From "Studies in mathematical learning theory", R. R. Bush and W. K. Estes (Eds.) Stanford University Press, 1959.

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A Theory of Serial Discrimination Learning[1]

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The models to be discussed apply to a class of learning situations in which the subject is presented on each trial with a fixed sequence of stimuli, or choice points. On each trial at a given choice point, one of two possible responses is reinforced. The two response alternatives are the same at every choice point of the sequence.

One common example is a multiple-unit T maze [4]. In each unit the subject is required to turn left or right, the correct turn taking him into the next choice point. A second example may be taken from verbal conditioning experiments with human subjects [1]. A panel of lights, arranged before the subject, may be lighted in different patterns to provide distinguishable choice points. With the presentation of each stimulus in the series, the subject is required to predict which of two events will occur. Following each prediction, the subject is informed of the correct prediction on that trial at that choice point.

The results of serial learning experiments indicate that learning at any particular choice point is influenced by what is being learned at neighboring choice points. The present models provide possible mechanisms to account for these interaction effects.

In the first section, we shall consider a common-elements model which is a simple extension of the Estes-Burke component model [2] to serial-learning phenomena. The interaction effects are handled by a stimulus-generalization mechanism. The generalization of response tendencies among the several choice points is assumed to be mediated by stimulus elements common to all the choice points.

After discussing some limitations of the common-elements model, we shall

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consider a stimulus-trace model which applies to a more general class of experimental situations. In this trace model it is assumed that stimulus elements sampled at one choice point may remain active at succeeding choice points. These trace elements have a double-barreled effect: first, they partly determine which response is performed at succeeding choice points; and second, they may become conditioned to the response that is reinforced at succeeding choice points. In this trace model, a parameter is introduced which is identified with the effectiveness of the conditions of reinforcement.

The following notation will be used throughout this paper. The total number of choice points in the series is denoted by K, and a particular choice point will be indexed by j. The jth choice point is identified with a population of stimulus elements, denoted by S_j^*. The set of stimulus elements that are common to all choice points will be denoted by S_c. The set of stimulus elements unique to the jth choice point will be denoted by S_j. We let N_j denote the total number of stimulus elements in the set S_j.

The response alternatives are called A_1 and A_2. The reinforcing events are E_1 (an A_1 is reinforced) and E_2 (an A_2 is reinforced). The probability of an E_1 event at the jth choice point is denoted by π_j, which is a constant throughout any given experiment.

Each stimulus element is assumed to be conditioned to A_1 or A_2. We let $F_{j,n}$ denote the probability that elements of the set S_j are conditioned to A_1 on trial n, and $F_{j,c,n}$ the expected probability that elements of S_c are conditioned to A_1 at the jth choice point on trial n. Finally, $p_{j,n}$ will denote the probability of an A_1 response at the jth choice point on trial n.

These models apply only to situations ensuring the reinforcement of the correct response at each choice point on every trial. This condition is necessary for the application of the conditioning rules given below. It is assumed that with animal subjects this condition is fulfilled by the use of a correction training procedure. With human subjects, information about the correct response may be sufficient.

The Common-Elements Model

The central assumption of this model is that the stimuli at the several choice points have common aspects or elements. The common elements, denoted by S_c, are those elements which may be sampled at any choice point of the series. These common elements may facilitate or retard learning at a particular choice point depending on whether the response conditioned to S_c is the same or different from the correct response at that choice point.

Assumptions about the Stimuli. At any choice point, j, there is a population of stimulus elements, S_j^*, which is composed of some elements unique to that choice point, S_j, and some elements common to all choice points, S_c. Thus, there is a total of $K+1$ nonoverlapping sets of stimulus elements: the set S_c and the K sets of stimulus elements unique to each of the K choice points.

These assumptions about the stimuli restrict the range of experimental situations to which the model is applicable. Without the simplification in the stimulus situation given by these assumptions, however, the derivation of the response probabilities in serial learning becomes exceedingly complicated.

The following situation will illustrate in simplified form the notions about the stimulus sets. Suppose that the stimulus panel in a verbal conditioning experiment contains seven small light bulbs arranged in a semicircle before the subject. Suppose also that two choice points are to be presented. The stimuli at the first choice point, S_1^*, might be the set of stimulus elements evoked by lighting bulbs 2, 3, and 4. The stimuli at the second choice point, S_2^*, might be those elements evoked by lighting bulbs 3, 4, 5, and 6. The common elements, S_c, include those elements evoked by lights 3 and 4. The set S_1 of unique stimuli in S_1^* includes those elements produced by lighting bulb 2; S_2, the set unique to S_2^*, includes those elements evoked by lights 5 and 6. On each trial the subject is first presented with S_1^* and required to predict A_1 or A_2, and then is presented with S_2^* and again required to predict A_1 or A_2. After each prediction the subject is informed of the correct prediction.

In this situation the experimenter can vary the size of any of the stimulus sets by adding or subtracting lights. For example, the number of elements in S_1 can be varied while the proportion of S_1 elements in S_1^* is held constant. In addition, the experimenter may vary the likelihood that particular lights in these stimulus sets will be lighted on a given trial. This latter manipulation will affect θ, the learning-rate parameter [1].

