On the Large Deviations of Resequencing Queue Size: 2-M/M/1 Case

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***Abstract*—In data communication networks, packets that arrive at the receiving host may be disordered for reasons such as retransmission of dropped packets or multipath routing. Reliable protocols such as the Transmission Control Protocol (TCP) require packets to be accepted, i.e., delivered to the receiving application, in the order they are transmitted at the sender. In order to do so, the receiver's transport layer is responsible for temporarily buffering out-of-order packets and resequencing them as more packets arrive. In this paper, we analyze a model where the disordering is caused by multipath routing. Packets are generated according to a Poisson process. Then, they arrive at a disordering network (DN) modeled by two parallel M/M/1 queues, and are routed to each of the queues according to an independent Bernoulli process. A resequencing buffer follows the DN. In such a model, the packet resequencing delay is known. However, the size of the resequencing queue (RSQ) is unknown. We derive the probability for the large deviations of the queue size.**

***Index Terms*—Large deviations, resequencing queue (RSQ), transport protocol.**

I. Introduction

DATA packets can be disordered by the communication networks for various reasons [1]. For instance, with the help of the destination address contained in every packet, the network can deliberately route packets via different paths to the destination, possibly for load balancing or for reducing transfer delay. Some packets may be dropped when the network is congested or when the packet is corrupted. For reliable communication, the sender must retransmit the dropped packet, possibly causing it to arrive out-of-order at the receiver.

Most applications can only accept packets (which contain application-level data) in the same order they are transmitted at the sender. They typically rely on reliable transport protocols, such as the Transmission Control Protocol (TCP), to temporarily buffer out-of-order packets and to resequence them as new packets arrive. The study of packet disordering and resequencing is important because of the following performance implications.

- Insufficient buffer size causes packet losses and reduced throughput.
- Even when the application can consume the packets infinitely fast, the packets may still suffer resequencing delay, which increases the response time of the application.
- The large number of queued packets create bursty load to the processor. Long queue length is typically the result of one or a few very late packets. During the time of queue buildup, the processor stays idle most of the time. When the late packets finally arrive, all queued packets are suddenly eligible for processing.
- The out-of-order packets that have arrived at the receiver must wait at the transport layer, consuming precious system resources such as memory and computation cycles. Since they are shared resources, an unusually large amount of out-of-order packets can negatively affect all applications in the same system.

In our earlier paper [2], we model packet disordering by adding an independent and identically distributed (i.i.d.) random propagation delay to each packet and derive simple expressions for the required buffer size and the resequencing delay. We demonstrate that these two quantities can be significant and show that the resequencing problem becomes worse as the link speed increases. In this paper, we analyze a model with correlated delays where the disordering is caused by multipath routing. Packets are generated according to a Poisson process. Then, they arrive at a disordering network (DN) modeled by two parallel M/M/1 queues, and are routed to each of the queues according to an independent Bernoulli process. A resequencing buffer follows the DN. In such a model, the packet resequencing delay is known. However, the size of the resequencing queue (RSQ) is unknown. We derive the probability for the large deviations of the queue size.

This paper is organized as follows. In Section II, we describe the resequencing model and give the main theorem of the paper. We also discuss the relation of this study with previous studies. Sections III–V constitute the bulk of the paper, which is a proof for the main theorem. We show some implications of the theorem in the concluding Section VI.

II. The Model and the Main Result

The detailed network and resequencing model is shown in Fig. 1. Sequentially numbered customers (or packets) arrive at the DN according to a Poisson process with rate λ. Each customer either enters queue 1 with probability p, or enters queue 2 with probability $1-p$, independent of other customers. Then, the arrival processes to the queues in the DN are independent Poisson processes with rate λ_i, $i \in \{1, 2\}$, where

$$\lambda_1 = p\lambda, \quad \lambda_2 = (1-p)\lambda.$$

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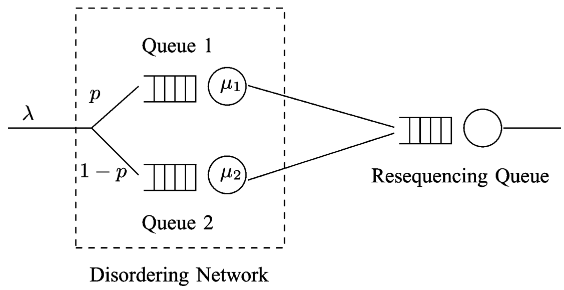

Fig. 1. Network and resequencing model.

The service times for the customers at queue i are i.i.d. exponentially distributed with mean $1/\mu_i, i = 1, 2$. Hence, we have two M/M/1 queues in the DN. Due to the multipath routing, customers may be disordered after the DN. They are resequenced at the RSQ that follows the DN. Customers immediately leave the RSQ after they are properly resequenced. That is, customer j leaves the RSQ as soon as all customers $i < j$ have arrived at the RSQ. Note that the server of the RSQ is assumed to have infinite processing capacity. We are interested in computing the stationary queue size of the RSQ. Let q^r be the stationary size of the RSQ. The main result of this paper is the following theorem. Without the loss of generality, let us assume $\mu_1 - \lambda_1 \leq \mu_2 - \lambda_2$. Then we have the following.

Theorem 1:

$$\lim_{n\to\infty} \frac{1}{n} \log P\{q^r(t) \geq n\} = \max\left\{\log \frac{\lambda_2}{\lambda_2 + \mu_1 - \lambda_1}, \log \frac{4\lambda_1\mu_1}{(\lambda_1 + \mu_1 + \mu_2 - \lambda_2)^2}\right\}. \quad (1)$$

The studies that deal with packet disordering due to multipath routing (also including parallel processing or load balancing, etc.) typically analyze an open queueing network, of which the model in Fig. 1 is a special case. In some models, a first-in first-out (FIFO) queue follows the resequencing buffer. The DN is also modeled as a queueing system, whose type typically distinguishes different studies. For instance, the DN is an M/M/∞ queue in [3], an M/GI/∞ queue in [4], a GI/GI/∞ queue in [5], an M/M/2 queue in [6], an M/M/K queue in [7], an M/H_2/K queue in [8], an M/M/2 queue with a threshold-type server assignment policy in [9], two parallel M/M/1 queues with additional fixed propagation delays in [10], and K parallel M/GI/1 queues in [11]. A survey is given in [12]. Most of these studies are concerned mostly with finding the distribution and/or mean of the resequencing delay or end-to-end delay. Several also give results about the number of packets in the RSQ. Among the previous studies reviewed here, the most relevant one is [11], where the DN consists of K parallel M/GI/1 queues. In [11], Jean-Marie and Gun derive the distribution of the resequencing delay. In contrast, our results are i) for the RSQ size, ii) of the large-deviations type, and iii) for the 2-M/M/1-queue case.

