theta)\biggr)\mathbf{1}(\cdot) \tag{4.9}$$

and let $\mathrm{E}_{\theta,n}(\cdot)$ be the expectation operator corresponding to the probability $\mathrm{P}_{\theta,n}$. It is well known that

$$\mathrm{E}_{\theta,n}\int_0^\infty H_n(u)\,\Gamma(\mathrm{d}u) = \psi_n'(\theta).$$

Hence, by choosing $\theta$ as the root $\tilde{\theta}_n$ to the equation

$$\psi_n'(\tilde{\theta}_n) = y, \tag{4.10}$$

we are modifying the distribution of $\int_0^\infty H_n(u)\,\Gamma(\mathrm{d}u)$ so that its mean is precisely equal to $y$. Given that we are interested in computing $\mathrm{P}(S_n^*(t) > y)$, it seems intuitively reasonable to use the distribution $\mathrm{P}_{\tilde{\theta}_n,n}$ to simulate the inflow process $\Gamma$; a substantial body of theory supports this choice of distribution in an asymptotic sense (as $n \to \infty$); see Bucklew *et al.* [4].

This leaves us with the question of how to generate variates with distribution $\mathrm{P}_{\theta,n}$. Let $(T_n : n \geq 1)$ be the jump times of the Poisson process $N$, and let $\psi_X(\theta) = \log m_X(\theta)$. Observe that

$$\begin{aligned}
\biggl[\frac{\mathrm{dP}_{\theta,n}}{\mathrm{dP}}\biggr] &= \exp\biggl(\theta\int_0^\infty H_n(u)\,\Gamma(\mathrm{d}u) - \psi_n(\theta)\biggr) \\
&= \exp\biggl(-\gamma\int_0^\infty (m_X(\theta H_n(s)) - 1)\,\mathrm{d}s + \theta\sum_{j=1}^\infty H_n(T_j)X_j\biggr) \\
&= \prod_{j=1}^\infty \exp(\theta H_n(T_j)X_j - \psi_X(\theta H_n(T_j)) \\
&\qquad\qquad \times \exp\biggl(\sum_{j=1}^\infty \psi_X(\theta H_n(T_j)) - \gamma\int_0^\infty (m_X(\theta H_n(s)) - 1\,\mathrm{d}s\biggr).
\end{aligned}$$

The final exponential above is the relative likelihood of a probability under which $N$ evolves according to a non-homogeneous Poisson process having rate $(\gamma m_X(\theta H_n(t)) : t \geq 0)$ (relative to a probability under which $N$ evolves according to a constant-rate Poisson process with rate $\gamma$); see [2] for similar calculations. The infinite product is the contribution to the relative likelihood from the $X_j$'s, conditional on $N$. Specifically, such an infinite product requires that the $X_j$'s, conditional on $N$, be independent under $\mathrm{P}_{\theta,n}$, with corresponding distributions

$$\mathrm{P}_{\theta,n}(X_j \in \cdot \mid N) = \mathrm{E}\,[\exp(\theta H_n(T_j)X_j - \psi_X(\theta H_n(T_j))\mathbf{1}(X_j \in \cdot) \mid N].$$

This suggests that to calculate $\mathrm{P}(S_n^*(t) > y)$, we should compute the root $\tilde{\theta}_n$ to (4.10), simulate $N$ according to the rate function $(\gamma m_X(\tilde{\theta}_n H_n(t)) : t \geq 0)$, and then generate $X_1, X_2, \dots$ independently from the distributions $F_1, F_2, \dots$ given by

$$F_j(\mathrm{d}x) = \exp(\tilde{\theta}_n H_n(T_j)x - \psi_X(\tilde{\theta}_n H_n(T_j)))\mathrm{P}(X_j \in \mathrm{d}x).$$

Observe that so long as $|t - m_n|/s_n$ is large, $|H_n(t)|$ is small and consequently the process $\Gamma$ evolves under $\mathrm{P}_{\tilde{\theta}_n,n}$ in effectively the same way as under P; it is only when $|t - m_n|/s_n$ is of small to moderate size that $\mathrm{P}_{\tilde{\theta}_n,n}$ substantially modifies the distribution of $\Gamma$.

Before stating our algorithm, we note that $\int_{|t-m_n|>\beta} H_n(t)\,\Gamma(\mathrm{d}t)$ goes to zero rapidly as $\beta \to \infty$ under $\mathrm{P}_{\tilde{\theta}_n,n}$, because $|H_n(\cdot)|$ is small over the region of integration, and $\Gamma$ evolves according to the 'normal' dynamics of P for such $t$-values. Consequently, we can reduce computation time by choosing $\beta$ sufficiently large, and then simulating $\Gamma$ only over $[m_n - \beta, m_n + \beta]$.