Sampling and Conditioning Assumptions. This part of the model is adapted directly from the stimulus-sampling model of Estes and Burke [3]. We let θ represent the probability that a stimulus element of S_j will occur in the active sample at the jth choice point on some trial n. The conditioning assumptions are

(1) $$F_{j,n+1} = (1-\theta)F_{j,n} + \theta \quad \text{if } E_1 \text{ at } S_j^* \text{ on trial } n;$$

(2) $$F_{j,n+1} = (1-\theta)F_{j,n} \quad \text{if } E_2 \text{ at } S_j^* \text{ on trial } n\,.$$

If the probability of an E_1 reinforcing event at choice j is π_j, then the average value of $F_{j,n+1}$ will be

(3) $$F_{j,n+1} = (1-\theta)F_{j,n} + \theta\pi_j\,.$$

The solution to this difference equation is the well-known growth curve

(4) $$F_{j,n} = \pi_j - (\pi_j - F_{j,1})(1-\theta)^{n-1},$$

where $F_{j,1}$ is the initial proportion of S_j elements conditioned to A_1. This result shows that the asymptotic proportion of elements in a stimulus set that are conditioned to A_1 equals π_j, the probability of an E_1 event in the presence of those stimuli.

Applications of the Theory to Serial Learning. Some derivations from the assumptions above will be made for serial-learning situations. The main interest is in finding an expression for $p_{j,n}$, the probability of an A_1 response at choice j on trial n. This derivation has three parts: (1) finding the proportion, $F_{j,n}$, of the unique cues in S_j^* conditioned to A_1 prior to trial n; (2) finding the proportion, $F_{j,c,n}$, of common elements conditioned to A_1 at choice point j on trial n; and (3) combining these two quantities, weighting each by the sampling probability of the corresponding stimulus set, to obtain $p_{j,n}$, the response probability at j on trial n.

We begin the derivation by noting that the unique stimuli at the jth choice point are presented just once on each trial of the series. An A_1 is reinforced in the presence of S_j stimuli with probability π_j. Hence, Equation 4 applies directly.

The derivation of $F_{j,c,n}$ is more difficult because the common elements are available for sampling and conditioning at each of the K choice points. The sample of S_c elements active at the jth choice point will be conditioned to A_1 or A_2 depending on which reinforcing event occurs. Therefore, from one choice point to the next within a trial, there will be successive changes in the proportion of common elements conditioned to A_1.

An easy way to follow successive changes in the conditioning of S_c elements is to construct a trials by choice-points matrix as in Table 1 below. The entry in cell j, n is the expected probability, $F_{j,c,n}$, that an element of S_c will be conditioned to A_1 at choice point j on trial n. The proportion of S_c elements initially conditioned to A_1 at the first choice point is $F_{1,c,1}$. The example illustrates the derivation for two choice points.

TABLE 1

Conditioning of Common Elements; First Two Trials at Two Choice Points

Choice Point	Pr(E_1)	Trial 1	Trial 2
1	π_1	$F_{1,c,1}$	$(1-\theta)^2F_{1,c,1} + \theta(1-\theta)\pi_1 + \theta\pi_2$
2	π_2	$(1-\theta)F_{1,c,1} + \theta\pi_1$	$(1-\theta)^3F_{1,c,1} + \theta(1-\theta)^2\pi_1 + \theta(1-\theta)\pi_2 + \theta\pi_1$

The entries in the first row of Table 1 are $F_{1,c,1}$ and $F_{1,c,2}$. We can write $F_{1,c,2}$ as a linear function of $F_{1,c,1}$:

(5) $$F_{1,c,2} = AF_{1,c,1} + B\,,$$

where $A = (1-\theta)^2$ and

$$B = \theta(1-\theta)\pi_1 + \theta\pi_2 = \theta\sum_{i=1}^{2}\pi_i(1-\theta)^{2-i}\,.$$

When there are K choice points the constants in Equation 5 are

(6) $$A = (1-\theta)^K, \qquad B = \theta\sum_{i=1}^{K}\pi_i(1-\theta)^{K-i}\,.$$

To return to our example, one can prove by induction that Equation 5 is just an instance of the general relation

(7) $$F_{1,c,n+1} = AF_{1,c,n} + B\,.$$

The solution to this difference equation is

(8) $$F_{1,c,n} = \frac{B}{1-A} - \left(\frac{B}{1-A} - F_{1,c,1}\right)A^{n-1} .$$

This equation gives an expression for the expected proportion of common elements conditioned to A_1 at the beginning of trial n.

The proportion of S_c elements conditioned after the first choice point is a linear function of $F_{1,c,n}$. As can be seen in Table 1, this relation is

(9) $$F_{2,c,n} = (1-\theta)F_{1,c,n} + \theta\pi_1 .$$

One can show that for any arbitrary choice point the general relation is

(10) $$F_{j+1,c,n} = (1-\theta)F_{j,c,n} + \theta\pi_j .$$

Equation 8 gives an expression for the expected proportion of S_c elements conditioned to A_1 at the beginning of any trial n. Equation 10 expresses the conditioning of S_c elements as the successive choice points are encountered on a given trial. Therefore, these two equations contain complete information about the conditioning of common elements at any given point in the experiment.

The derivations above provide expressions for the conditioning of S_j and S_c elements that has occurred before choice point j is presented on trial n. To relate these quantities to $p_{j,n}$, we note that the average sample of stimuli active at choice point j includes θN_j elements from S_j and θN_c elements from S_c. The proportion of S_j elements in the sample will define a parameter, ρ_j.

(11) $$\rho_j = \frac{\theta N_j}{\theta N_j + \theta N_c} = \frac{N_j}{N_j + N_c} .$$

The parameter ρ_j measures the relative distinctiveness of the stimuli at the jth choice point. The average proportion of common elements in the sample at S_j^* is $1-\rho_j$. Since the proportions of S_j and S_c elements conditioned to A_1 are $F_{j,n}$ and $F_{j,c,n}$, respectively, the total proportion of sampled elements that are conditioned to A_1 is

(12) $$p_{j,n} = \rho_j F_{j,n} + (1-\rho_j)F_{j,c,n} .$$

This result exhibits the assumption that the probability of A_1 to the compound stimulus S_j^* is a weighted composite of the conditioning of A_1 to the subsets, S_j and S_c, of the compound stimulus. By substituting the quantities $F_{j,n}$ and $F_{j,c,n}$ derived above, Equation 12 provides an expression for the expected probability of A_1 at choice point j on trial n. This concludes the derivation.