Packet disordering caused by the retransmissions of dropped packets is studied within the context of automatic repeat request (ARQ) protocols [13]–[19]. In these studies, ARQ is typically considered as a link-layer protocol running between a sender–receiver pair over a noisy link with constant propagation delay. The sender must retransmit corrupted or dropped packets based on the feedback information it gets from the receiver. Models in this family cannot be easily combined into a generic model. Their details and analytical techniques involved differ greatly. Their strength lies in that they typically can model the feedback from the receiver to the sender.

Many previous studies on ARQ models focused on the throughput of the ARQ protocol, or the delay and queue size at the sender side. For instance, Miller and Lin [15] analyzed the throughput for certain Selective-Repeat ARQ schemes. Towsley and Wolf analyzed the queue size and delay at the sender side for the Stop-and-Wait ARQ and the Go-Back-N ARQ in [13], and mean queue length for the Stutter-Go-Back-N ARQ in [20]. Konheim [14] analyzed a Go-Back-N ARQ and a Selective-Repeat ARQ. Anagnostou and Protonotarios [17] analyzed the queue size and delay at the sender side in a Selective-Repeat ARQ model. There are also several studies on the resequencing delay and queue size at the receiver in the ARQ literature. Rosberg and Shacham [18] analyzed a specific Selective-Repeat ARQ protocol over a noisy forward channel from the sender to the receiver and a perfect feedback channel. The distributions of the buffer occupancy and the resequencing delay at the receiver were derived. Rosberg and Sidi [19] extended the above model to allow nongreedy source. In several other studies, Shacham and Towsley [21] considered the resequencing problem for a multicast Selective-Repeat ARQ. Shacham and Shin [22] analyzed the resequencing problem of a Selective-Repeat-ARQ with parallel channels, using a discrete-time model. Varma [23], Ayoun and Rosberg [24] considered optimal control problems in a queue with two servers of different service rates. The question is how to assign the customers to the servers so as to minimize the end-to-end delay [23] or the long-run average holding costs of the customers [24]. Packets get disordered at the server-assignment stage and are required to be resequenced after leaving the two-server queue.

In the remaining part of the paper, we will prove Theorem 1. The basic argument of the proof is as follows. Suppose the oldest customer in the DN is C_* and is being serviced at queue 1 in the DN. We wish to find out the probability that the RSQ has at least n customers. The customers in the RSQ must have all arrived at the DN after C_*, and all gone through queue 2 in the DN during the time C_* spent in queue 1, which is (roughly) an exponential random variable, independent of the queue 2 process. Therefore, the probability that the RSQ has at least n customers is the same as the probability that at least n customers arrive at queue 2, an M/M/1 queue, and at least n of those customers depart the queue during an exponential random time T that is independent of the queue 2 process. There is also the symmetric case where the oldest customer is in queue 2 and all customers in the RSQ come from queue 1. In Section III, we set up the two different cases and write the quantities to be computed. In Section IV, we compute the key quantity, $P\{M(T) \geq n\}$, where the function $M(t)$ is the number of those customers who arrived at the M/M/1 queue on the interval $[0, t]$ and who departed by time t, and T is an exponential random variable independent of the M/M/1 queue. In Section V, we combine results of the previous two sections and give the proof for Theroem 1.

III. The Setup

At time t, let $V(t)$ be the event {the DN is empty at time t}. If $\bar{V}(t)$, let $C_*(t)$ be the oldest customer in the DN, let $W_*(t)$ be the time $C_*(t)$ has spent in the DN, and let $I_*(t)$ be the queue in the DN which $C_*(t)$ goes through. For $n \geq 0$, let

$$E(t,s,n) = \{\text{at least } n \text{ customers arrived at the DN on the interval } (t-s,\ t],\ \text{out of which at least } n \text{ have left the DN by } t\}.$$

Let the size of the RSQ at time t be $q^r(t)$, and let $q_i(t)$ be the size of queue i at time t, where $i = 1$ or 2. Then, for $n > 0$

$$P\{q^r(t) \geq n\} = P\{\bar{V}(t) \text{ and } E(t, W_*(t), n)\}. \tag{2}$$

Next, we will explain equality (2). When the RSQ size is greater than or equal to n, where $n > 0$, it must be waiting for some customer still in the DN. In particular, the next packet gap the RSQ is trying to fill is $C_*(t)$. The customers in the RSQ are exactly those who arrived at the DN later than $C_*(t)$, but who have left the DN by time t. We are interested in computing $\lim_{t\to\infty} P\{q^r(t) \geq n\}$. Alternatively, let us assume all relevant processes are stationary.