**Algorithm** (for computing $P(S_n^*(t) > y)$, $y > \gamma\,\mathrm{E}\,X_1/\alpha_n$):

1. Select a sample size $n \geq 1$ and tolerances $\varepsilon$, $\delta$ (see (3) below).

2. Compute the root $\tilde{\theta}_n$ of (4.10). (An approximation to $\tilde{\theta}_n$, valid for large $n$, is $\tilde{\theta}_n \approx \theta_n^* \alpha_n s_n$, where $\psi'(\theta_n^*) = \alpha_n y$.)

3. Select $\beta$ so that $\mathrm{P}_{\tilde{\theta}_n,n}(|\int_{|u-m_n|>\beta} H_n(u)\,\Gamma(\mathrm{d}u)| > \varepsilon) < \delta$. One can use the exponential inequality, valid for $x > 0$,

$$\begin{aligned}
&\mathrm{P}_{\tilde{\theta}_n,n}\left(\left|\int_{|u-m_n|>\beta} H_n(u)\,\Gamma(\mathrm{d}u)\right| > \varepsilon\right)\\
&\quad\leq \exp(-x\varepsilon)\,\mathrm{E}_{\tilde{\theta}_n,n} \exp\left(x \int_{|t-m_n|>\beta} H_n(t)\,\Gamma(\mathrm{d}t)\right)\\
&\quad= \exp(-x\varepsilon)\,\mathrm{E}\,\exp\left(\tilde{\theta}_n \int_0^\infty H_n(t)\,\Gamma(\mathrm{d}t) + x \int_{|t-m_n|>\beta} H_n(t)\,\Gamma(\mathrm{d}t) - \psi_n(\tilde{\theta}_n)\right)\\
&\quad= \exp(-x\varepsilon)\exp\left(\gamma \int_{|t-m_n|\leq\beta} (m_X(\tilde{\theta}_n H_n(s)) - 1)\,\mathrm{d}s\right.\\
&\qquad\qquad \left. + \gamma \int_{|t-m_n|>\beta} (m_X((\tilde{\theta}_n + x)H_n(s)) - 1)\,\mathrm{d}s - \psi_n(\tilde{\theta}_n)\right)\\
&\quad= \exp\left(-x\varepsilon + \gamma \int_{|t-m_n|>\beta} (m_X(\tilde{\theta}_n + x)H_n(s)) - m_X(\tilde{\theta}_n H_n(s))\,\mathrm{d}s\right);
\end{aligned}$$

   a similar inequality can be deduced for $\mathrm{P}_{\tilde{\theta}_n,n}(\int_{|t-m_n|>\beta} H_n(t)\,\Gamma(\mathrm{d}t) < -\varepsilon)$.

4. Simulate a non-homogeneous Poisson process $N$ over $[m_n - \beta, m_n + \beta]$ according to the rate function $(\gamma m_X(\tilde{\theta}_n H_n(t)) : m_n - \beta \leq t \leq m_n + \beta)$, with corresponding jump times $T_1, T_2, \dots, T_L$.

5. For each jump time $T_j$, simulated in step 4, generate $X_j$ from the distribution

$$\exp(\tilde{\theta}_n H_n(T_j)x - \psi_X(\tilde{\theta}_n H_n(T_j)))\mathrm{P}(X_j \in \mathrm{d}x).$$

6. Compute

$$W = \mathbf{1}\Big(\sum_{j=1}^{L} H_n(T_j)X_j > y\Big) \times \exp\Big(-\sum_{j=1}^{L} \tilde{\theta}_n H_n(T_j)X_j + \gamma \int_{m_n-\beta}^{m_n+\beta} (m_X(\theta H_n(s)) - 1)\,\mathrm{d}s\Big).$$

7. Replicate steps 4–6 $m$ independent times, thereby generating $W_1, \dots, W_m$ and form the estimator

$$\hat{\alpha}_m = \frac{1}{m}\sum_{i=1}^{m} W_i.$$

Then, $\hat{\alpha}_m$ is the desired estimator for $\alpha = \mathrm{P}(S_n^*(t) > y)$. Note that if one desires bounds on the bias of $\hat{\alpha}_m$ induced via the $\beta$ truncation of step 4 (rather than the probabilistic bounds computed in step 3), these can be readily obtained by integrating the exponential inequality of step 3.

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