To illustrate this result, consider a situation with four choice points where the E_1 probabilities are $\pi_1 = .10$, $\pi_2 = .30$, $\pi_3 = .60$, and $\pi_4 = .90$. For this example, all the ρ_j were set equal to .50 and θ equal to .10. In Fig. 1 the asymptotes of $F_{j,n}$, $F_{j,c,n}$, and $p_{j,n}$ are plotted for the four choice points.

Since $\rho = .50$ in this example, p_j is just the mean of F_j and $F_{j,c}$. Notice that the values of $F_{j,c,\infty}$ do not vary appreciably from the average π value of .475. If the choice points were presented in random order, then $F_{j,c,\infty}$

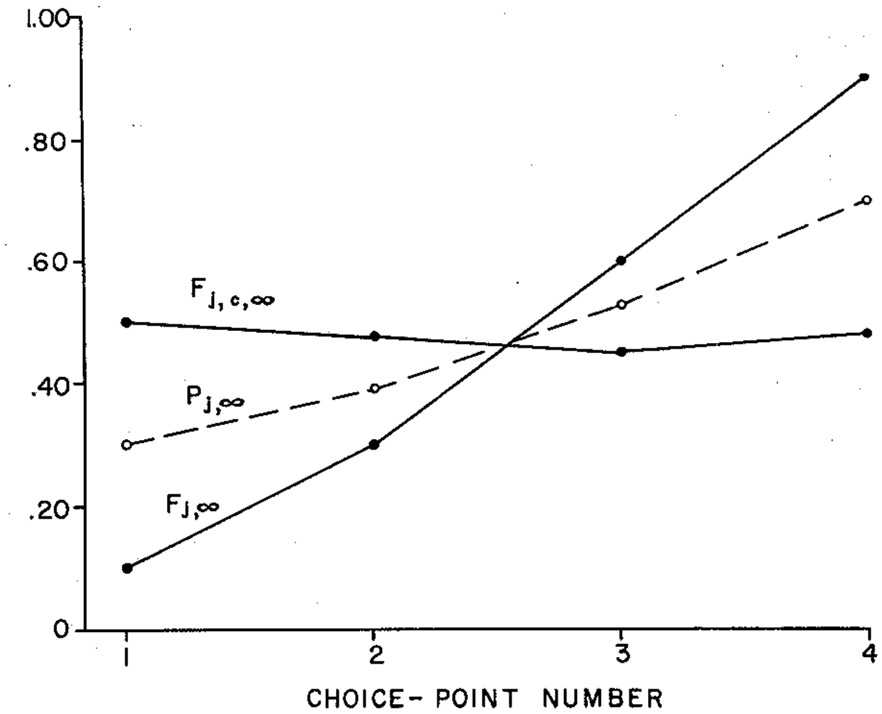

FIGURE 1. Asymptotes of F_j, $F_{j,c}$, and p_j.

would be a horizontal line at .475. This result indicates that the model will not predict large serial-position effects.

To illustrate this point with a second example, consider the case where all $\pi_j = 1$ (i.e., A_1 is reinforced at every choice point), which Hull [4] has called the "homogeneous" case. It can be shown that for this situation the expected total number of errors at the jth choice point is

(13) $$\sum_{n=1}^{\infty}(1-p_{j,n}) = \rho_j \frac{1-F_{j,1}}{\theta} + (1-\rho_j)\frac{(1-F_{1,c,1})(1-\theta)^{j-1}}{1-(1-\theta)^K} .$$

In Fig. 2 the expected number of errors is plotted as a function of the ordinal position of the four choice points. For this example, $F_{1,c,1} = F_{j,1} = .50$, $\theta = .10$, and the ρ_j for each curve equal a common value ρ. In the two curves the values of ρ are .70 and .30.

When $\rho = 1$ there are no common elements, and thus learning at one choice point can have no influence on learning at any other choice point. When there are common elements ($\rho < 1$), learning in the homogeneous case is facilitated at all choice points and a serial-position effect appears. As Fig. 2 shows, the amount of facilitation is directly related to the relative size of the set of common elements. It may also be seen that the slope of the serial-position