Let us extend the definition of $W_*(t)$, $W_*(t) = 0$ if $V(t)$. Then, when $n = 0$

$$\begin{aligned}
&P\{q^r(t) \geq n\} = 1.\\
&P\{\bar{V}(t) \text{ and } E(t, W_*(t), n)\}\\
&\quad = P\{E(t, W_*(t), n)|\bar{V}(t)\}P\{\bar{V}(t)\} = P\{\bar{V}(t)\}\\
&P\{V(t) \text{ and } E(t, W_*(t), n)\}\\
&\quad = P\{E(t, W_*(t), n)|V(t)\}P\{V(t)\} = P\{V(t)\}.
\end{aligned}$$

Hence, for $n = 0$

$$P\{q^r(t) \geq n\} = P\{E(t, W_*(t), n)\}. \tag{3}$$

For $n > 0$, (3) is still true because

$$\begin{aligned}
&P\{V(t) \text{ and } E(t, W_*(t), n)\}\\
&\qquad = P\{E(t, 0, n)|V(t)\}P\{V(t)\} = 0.
\end{aligned}$$

Note that, because customers are served on first-come first-serve basis in each of the queues, the oldest customers in the nonempty DN must be in service at one of the queues. If queue i is not empty, $i \in \{0, 1\}$, let $W_i(t)$ be the duration for which the customer in service at queue i has stayed in the queue. If queue i is empty, let $W_i(t) = 0$. By using a simple reversibility argument, $W_i(t)$ has the same distribution as the waiting time in queue i (not including the service time) by an arbitrary customer. This distribution and the density are [25, p. 213], for $x \geq 0$

$$F_{W_i}(x) = P\{W_i(t) \leq x\} = 1 - \rho_i e^{-(\mu_i - \lambda_i)x} \tag{4}$$

$$f_{W_i}(x) = (1 - \rho_i)\delta(x) + \lambda_i(1 - \rho_i)e^{-(\mu_i - \lambda_i)x} \tag{5}$$

where $\rho_i = \lambda_i/\mu_i$, and $\delta(x)$ is the Dirac delta function, representing the point probability mass at $x = 0$. We will occasionally omit the dependency on t for brevity.

Let $\hat{M}_i(t,s)$ be the number of those customers who arrived at queue i on the interval $(t-s, t]$ and who departed by time t. Note that for $n > 0$

$$\begin{aligned}
&P\{\hat{M}_1(t, W_*(t)) \geq n \mid W_1(t) = W_2(t) = 0\}\\
&\quad = P\{\hat{M}_1(t, 0) \geq n \mid W_1(t) = W_2(t) = 0\} = 0.
\end{aligned}$$

Also

$$P\{W_1(t) = W_2(t) \neq 0\} = 0.$$

Therefore

$$\begin{aligned}
&P\{\hat{M}_1(t, W_*(t)) \geq n \mid W_1(t) = W_2(t)\}\\
&\qquad \cdot P\{W_1(t) = W_2(t)\} = 0.
\end{aligned}$$

Then, for $n > 0$

$$\begin{aligned}
&P\{q^r(t) \geq n\}\\
&\quad = P\{E(t, W_*(t), n)\}\\
&\quad = P\{\hat{M}_2(t, W_*(t)) \geq n \mid W_1(t) > W_2(t)\}\\
&\qquad \cdot P\{W_1(t) > W_2(t)\}\\
&\qquad + P\{\hat{M}_1(t, W_*(t)) \geq n \mid W_2(t) > W_1(t)\}\\
&\qquad \cdot P\{W_2(t) > W_1(t)\}.
\end{aligned} \tag{6}$$

This can be explained as follows. If $W_1(t) > W_2(t)$, then the oldest customer, $C_*(t)$, in the DN must be in service at queue 1. Hence, $W_1(t) = W_*(t)$. All customers who came to the DN after $C_*(t)$ and who have left the DN by time t must have been routed to the RSQ via queue 2.

For $n > 0$

$$\begin{aligned}
&P\{\hat{M}_2(t, W_*(t)) \geq n \mid W_1(t) > W_2(t)\}\\
&\quad = \int_{0^+}^{\infty} P\{\hat{M}_2(t, s) \geq n \mid W_1(t) = s, W_1(t) > W_2(t)\}\\
&\qquad \cdot f_{W_1|W_1>W_2}(s)ds\\
&\quad = \int_{0^+}^{\infty} P\{\hat{M}_2(t, s) \geq n \mid W_1(t) = s, W_2(t) < s\}\\
&\qquad \cdot f_{W_1|W_1>W_2}(s)ds\\
&\quad = \int_{0^+}^{\infty} P\{\hat{M}_2(t, s) \geq n \mid W_2(t) < s\}\\
&\qquad \cdot f_{W_1|W_1>W_2}(s)ds.
\end{aligned} \tag{7}$$

In the above, $f_{W_1|W_1>W_2}(s)$ denotes the conditional density of $W_1(t)$ given $\{W_1(t) > W_2(t)\}$. In the last step, we used the fact that the two queue processes are independent. Note that, in the integral, the (conditional) probability mass at $s = 0$ does not contribute to the probability on the left-hand side.

We will compute the conditional density by starting with the joint probability. For $x \geq 0$

$$\begin{aligned}
&P\{W_1 > x, W_1 > W_2\}\\
&\quad = \rho_1 e^{-(\mu_1-\lambda_1)x} - \rho_1\rho_2 \frac{\mu_1 - \lambda_1}{\mu_1 - \lambda_1 + \mu_2 - \lambda_2}\\
&\qquad \cdot e^{-(\mu_1-\lambda_1+\mu_2-\lambda_2)x}.
\end{aligned} \tag{8}$$

From (8), we have

$$\begin{aligned}
P\{W_1 > W_2\} &= P\{W_1 > 0, W_1 > W_2\}\\
&= \rho_1 - \rho_1\rho_2 \frac{\mu_1 - \lambda_1}{\mu_1 - \lambda_1 + \mu_2 - \lambda_2}.
\end{aligned} \tag{9}$$

From (8) and (9), we get the conditional density for $x \geq 0$

$$f_{W_1|W_1>W_2}(x) = K_1 e^{-(\mu_1-\lambda_1)x} - K_2 e^{-(\mu_1-\lambda_1+\mu_2-\lambda_2)x} \quad (10)$$

where K_1 and K_2 are constants, given by

$$K_1 = \frac{\mu_1-\lambda_1}{1-\rho_2\frac{\mu_1-\lambda_1}{\mu_1-\lambda_1+\mu_2-\lambda_2}} \quad (11)$$

$$K_2 = \frac{\rho_2(\mu_1-\lambda_1)}{1-\rho_2\frac{\mu_1-\lambda_1}{\mu_1-\lambda_1+\mu_2-\lambda_2}}. \quad (12)$$

Note that the second term in (10) decays much faster than the first term. If we ignore it, the conditional probability density decays exponentially.