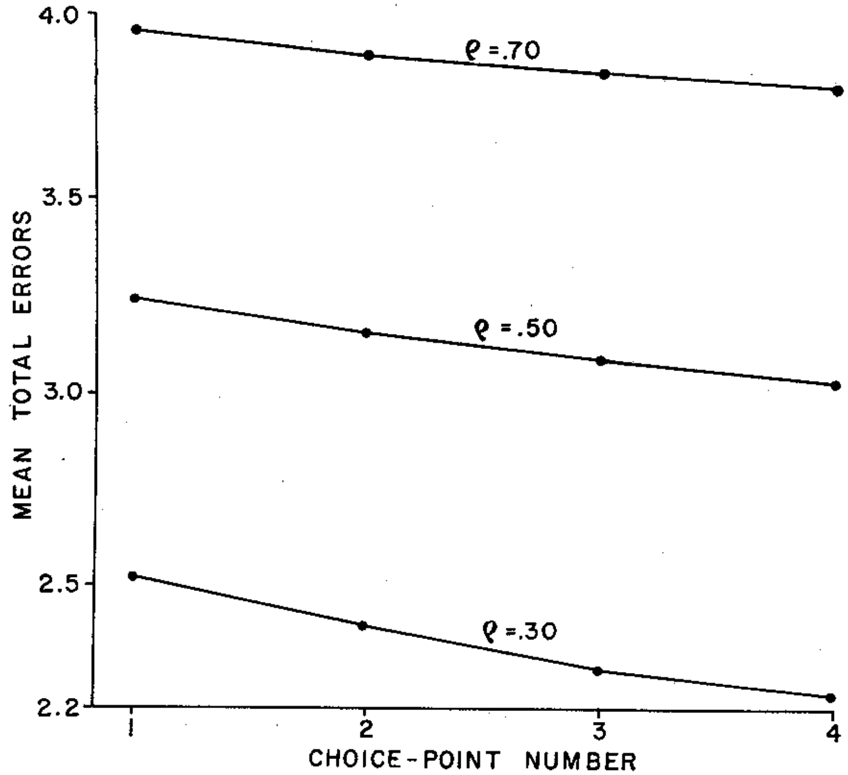

FIGURE 2. Serial position error curve for three values of the ρ parameter.

error curve increases with the proportion of common elements in the choice-point stimuli. The reason for this result is clear. The extent to which the common elements are conditioned differs at the successive choice points. The greater the weight of the common elements in the stimulus sample, the more prominent the serial-position differences in the proportion of S_c elements conditioned to A_1.

Remarks about the Common-Elements Model. The assumptions of the common-elements model, namely, the partitioning of choice-point stimuli into common and unique elements and the rules for conditioning responses to these stimuli, are the axioms used by Burke and Estes [2]. In their discrimination model, they derive some implications of these axioms for situations in which the order of presentation of choice points is randomized. The present paper has derived some implications of these same axioms for situations in which the choice points are presented in a fixed serial order.

Despite the attractive parsimony and consistency of this approach to serial learning, it is felt that the common-elements model has certain limitations. First, it does not predict a very large serial-position effect with reasonable

values of the parameters. The smallness of the effect is seen, for example, in Fig. 1, where the values of $F_{j,c,\infty}$ do not deviate appreciably from the mean of the π_j values. Also, in Fig. 2 the serial-position effect is slight. The fact that the predicted effects are small suggests that some factor in addition to a generalization mechanism is needed to account for the size of serial-position effects frequently found experimentally [4].

A second limitation of the common-elements model is that it does not give a credible account of anticipatory errors. This point may be made clear by the following example. Consider a serial-learning situation involving nine choice points; an A_2 is correct at the fifth choice point while an A_1 is correct at the remaining eight choice points. It seems reasonable to expect that erroneous anticipations of A_2 would occur most frequently at those choice points just preceding the fifth choice point. However, the common-elements model predicts just the opposite, namely, that the fewest errors will occur at the fourth choice point. This result is predicted because the fourth is the choice point farthest removed in time from the last occurrence of E_2 at the fifth choice point; the effect on common elements of reinforcing A_2 in the fifth choice point will be maximally reduced by the time the fourth choice point is presented on the next trial.

To summarize, the common-elements model encounters two difficulties: (1) it provides for only a small serial-position effect, and (2) it does not handle anticipatory effects. The trace model presented below surmounts these problems. First, it predicts a sizable serial-position effect even when the choice points have no stimulus elements in common. Second, it provides a mechanism which accounts for both anticipatory and perseverative effects in serial learning.

The Trace Model

The central concept of this model is the perseverative stimulus trace. The usual notion of stimulus trace refers to the neural events assumed to remain active for some time after the termination of some evoking external stimulation. Pavlov [5] and Hull [4] have used this concept extensively in their discussions of "trace" conditioning and serial learning. In the following, the concept of the stimulus trace is given an interpretation consistent with a statistical representation of the stimulating environment.

A basic assumption of statistical learning theory [3] is that the stimulating conditions defined by an experimental situation represent a population of potential stimulus events or elements. A given stimulus event may occur on some experimental trials and fail to occur on others. The parameter θ represents the average probability that any particular event, of the population of potential events, will occur on some trial. The stimulus events that occur on some trial are assumed to be a random sample from the population of potential stimulus events defined by the experimental situation. Potential stimulus events that do not occur on a given trial may be described as ineffective or inactive on that trial.

When a stimulus event occurs, it will presumably remain effective over some small unit of time. The word "trace" will be used generally to describe a stimulus element that remains effective for some time after the occurrence of the evoking stimulating event in the subject's environment. When a stimulus trace becomes ineffective, we say that it is returned to an inactive state.

The trace model elaborates this process of returning sampled elements to the inactive state. It is assumed specifically that the number of sampled elements active as traces after sampling decays exponentially with time. This hypothesis implies, in serial-learning situations, that traces evoked by the choice-point stimuli early in the series may still be active when later choice points are presented. In the following we show some implications of this observation. First, however, we explicitly assume that trace elements have no special properties and are essentially like "ordinary" stimulus elements: i.e., that they partly determine which response is performed in their presence, and that they may become conditioned to responses reinforced in their presence. We then illustrate these assumptions by deriving $p_{j,n}$ in a simple situation.

Role of Traces in Learning and Performance. In the trace model, it is assumed that trace elements active at a given choice point may become conditioned to the response reinforced at that choice point. For example, if A_1 is correct at S_1 and A_2 is correct at S_2, then traces from S_1 that are active when S_2 is presented and E_2 occurs may become conditioned to A_2. If these elements are sampled on the next trial at S_1, they will tend to elicit A_2, which would be an "anticipatory" error. Hence, the conditioning of traces produces anticipatory effects. Whether the anticipatory effect will facilitate learning at a particular choice point depends on whether succeeding correct responses are the same as the correct response at that particular choice point.

It is assumed further that traces influence performance at succeeding choice points. Specifically, the performance at some choice point j depends on the relative numbers or weights of the traces from $S_{j-1}, S_{j-2}, \cdots$, that are active in the total sample at the time choice point j is presented to the subject. In the preceding example, traces from S_1 that are active when S_2 is presented will tend to elicit A_1. An A_1 occurring at the second choice point is called a "perseverative" error. Hence, traces produce a perseverative effect on performance.

Auxiliary Assumptions. The conditioning assumptions are the same as before except for one change. It is assumed that a certain proportion β of the stimulus elements sampled on a trial are conditioned to the reinforced response, while a proportion $1-\beta$ of the sampled elements remain conditioned as they were before the reinforcing event occurred. The parameter β represents the effectiveness of the conditions of reinforcement, and is identical in its properties with the parameter c discussed in Chapter 1.

In the following, we consider the case where the choice-point stimuli have no elements in common. Besides simplifying the exposition considerably, the study of the nonoverlap case isolates and clarifies the influence of the trace mechanism in serial learning.

For convenience, we shall consider experimental situations in which the interval between successive presentations of the entire series is sufficiently long so that all traces from one series are inactive at the beginning of the next presentation. The derivations below are appropriate for those arrangements in which the time between successive choices within a trial is relatively constant. For some particular interchoice time, we let λ represent the probability that elements sampled at one choice point become inactive before the next choice point is presented in the same series. Hence, $1-\lambda$ represents the proportion of sampled elements remaining as active traces when the next choice point is presented. The experimenter may manipulate λ by varying the duration of the interchoice interval.

The simple case involving two choice points suffices to illustrate the principal features of the model. The application of the model to situations involving more choice points is straightforward and involves no special problems. Two cases will be discussed separately: the homogeneous case, where the same response is always reinforced at every choice point; and the heterogeneous case, where different responses are reinforced at the various choice points.

The Homogeneous Case. Suppose that the experimenter always reinforces A_1 at the first and second choice points. Let us consider the conditioning of S_1 elements that occurs on some trial n of the experiment. A proportion $1-\theta$ of the S_1 elements are not sampled and thus remain conditioned as before. Of the sampled S_1 elements, a proportion β are conditioned when A_1 is reinforced at the first choice point, while a proportion $1-\beta$ of the S_1 sample are unchanged by this event. A fraction $1-\lambda$ of the sample from S_1 (including those that were conditioned and those that were unchanged by E_1 at the first choice point) remain active until the next choice point is presented. Of these active traces from S_1, a proportion β become conditioned (if they are not already conditioned) when A_1 is reinforced at the second choice point. These traces from S_1 that are active when E_1 occurs at the second choice point have, effectively, a second opportunity to become conditioned to A_1 on that trial. These considerations imply that the conditioning of S_1 elements proceeds according to

$$(14)\quad F_{1,n+1}=(1-\theta)F_{1,n}+\theta\{\beta+(1-\beta)[\lambda F_{1,n}+\beta(1-\lambda)+(1-\lambda)(1-\beta)F_{1,n}]\}.$$

Equation 14 can be simplified to

$$(15)\qquad F_{1,n+1}=(1-A)F_{1,n}+A\,,$$

where $A=\theta\beta[1+(1-\lambda)(1-\beta)]$. The solution of the difference equation 15 is

$$(16)\qquad F_{1,n}=1-(1-F_{1,1})(1-A)^{n-1}\,.$$

Equation 16 provides an expression for the proportion of S_1 elements conditioned before trial n. The constant A is the effective learning rate. If the experiment had involved S_1 as the only choice point, then the effective learning rate would have been $\theta\beta$. By adding a second choice point in the homogeneous case, we increase the learning rate at S_1 by the amount $\theta\beta(1-\lambda)(1-\beta)$. This quantity is precisely the probability that an element of S_1 becomes conditioned to A_1 on its second opportunity of any given trial.

The conditioning of S_2 elements may be described more easily. A proportion $1-\theta$ are not sampled and thus remain unchanged. Of the sampled elements from S_2 a proportion β are conditioned when E_1 occurs at the second choice point, while a proportion $1-\beta$ are unchanged by this event. Since the choice at S_2 terminates the trial and since, by assumption, traces from S_2 become inactive during the interval between trials, no further conditioning of S_2 elements occurs on that trial. In short, S_2 elements do not have a second opportunity to become conditioned to A_1 on any trial. From these considerations, we see that the conditioning of S_2 elements on trial n may be described by the relation

(17) $$F_{2,n+1} = (1-\theta)F_{2,n} + \theta[\beta + (1-\beta)F_{2,n}],$$

which simplifies to

(18) $$F_{2,n+1} = (1-\theta\beta)F_{2,n} + \theta\beta .$$

The difference equation 18 may be solved to give

(19) $$F_{2,n} = 1 - (1-F_{2,1})(1-\theta\beta)^{n-1} .$$

The effective learning rate at S_2 is seen to be $\theta\beta$. Thus, the difference in learning rates at S_1 and S_2 is just the "second opportunity factor," $\theta\beta(1-\lambda)(1-\beta)$.