Next, we will bound (7) from above and below.

$$\begin{aligned}
&\int_{0^+}^{\infty} P\{\hat{M}_2(t,s)\geq n \mid W_2(t)<s\} f_{W_1|W_1>W_2}(s)ds\\
&= \int_{0^+}^{\infty} P\{\hat{M}_2(t,s)\geq n, W_2(t)<s\}\frac{f_{W_1|W_1>W_2}(s)}{P\{W_2(t)<s\}}ds\\
&\leq \int_{0^+}^{\infty} P\{\hat{M}_2(t,s)\geq n\}\frac{f_{W_1|W_1>W_2}(s)}{P\{W_2(t)=0\}}ds\\
&\leq \frac{1}{1-\rho_2}\int_{0^+}^{\infty} P\{\hat{M}_2(t,s)\geq n\} f_{W_1|W_1>W_2}(s)ds. \quad (13)
\end{aligned}$$

For a lower bound

$$\begin{aligned}
&\int_{0^+}^{\infty} P\{\hat{M}_2(t,s)\geq n \mid W_2(t)<s\} f_{W_1|W_1>W_2}(s)ds\\
&= \int_{0^+}^{\infty} P\{\hat{M}_2(t,s)\geq n, W_2(t)<s\}\frac{f_{W_1|W_1>W_2}(s)}{P\{W_2(t)<s\}}ds\\
&\geq \int_{0^+}^{\infty} P\{\hat{M}_2(t,s)\geq n, W_2(t)=0\} f_{W_1|W_1>W_2}(s)ds\\
&= \int_{0^+}^{\infty} P\{\hat{M}_2(t,s)\geq n, q_2(t)=0\} f_{W_1|W_1>W_2}(s)ds. \quad (14)
\end{aligned}$$

In Section IV, we will prepare to compute the upper and lower bounds.

IV. Computation of $P\{M(T)\geq n\}$

In this section, we consider a stationary M/M/1 queue whose arrival rate is λ_1 and whose departure rate is μ_1. We assume $\lambda_1 < \mu_1$ so that the queue is stable. Let T be an exponential random variable independent of the queue process with mean $1/(\mu_2-\lambda_2)$, where $\lambda_2<\mu_2$. Let $M(t)$ be the number of those customers who arrived on the interval $[0,t]$ and who departed by time t. We wish to compute $P\{M(T)\geq n\}$ for large n. The main result of this section is Theorem 2. A similar result is Lemma 4.

Theorem 2:

$$\lim_{n\to\infty}\frac{1}{n}\log P\{M(T)\geq n\} = \begin{cases}\log\frac{\lambda_1}{\lambda_1+\mu_2-\lambda_2}, & \text{if } \mu_1-\lambda_1\geq\mu_2-\lambda_2\\ \log\frac{4\lambda_1\mu_1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^2}, & \text{if } \mu_1-\lambda_1<\mu_2-\lambda_2.\end{cases} \quad (15)$$

In Sections IV-A1 and -A2, we will prove Theorem 2. We will frequently use the following fact. For $a>0$ and integer $k\geq 0$, we get the following.

Fact 3:

$$\int_0^{\infty}\frac{e^{-at}t^k}{k!}dt = \left(\frac{1}{a}\right)^{k+1}. \quad (16)$$

A. Case of $\mu_1-\lambda_1\geq\mu_2-\lambda_2$

1) The Upper Bound:

$$\begin{aligned}
&P\{M(T)\geq n\}\\
&\leq P\{\text{the number of customer arrivals on the interval } [0,T] \text{ is at least } n\}\\
&= \sum_{k=n}^{\infty}\int_0^{\infty}\frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}(\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt\\
&= \sum_{k=n}^{\infty}(\mu_2-\lambda_2)\int_0^{\infty}\frac{e^{-(\lambda_1+\mu_2-\lambda_2)t}(\lambda_1 t)^k}{k!}dt\\
&= \sum_{k=n}^{\infty}\frac{\mu_2-\lambda_2}{\lambda_1+\mu_2-\lambda_2}\left(\frac{\lambda_1}{\lambda_1+\mu_2-\lambda_2}\right)^k \quad \text{(by (16))}\\
&= \frac{\mu_2-\lambda_2}{\lambda_1+\mu_2-\lambda_2}\frac{\left(\frac{\lambda_1}{\lambda_1+\mu_2-\lambda_2}\right)^n}{1-\frac{\lambda_1}{\lambda_1+\mu_2-\lambda_2}}\\
&= \left(\frac{\lambda_1}{\lambda_1+\mu_2-\lambda_2}\right)^n. \quad (17)
\end{aligned}$$

2) The Lower Bound:

For a function that grows as $\exp(\alpha n + o(n))$ when n increases, α is the rate of growth. The method that estimates the rate of growth of an integral by that of the maximum of the integrand is known as the Laplace principle (see [26, p. 12]). In our case, we will consider the following integral, as n gets large

$$\int_0^{\infty}\frac{e^{-\lambda_1 t}(\lambda_1 t)^n}{n!}(\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt.$$

It can be shown easily that the integrand is maximized at

$$t_o = n/(\lambda_1+\mu_2-\lambda_2). \quad (18)$$

This information will be useful in the proof for the lower bound.