The probability of an A_1 response at any choice point is a weighted composite of the proportions of S_1 and S_2 elements, including traces, in the total sample at that choice point times the probabilities that these stimulus elements are conditioned to A_1. At the first choice point, only elements of S_1 are in the sample since, by the assumption of a long interval between trials, no traces from S_2 are active. Hence, the probability of A_1 at the first choice point is

(20) $$p_{1,n} = F_{1,n} = 1 - (1-F_{1,1})(1-A)^{n-1} .$$

The response probability, $p_{2,n}$, at the second choice point is readily determined. The probability of an S_2 element being sampled is θ. The probability that an S_1 element is an active trace at the time the second choice point is presented is $\theta(1-\lambda)$. Of these S_1 traces, a proportion β have definitely been conditioned by the prior E_1 event at the first choice point; the remaining proportion, $1-\beta$, are conditioned to A_1 with probability $F_{1,n}$. If we assume that the total number of stimulus elements in S_1 and S_2 is the same, then $p_{2,n}$ can be written as

(21) $$p_{2,n} = \frac{\theta F_{2,n} + \theta(1-\lambda)[\beta + (1-\beta)F_{1,n}]}{\theta + \theta(1-\lambda)},$$

which simplifies to

(22) $$p_{2,n} = \frac{F_{2,n} + (1-\lambda)[\beta + (1-\beta)F_{1,n}]}{2-\lambda} .$$

When expressions for $F_{2,n}$ and $F_{1,n}$ are substituted in Equation 22, $p_{2,n}$ has been expressed as a function of n and the parameters.

To illustrate some implications of these equations, let us consider the expected total number of errors at S_1 and S_2, which are denoted by W_1 and W_2, respectively:

(23) $$W_1 = \sum_{n=1}^{\infty}(1-p_{1,n}) = \frac{1-F_{1,1}}{\theta\beta[1+(1-\lambda)(1-\beta)]},$$

(24) $$W_2 = \sum_{n=1}^{\infty}(1-p_{2,n}) = \frac{1}{2-\lambda}\left[\frac{1-F_{2,1}}{\theta\beta} + (1-\lambda)(1-\beta)W_1\right].$$

Equation 24 relates the expected errors at the second choice point to W_1, the expected errors at the first choice point. It is frequently of interest to know the difference between expected errors at the first and second choice points. If the initial conditions are $F_{1,1} = F_{2,1} = F_1$ and Equation 24 is subtracted from Equation 23, the result, after simplification, is

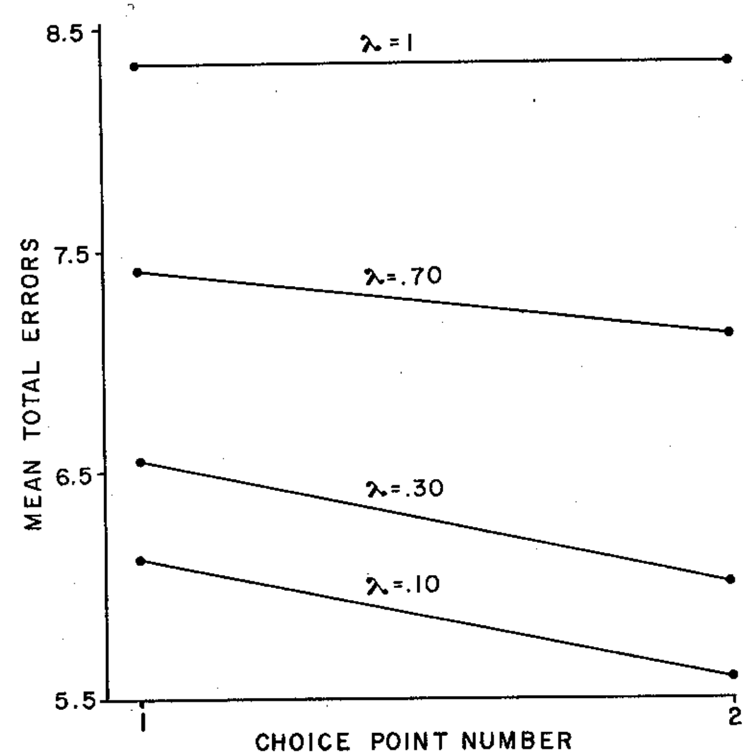

FIGURE 3. Serial position error curve for four values of the λ parameter.

(25) $$W_1 - W_2 = \frac{(1 - F_1)(1 - \lambda)(2\beta - 1)}{\theta\beta(2 - \lambda)[1 + (1 - \lambda)(1 - \beta)]} .$$

This result shows that the ordering of expected errors at S_1 and S_2 depends on β, the reinforcement parameter: if $\beta > .50$, then the model predicts more errors at the first choice point; if $\beta < .50$, then the model predicts more errors at the second choice point.

Equations 23 and 24 also show that W_1 and W_2 vary inversely with λ. This relation is illustrated in Fig. 3 for several values of λ. The parameter β was set equal to .60; hence $W_1 > W_2$, in accordance with Equation 25. The other parameter values for Fig. 3 are $F_{1,1} = F_{2,1} = .50$ and $\theta = .10$. In an experimental situation λ should be expected to vary directly with the time between presentations of S_1 and S_2. In the homogeneous case the model predicts that errors at both S_1 and S_2 will decrease with faster presentation rates. In contrast, when the correct responses at S_1 and S_2 differ, the model predicts that the total errors will increase with faster presentation rates.

There is a simple theorem that can be derived for the homogeneous case. The theorem states that the expected number of errors at the first choice point is a decreasing function of the total number of choice points in the series. When there are K homogeneous choice points, the expression for $p_{1,n}$ is

(26) $$p_{1,n} = 1 - (1 - F_{1,1})(1 - \theta\beta\alpha_1)^{n-1} ,$$

where

$$\alpha_1 = \frac{1 - [(1 - \lambda)(1 - \beta)]^K}{1 - [(1 - \lambda)(1 - \beta)]} .$$

The expected total number of errors at the first choice point is given by

(27) $$W_1(K) = \frac{1 - F_{1,1}}{\theta\beta\alpha_1} = \frac{(1 - F_{1,1})[1 - (1 - \lambda)(1 - \beta)]}{\theta\beta[1 - (1 - \lambda)^K(1 - \beta)^K]} .$$

Equation 27 shows that $W_1(K)$ is a decreasing function of K, as would be expected since the learning rate at S_1 is an increasing function of K. In Fig. 4, $W_1(K)$ is plotted as a function of K, the length of the homogeneous series, with λ as a parameter in the several curves. The values of the other parameters are $F_{1,1} = .50$, $\beta = .60$, and $\theta = .10$.

This theorem about errors at the first choice point is quite reasonable. When more choice points are added, the trace elements from S_1 have just that many more opportunities to become conditioned to A_1 on each trial. However, as Fig. 4 shows, one quickly reaches a point of diminishing returns in W_1 by adding more choice points. If β were smaller than .60, then there would be a larger downward slope to the curves in Fig. 4, the slope being exaggerated for smaller values of λ.

Actually, the theorem just cited is a special case of a more general theorem which can be proved for any arbitrary choice point j. For convenience, let us set $a = 1 - \lambda$ and $b = 1 - \beta$ and assume the initial conditions $F_{j,1} = F_1$ for all j. The expected total errors at choice point j in a homogeneous

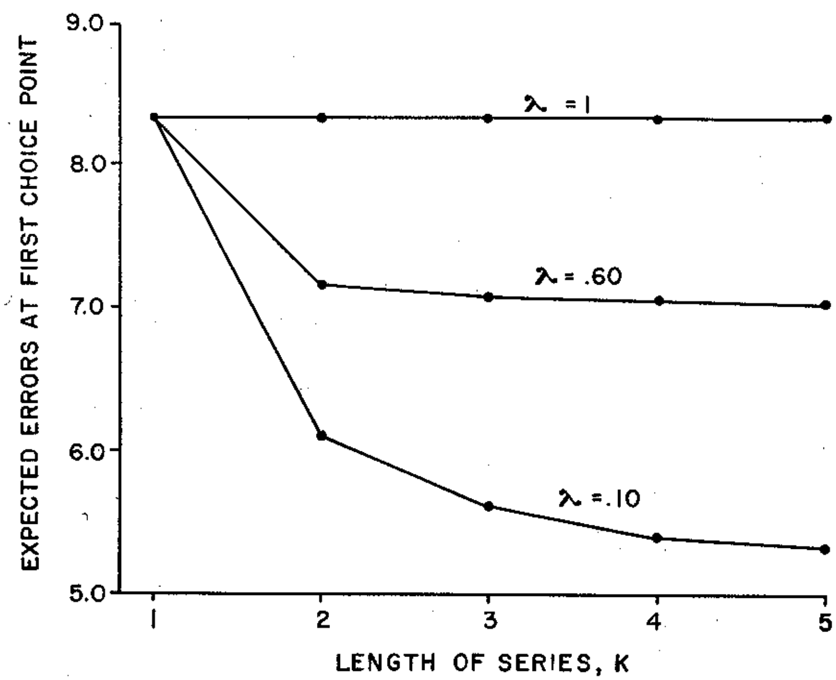

FIGURE 4. Errors at S_1 as a function of length of series for three values of the λ parameter.

series may then be expressed as

(28) $$W_j(K) = \frac{(1 - a)(1 - F_1)}{\theta\beta(1 - a^j)}\left[\frac{1}{\alpha_j} + \frac{ab}{\alpha_{j-1}} + \frac{(ab)^2}{\alpha_{j-2}} + \cdots + \frac{(ab)^{j-1}}{\alpha_1}\right],$$

where

$$\alpha_i = \frac{1 - (ab)^{K+1-i}}{1 - ab} .$$

The reader may verify Equation 28 by obtaining Equations 23 and 24 when $K = 2$ and $j = 1, 2$, respectively. Also, when $j = 1$ and K is unspecified, Equation 28 reduces to Equation 27 for $W_1(K)$.

Some clarification of Equation 28 is needed. The effective learning rate for elements of the set S_i is $\theta\beta\alpha_i$, where α_i expresses the fact that sampled elements from S_i have $K + 1 - i$ opportunities on each trial to become conditioned to A_1. Because traces from $S_{j-1}, S_{j-2}, \cdots, S_1$ influence performance at choice point j, the learning rates for these stimulus sets enter into the equation for $W_j(K)$. The factor $(1 - F_1)(\theta\beta\alpha_{j-i})^{-1}$ is weighted by the likelihood that traces from S_{j-i} will contribute to errors at choice point j. The factor $(ab)^i$ represents the probability that when choice point j is presented, traces from S_{j-i} are active but still conditioned as on the preceding trial. The proportion, denoted by c_{j-i}, of such traces in the total active sample at choice point j is

$$c_{j-i} = \frac{[(1-\lambda)(1-\beta)]^i}{\sum_{m=0}^{j-1}(1-\lambda)^m} = \frac{\lambda[(1-\lambda)(1-\beta)]^i}{1-(1-\lambda)^j}. \tag{29}$$

Accordingly, it is the weighting coefficient of $(1-F_1)(\theta\beta\alpha_{j-i})^{-1}$ in Equation 28.

Equation 28 is general and gives complete information about the predicted errors at j as a function of the parameters. We have mentioned the fact that if $j = 1$, then the theorem about $W_1(K)$ follows directly. More generally, Equation 28 implies that for any arbitrary j, $W_j(K)$ is a decreasing function of K, provided that $\lambda < 1$.

Equation 28 also implies various features of the serial-position error curve which depend on the parameters. If, for example, $\beta = 1$ (i.e., all sampled elements are conditioned to the reinforced response), then Equation 28 reduces to

$$W_j(K) = \frac{\lambda(1-F_1)}{\theta[1-(1-\lambda)^j]}. \tag{30}$$

Equation 30 gives $W_j(K)$ as a decreasing function of j; hence, the fewest errors are expected at the last choice point of the series. If, on the contrary, β is very small (of the order of .001), then the predicted serial-position error curve would be lowest near the beginning of the series and would increase progressively over serial position. For intermediate values of β, the point of fewest errors occurs somewhere near the middle of the series. Where it occurs depends on the behavior of the sum on the right-hand side of Equation 28. Unfortunately, there is no compact expression for the value of this sum, and one cannot find by simple methods the j which minimizes $W_j(K)$.