Let $q(t)$ be the queue size at time t. Let $D(t)$ be the number of departures on the interval $[0,t]$.

$$\begin{aligned}
&P\{M(t)=k\}\\
&= \sum_{m=0}^{\infty}P\{M(t)=k|q(0)=m\}P\{q(0)=m\}\\
&\geq P\{M(t)=k|q(0)=0\}P\{q(0)=0\}\\
&= (1-\rho_1)P\{D(t)=k|q(0)=0\}\\
&\geq (1-\rho_1)P\{D(t)=k, q(t)=0|q(0)=0\}. \quad (19)
\end{aligned}$$

From [27, p. 199]

$$\begin{aligned}
&P\{D(t)=k, q(t)=0|q(0)=0\}\\
&= \sum_{i=0}^{\infty}\frac{(1+i)\rho_1^k}{k!(k+i+1)!}(\mu_1 t)^{2k+i}e^{-(\lambda_1+\mu_1)t}\\
&= \frac{(\lambda_1 t)^k e^{-\lambda_1 t}}{k!}\sum_{i=0}^{\infty}\frac{1+i}{(k+i+1)!}(\mu_1 t)^{k+i}e^{-\mu_1 t}
\end{aligned}$$

$$\geq \frac{1}{k+1}\frac{(\lambda_1 t)^k e^{-\lambda_1 t}}{k!}\sum_{i=0}^{\infty}\frac{1}{(k+i)!}(\mu_1 t)^{k+i}e^{-\mu_1 t}$$
$$= \frac{1}{k+1}\frac{(\lambda_1 t)^k e^{-\lambda_1 t}}{k!}P\{Y_{(\mu_1 t)} \geq k\} \tag{20}$$

where $Y_{(\mu_1 t)}$ is a Poisson random variable with mean $\mu_1 t$. Now, with the definition of t_o as in (18)

$$\begin{aligned}
&P\{M(T) \geq n\}\\
&\quad \geq P\{M(T) \geq n, T \geq t_o\}\\
&\quad \geq P\{M(t_o) \geq n, T \geq t_o\}\\
&\quad = P\{M(t_o) \geq n\}P\{T \geq t_o\} &(21)\\
&\quad = \sum_{k=n}^{\infty} P\{M(t_o) = k\}P\{T \geq t_o\}. &(22)
\end{aligned}$$

The equality in (21) is because of independence between the queue process and the random variable T. Then, by (22), (19), and (20)

$$\begin{aligned}
&P\{M(T) \geq n\}\\
&\quad \geq (1-\rho_1)\sum_{k=n}^{\infty}\frac{1}{k+1}\frac{(\lambda_1 t_o)^k e^{-\lambda_1 t_o}}{k!}\\
&\qquad \cdot P\{Y_{(\mu_1 t_o)} \geq k\}e^{-(\mu_2-\lambda_2)t_o}\\
&\quad \geq (1-\rho_1)\frac{1}{n+1}\frac{(\lambda_1 t_o)^n e^{-\lambda_1 t_o}}{n!}\\
&\qquad \cdot P\{Y_{(\mu_1 t_o)} \geq n\}e^{-(\mu_2-\lambda_2)t_o}. &(23)
\end{aligned}$$

We will show $P\{Y_{(\mu_1 t_o)} \geq n\}$ is greater than a constant as n tends to infinity. By the definition of t_o and by the assumption $\mu_1 - \lambda_1 \geq \mu_2 - \lambda_2$

$$\mu_1 t_o = \frac{\mu_1}{\lambda_1 + \mu_2 - \lambda_2}n \geq n.$$

Let $n_o = \lfloor \mu_1 t_o \rfloor$. Then, $n_o \geq n$. Let $X_1, X_2, \ldots, X_{n_o}$ be i.i.d. Poisson random variables with mean 1. Then

$$\begin{aligned}
P\{Y_{(\mu_1 t_o)} \geq n\} &\geq P\left\{\frac{X_1 + X_2 + \cdots + X_{n_o}}{n_o} \geq \frac{n}{n_o}\right\}\\
&\geq P\left\{\frac{X_1 + X_2 + \cdots + X_{n_o}}{n_o} \geq 1\right\}\\
&= P\left\{\frac{X_1 + X_2 + \cdots + X_{n_o} - n_o}{\sqrt{n_o}\sqrt{n_o}} \geq 0\right\}\\
&= P\left\{\frac{X_1 + X_2 + \cdots + X_{n_o} - n_o}{\sqrt{n_o}} \geq 0\right\}.
\end{aligned}$$

By the central limit theorem

$$\lim_{n_o \to \infty} P\left\{\frac{X_1 + X_2 + \cdots + X_{n_o} - n_o}{\sqrt{n_o}} \geq 0\right\} = \int_0^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = \frac{1}{2}.$$

Therefore, for any $\epsilon > 0$, there exists some integer $N > 0$ such that for all $n > N$

$$P\{Y_{(\mu_1 t_o)} \geq n\} \geq \frac{1}{2} - \epsilon. \tag{24}$$

Continuing from (23), for all $n > N$

$$\begin{aligned}
&P\{M(T) \geq n\}\\
&\geq (1-\rho_1)\left(\frac{1}{2} - \epsilon\right)\frac{1}{n+1}\frac{(\lambda_1 t_o)^n e^{-\lambda_1 t_o}}{n!}e^{-(\mu_2-\lambda_2)t_o}. &(25)
\end{aligned}$$

By Stirling's approximation

$$n! = \sqrt{2\pi n}\,n^n e^{-n}(1 + O(1/n)).$$

For sufficiently large n

$$n! \leq 2\sqrt{2\pi n}\,n^n e^{-n}.$$

Therefore, for large enough n, using the definition for t_o in (18), we have

$$\begin{aligned}
&P\{M(T) \geq n\}\\
&\quad \geq \frac{1}{4}(1-\rho_1)(1-2\epsilon)\frac{1}{n+1}\left(\frac{\lambda_1}{\lambda_1 + \mu_2 - \lambda_2}\right)^n\\
&\qquad \cdot \frac{n^n \exp\left(-\frac{\lambda_1}{\lambda_1+\mu_2-\lambda_2}n\right)}{\sqrt{2\pi n}\,n^n e^{-n}}\exp\left(-\frac{\mu_2 - \lambda_2}{\lambda_1 + \mu_2 - \lambda_2}n\right)\\
&\quad = \frac{(1-\rho_1)(1-2\epsilon)}{4\sqrt{2\pi n}(n+1)}\left(\frac{\lambda_1}{\lambda_1 + \mu_2 - \lambda_2}\right)^n. &(26)
\end{aligned}$$

B. *Case of* $\mu_1 - \lambda_1 < \mu_2 - \lambda_2$

1) The Lower Bound: By (19) and (20)

$$\begin{aligned}
&P\{M(T) \geq n\}\\
&\quad \geq \int_0^{\infty}\sum_{k=n}^{\infty}(1-\rho_1)P\{D(t) = k, q(t) = 0 | q(0) = 0\}\\
&\qquad \cdot (\mu_2 - \lambda_2)e^{-(\mu_2-\lambda_2)t}dt\\
&\quad \geq (1-\rho_1)(\mu_2-\lambda_2)\frac{1}{n+1}\\
&\qquad \cdot \int_0^{\infty}\frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!}P\{Y_{(\mu_1 t)} = n\}e^{-(\mu_2-\lambda_2)t}dt\\
&\quad = (1-\rho_1)(\mu_2-\lambda_2)\frac{1}{n+1}\\
&\qquad \cdot \int_0^{\infty}\frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!}\frac{(\mu_1 t)^n e^{-\mu_1 t}}{n!}e^{-(\mu_2-\lambda_2)t}dt\\
&\quad = (1-\rho_1)(\mu_2-\lambda_2)\frac{1}{n+1}\frac{(2n)!}{n!n!}(\lambda_1\mu_1)^n\\
&\qquad \cdot \int_0^{\infty}\frac{t^{2n}e^{-(\lambda_1+\mu_1+\mu_2-\lambda_2)t}}{(2n)!}dt\\
&\quad = (1-\rho_1)\frac{\mu_2-\lambda_2}{\lambda_1+\mu_1+\mu_2-\lambda_2}\frac{1}{n+1}\frac{(2n)!}{n!n!}\\
&\qquad \cdot (\lambda_1\mu_1)^n\frac{1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^{2n}}. &(27)
\end{aligned}$$

In deriving the last step, (16) has been used. By Stirling's approximation, for large enough n

$$\frac{(2n)!}{n!n!} = \frac{\sqrt{4\pi n}(2n)^{2n}e^{-2n}(1+O(1/n))}{(\sqrt{2\pi n}(n)^n e^{-n}(1+O(1/n)))^2} \geq \frac{C_1}{\sqrt{n}}4^n$$

for some constant $C_1 > 0$. Therefore

$$P\{M(T) \geq n\} \geq C_1(1-\rho_1)\frac{\mu_2-\lambda_2}{\lambda_1+\mu_1+\mu_2-\lambda_2} \cdot \frac{1}{\sqrt{n}(n+1)}\left(\frac{4\lambda_1\mu_1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^2}\right)^n. \tag{28}$$

2) The Upper Bound: The computation for the upper bound in the previous case does not apply here. To see the reason, consider the integral in the lower bound calculation. Suppose, as n becomes large, that

$$\int_0^\infty \frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!}\frac{(\mu_1 t)^n e^{-\mu_1 t}}{n!}e^{-(\mu_2-\lambda_2)t}dt \approx \max_{t\geq 0}\frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!}\frac{(\mu_1 t)^n e^{-\mu_1 t}}{n!}e^{-(\mu_2-\lambda_2)t}.$$

It can be shown easily the above maximum is achieved at

$$t_o = \frac{2n}{\lambda_1+\mu_1+\mu_2-\lambda_2}. \tag{29}$$

Note that when $\mu_1-\lambda_1 < \mu_2-\lambda_2$

$$\mu_1 t_0 = \frac{2\mu_1 n}{\lambda_1+\mu_1+\mu_2-\lambda_2} < n.$$

Therefore, $\{Y_{(\mu_1 t_o)} \geq n\}$ is a large deviations type of event instead of an event with constant probability, as n becomes large. It is not tight enough to bound $P\{M(t)\geq n\}$ from above by only looking at the arrival processes, as was done in (17).

We will next carry out the analysis on the upper bound.

$$\begin{aligned}P\{M(t)\geq n\} \leq P\{&\text{at least } n \text{ customers arrived}\\ &\text{on the interval } [0,t], \text{ and}\\ &\text{at least } n \text{ customers are served}\\ &\text{on the same interval}\}\end{aligned}$$

$$\leq \sum_{k=n}^\infty \frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}P\left\{\sum_{i=1}^n X_i \leq t\right\} \tag{30}$$

where $\{X_1, X_2, \ldots, X_n\}$ are i.i.d. service times. The sum $\sum_{i=1}^n X_i$ has the Gamma distribution with density

$$f(t) = \frac{\mu_1 e^{-\mu_1 t}(\mu_1 t)^{n-1}}{(n-1)!}.$$

Hence

$$\begin{aligned}P\{M(T)\geq n\} &\leq \sum_{k=n}^\infty \int_0^\infty \frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!} \int_0^t \frac{\mu_1 e^{-\mu_1\tau}(\mu_1\tau)^{n-1}}{(n-1)!}d\tau(\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt\\ &= \mu_1(\mu_2-\lambda_2)\sum_{k=n}^\infty\int_0^\infty\int_\tau^\infty \frac{e^{-(\lambda_1+\mu_2-\lambda_2)t}(\lambda_1 t)^k}{k!}dt \cdot \frac{e^{-\mu_1\tau}(\mu_1\tau)^{n-1}}{(n-1)!}d\tau.\end{aligned}$$