A final point to be noted regarding Equation 28 is that if $\lambda = 1$ (i.e., no traces are effective at succeeding choice points), the equation reduces to

$$W_j(K) = \frac{1-F_1}{\theta\beta}, \tag{31}$$

which is a constant for all j. Obviously, a trace model cannot predict a serial-position effect in an experimental situation where trace effects are absent.

The Heterogeneous Case. In the heterogeneous case, where different responses are reinforced at different choice points, no simple general functions can be shown. The results depend on the particular series of π values. It is evident that this must be so if the model is to be sensitive to the reinforcement contingencies.

Nevertheless, a few general statements can be made. For example, the asymptotic probability of the correct response at any choice point is less

than unity. The asymptote of p_j is a linear combination of the asymptotes of $F_j, F_{j-1}, \cdots, F_1$, and these in turn depend on the π values.

It is possible to derive a theorem concerning errors at the first choice point comparable to the previous theorem in the homogeneous case. If A_1 is reinforced at S_1 and A_2 is reinforced at the remaining $K-1$ choice points, then the predicted error frequency at the first choice point increases with K. If we add more choice points at which A_2 is reinforced, the traces of elements sampled from S_1 have just that many more opportunities to become conditioned to A_2, which is the wrong response at the first choice point.

If an "odd" choice point, S_j, with $\pi_j = 0$ is inserted into a series with all other $\pi_i = 1$, the number of errors increases throughout the series relative to the strict homogeneous situation. Moreover, the increment in total errors is maximal when the odd choice point is placed in the middle of the series. The reasons for this prediction may be described generally. Traces from preceding choice points that are active when E_2 occurs at S_j may become conditioned to A_2. The anticipatory errors produced by such conditioning would be most frequent at S_{j-1} and progressively less at $S_{j-2}, \cdots, S_1$. The total effect of the odd choice point in producing anticipatory errors would be maximal if it were placed at the end of the series, since in that case traces from all the preceding $K-1$ choice points could become conditioned to A_2, the "wrong" response.

However, the odd choice point may contribute to total errors in another way; traces from the odd choice point stimuli will tend to elicit erroneous A_2 responses at choice points following S_j. The perseverative errors produced by the odd choice point would be most frequent at S_{j+1} and progressively less at $S_{j+2}, \cdots, S_K$. The total effect of the odd choice point in producing perseverative errors would be maximal if the odd choice point were placed at the beginning of the series.

Since the problem is to maximize the total errors (anticipatory and perseverative combined), we are required to find that point in the series at which the odd choice point would maximize the sum of the anticipatory and perseverative error effects. Since the anticipatory and perseverative error gradients decrease exponentially with distance from the odd choice point, it turns out that the area under the two gradients is maximized by placing the odd choice point in the middle of the series. Thus, the increase in total errors throughout the series is maximal when the odd choice point is in this position. The total number of errors at the odd choice point is also maximized by placing it in the middle of the series.

Comments on the Trace Model. In developing the trace model, it was assumed that the sets of stimulus elements at the several choice points have no elements in common. If we relax this condition, we obtain a model combining traces and common-element generalization effects. Derivations within the combined model are exceedingly complicated. A wise strategy, then, when one wishes to test assumptions concerning either trace or generalization effects in serial learning, is to employ experimental arrangements in which either trace or common-element effects may be maximized or minimized independently.

In the preceding sections, we have assumed a sufficiently long intertrial interval so that all traces are inactive at the beginning of the next series. If this condition is relaxed, the model may be used to predict trial distribution phenomena. In the example of two choice points, if trials are spaced, for example, two seconds apart, there will be many S_2 traces active when S_1 is presented. Hence, the subject will tend to make the response appropriate to the second choice point when he is presented with the first one. From these considerations, we would expect performance in a homogeneous series to be facilitated by massed practice, while performance in a heterogeneous series would be impaired by massed practice.

Finally, throughout this paper we have required that reinforcement be provided for the correct response at each choice point. In practice, many studies of serial learning with animals have provided reinforcement only at the terminal choice point. For example, a hungry rat may be trained to perform correctly in a five-unit T maze although primary reinforcement (food) is given only at the last unit of the maze.

Spence [6] has proposed the hypothesis that the responses early in the series are learned through secondary reinforcement which develops in the situation. Stimuli produced by the subject when he makes the correct response at the various choice points acquire secondary reinforcing properties through their temporal proximity to the primary reinforcement in the last maze unit. The amount of secondary reinforcement developed at any point in the series depends directly upon the temporal proximity of that point to the reinforcing event. From this hypothesis one can derive the result that learning should proceed faster at choice points closer to the goal, the so-called "goal gradient."

Within the present trace model, differences in the amount of secondary reinforcement at the various choice points can be reflected by differences in β, the reinforcement parameter. If β_j denotes the reinforcement parameter at the jth choice point, then the hypothesis of graded reinforcing value would state that β_j is an increasing function of serial position (j) for situations involving terminal primary reinforcement.

To see one of the implications of this hypothesis, first recall that with serial reinforcement in a homogeneous series, the trace model, for intermediate values of β, predicts that the point of fewest errors will occur near the middle of the series. In a terminal reinforcement situation, the effect of the hypothesis about β_j is (1) to increase the total errors throughout the series, and (2) to move the point of fewest errors toward the end of the series. Both of these predictions are in agreement with data reported by Hull [4].

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