Let $t=\tau+u$. The above becomes (31) shown at the bottom of the page. For $i=0,1,\ldots,k$, define

$$a(k,i) = \frac{(n-1+k-i)!}{2^k(k-i)!(n-1)!}.$$

Let

$$\beta = \frac{\lambda_1+\mu_1+\mu_2-\lambda_2}{\lambda_1+\mu_2-\lambda_2}.$$

Note that for $\mu_1-\lambda_1 < \mu_2-\lambda_2$, $\beta < 2$

$$\frac{a(k+1,i)}{a(k,i)} = \frac{n+k-i}{2(k+1-i)} = \frac{1+\frac{n-1}{k+1-i}}{2}.$$

$$\begin{aligned}P\{M(T)\geq n\} &\leq \mu_1(\mu_2-\lambda_2)\sum_{k=n}^\infty\int_0^\infty\int_0^\infty \frac{e^{-(\lambda_1+\mu_2-\lambda_2)(\tau+u)}(\lambda_1(\tau+u))^k}{k!}du\frac{e^{\mu_1\tau}(\mu_1\tau)^{n-1}}{(n-1)!}d\tau\\ &= \mu_1(\mu_2-\lambda_2)\sum_{k=n}^\infty\lambda_1^k\int_0^\infty\int_0^\infty \frac{e^{-(\lambda_1+\mu_2-\lambda_2)u}\sum_{i=0}^k\frac{k!}{i!(k-i)!}u^i\tau^{k-i}}{k!}du\frac{e^{-(\lambda_1+\mu_1+\mu_2-\lambda_2)\tau}(\mu_1\tau)^{n-1}}{(n-1)!}d\tau\\ &= \mu_1(\mu_2-\lambda_2)\sum_{k=n}^\infty\lambda_1^k\sum_{i=0}^k\frac{1}{(k-i)!}\int_0^\infty\int_0^\infty\frac{e^{-(\lambda_1+\mu_2-\lambda_2)u}u^i}{i!}du\frac{e^{-(\lambda_1+\mu_1+\mu_2-\lambda_2)\tau}\tau^{k-i}(\mu_1\tau)^{n-1}}{(n-1)!}d\tau\\ &= (\mu_2-\lambda_2)\mu_1^n\sum_{k=n}^\infty\lambda_1^k\sum_{i=0}^k\frac{1}{(k-i)!}\frac{1}{(\lambda_1+\mu_2-\lambda_2)^{i+1}}\int_0^\infty\frac{e^{-(\lambda_1+\mu_1+\mu_2-\lambda_2)\tau}\tau^{n-1+k-i}}{(n-1)!}d\tau\\ &= (\mu_2-\lambda_2)\mu_1^n\sum_{k=n}^\infty\lambda_1^k\sum_{i=0}^k\frac{(n-1+k-i)!}{(k-i)!(n-1)!}\frac{1}{(\lambda_1+\mu_2-\lambda_2)^{i+1}}\frac{1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^{n+k-i}}\\ &= \frac{\mu_2-\lambda_2}{\lambda_1+\mu_2-\lambda_2}\left(\frac{\mu_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^n\sum_{k=n}^\infty\left(\frac{\lambda_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^k\\ &\quad\cdot\sum_{i=0}^k\frac{(n-1+k-i)!}{(k-i)!(n-1)!}\left(\frac{\lambda_1+\mu_1+\mu_2-\lambda_2}{\lambda_1+\mu_2-\lambda_2}\right)^i.\end{aligned} \tag{31}$$

Then, for each fixed $i \in \{0, 1, \ldots, k\}$

$$\frac{a(k+1,i)}{a(k,i)} \begin{cases} \geq 1, & \text{if } k \leq n+i-2 \\ < 1, & \text{if } k > n+i-2. \end{cases}$$

Therefore, $a(k,i)$ is maximized at $k = n+i-1$ for each i.[1] Then

$$a(n+i-1,i) = \frac{(2n-2)!}{(n-1)!(n-1)!2^{n+i-1}}.$$

Then, the sum in (31) index by i becomes

$$\begin{aligned} &\sum_{i=0}^{k} \frac{(n-1+k-i)!}{(k-i)!(n-1)!} \left(\frac{\lambda_1+\mu_1+\mu_2-\lambda_2}{\lambda_1+\mu_2-\lambda_2}\right)^i \\ &= 2^k \sum_{i=0}^{k} a(k,i)\beta^i \\ &\leq 2^{k-n} \sum_{i=0}^{k} \frac{2(2n-2)!}{(n-1)!(n-1)!} \left(\frac{\beta}{2}\right)^i \\ &\leq 2^{k-n} \frac{2(2n-2)!}{(n-1)!(n-1)!} \sum_{i=0}^{\infty} \left(\frac{\beta}{2}\right)^i \\ &= 2^{k-n} \frac{2(2n-2)!}{(n-1)!(n-1)!} \frac{2}{2-\beta}. \end{aligned}$$

The infinite sum above is finite because $\beta/2 < 1$. Going back to (31), we get (32) and (33) at the bottom of the page. The sum in (32) is finite because, for $\mu_1 - \lambda_1 < \mu_2 - \lambda_2$ and $\lambda_1 < \mu_1$

$$\frac{2\lambda_1}{\lambda_1+\mu_1+\mu_2-\lambda_2} < 1. \tag{34}$$

By Stirling's approximation

$$\frac{(2n)!}{n!n!} = \frac{\sqrt{4\pi n}(2n)^{2n}e^{-2n}(1+O(1/n))}{(\sqrt{2\pi n}(n)^n e^{-n}(1+O(1/n)))^2} \leq \frac{C_2}{\sqrt{n}}4^n. \tag{35}$$

Combining (33) and (35), we have, for some constant $C_4 > 0$

$$P\{M(T) \geq n\} \leq \frac{C_4}{\sqrt{n}} \left(\frac{4\lambda_1\mu_1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^2}\right)^n.$$

This completes the analysis on the upper bound.

C. A Related Lemma to Theorem 2

The following lemma is also used in the proof of Theorem 1. Its proof is directly related to that of Theorem 2. The notations will be the same as those used in the proof of the lower bound in Theorem 1.

[1]We assume that n is large enough when necessary. In this case, $n \geq 1$.

Lemma 4:

$$\lim_{n\to\infty} \frac{1}{n} \log P\{M(T) \geq n, q(T) = 0\} = \begin{cases} \log \frac{\lambda_1}{\lambda_1+\mu_2-\lambda_2}, & \text{if } \mu_1 - \lambda_1 \geq \mu_2 - \lambda_2 \\ \log \frac{4\lambda_1\mu_1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^2}, & \text{if } \mu_1 - \lambda_1 < \mu_2 - \lambda_2. \end{cases} \tag{36}$$

Proof: Since

$$P\{M(T) \geq n, q(T) = 0\} \leq P\{M(T) \geq n\}$$

the upper bound is immediate from Theorem 2. We only need to show the left-hand side of (36) is not less than the right-hand side.

Let us first consider the case where $\mu_1 - \lambda_1 \geq \mu_2 - \lambda_2$. Conditional on T, which is independent of the queue process, we have

$$\begin{aligned} &P\{M(T) \geq n, q(T) = 0\} \\ &= \int_0^\infty P\{M(t) \geq n, q(t) = 0\}(\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \\ &= \sum_{k=n}^{\infty} \int_0^\infty P\{M(t) = k, q(t) = 0\}(\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \\ &\geq \sum_{k=n}^{\infty} \int_0^\infty P\{M(t) = k, q(t) = 0 | q(0) = 0\}P\{q(0) = 0\} \\ &\quad \cdot (\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \\ &= \sum_{k=n}^{\infty} \int_0^\infty (1-\rho_1)P\{D(t) = k, q(t) = 0 | q(0) = 0\} \\ &\quad \cdot (\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \qquad (37) \\ &\geq \sum_{k=n}^{\infty} \int_0^\infty (1-\rho_1)\frac{1}{k+1}\frac{(\lambda_1 t)^k e^{-\lambda_1 t}}{k!}P\{Y_{(\mu_1 t)} \geq k\} \\ &\quad \cdot (\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \qquad (38) \\ &\geq \sum_{k=n}^{\infty} \int_{t_o}^\infty (1-\rho_1)\frac{1}{k+1}\frac{(\lambda_1 t)^k e^{-\lambda_1 t}}{k!}P\{Y_{(\mu_1 t)} \geq k\} \\ &\quad \cdot (\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \\ &\geq \int_{t_o}^\infty (1-\rho_1)\frac{1}{n+1}\frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!}P\{Y_{(\mu_1 t)} \geq n\} \\ &\quad \cdot (\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt. \qquad (39) \end{aligned}$$

To obtain ((38), we have used (20). In the last two steps above, t_o is as given in (18). Note that, for all $t \geq t_o$

$$P\{Y_{(\mu_1 t)} \geq n\} \geq P\{Y_{(\mu_1 t_o)} \geq n\}. \tag{40}$$

$$\begin{aligned} P\{M(T) \geq n\} &\leq \frac{4(\mu_2-\lambda_2)}{(2-\beta)(\lambda_1+\mu_2-\lambda_2)} \frac{(2n-2)!}{2^n(n-1)!(n-1)!} \left(\frac{\mu_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^n \sum_{k=n}^{\infty} \left(\frac{2\lambda_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^k \qquad (32) \\ &= \frac{4(\mu_2-\lambda_2)}{\mu_2-\lambda_2-(\mu_1-\lambda_1)} \frac{(2n-2)!}{2^n(n-1)!(n-1)!} \left(\frac{\mu_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^n \\ &\quad \cdot \left(\frac{2\lambda_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^n \Big/ \left(1 - \frac{2\lambda_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right). \qquad (33) \end{aligned}$$

By (40) and (24), we obtain that, for any $\epsilon > 0$, there exists some integer $N > 0$ such that for all $n \geq N$ and for all $t \geq t_o$

$$P\{Y_{(\mu_1 t)} \geq n\} \geq \frac{1}{2} - \epsilon.$$

Continuing from (39), we get

$$P\{M(T) \geq n, q(T) = 0\} \geq (1-\rho_1)\left(\frac{1}{2} - \epsilon\right)\frac{(\mu_2 - \lambda_2)}{n+1}\int_{t_o}^{\infty}\frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!}e^{-(\mu_2-\lambda_2)t}dt. \tag{41}$$

As noted in (18), the above integrand achieves the maximum value at t_o. Furthermore, it is easy to show that, for $t \geq t_o$, the integrand is a decreasing function of t. We must have

$$\begin{aligned}\int_{t_o}^{\infty}&\frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!}e^{-(\mu_2-\lambda_2)t}dt \\ &\geq \int_{t_o}^{t_o+1}\frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!}e^{-(\mu_2-\lambda_2)t}dt \\ &\geq \frac{(\lambda_1(t_o+1))^n e^{-\lambda_1(t_o+1)}}{n!} \\ &\quad\times e^{-(\mu_2-\lambda_2)(t_o+1)} \\ &\geq \frac{(\lambda_1 t_o)^n e^{-\lambda_1 t_o}}{n!}e^{-(\mu_2-\lambda_2)t_o}e^{-(\lambda_1+\mu_2-\lambda_2)}.\end{aligned} \tag{42}$$

Combining (41) and (42), for sufficiently large n, we get

$$\begin{aligned}P\{M(T) &\geq n, q(T) = 0\} \\ &\geq (1-\rho_1)\left(\frac{1}{2} - \epsilon\right)\frac{(\mu_2 - \lambda_2)}{n+1}e^{-(\lambda_1+\mu_2-\lambda_2)} \\ &\quad\cdot\frac{(\lambda_1 t_o)^n e^{-\lambda_1 t_o}}{n!}e^{-(\mu_2-\lambda_2)t_o}.\end{aligned} \tag{43}$$

Following the steps in (25) to (26), for sufficiently large n, we get the lower bound

$$P\{M(T) \geq n\} \geq \frac{(1-\rho_1)(1-2\epsilon)(\mu_2-\lambda_2)}{4\sqrt{2\pi n}(n+1)}e^{-(\lambda_1+\mu_2-\lambda_2)}\left(\frac{\lambda_1}{\lambda_1+\mu_2-\lambda_2}\right)^n. \tag{44}$$

Next, let us consider the case where $\mu_1 - \lambda_1 < \mu_2 - \lambda_2$. From (37), we have

$$\begin{aligned}P\{M(T) &\geq n, q(T) = 0\} \\ &\geq \sum_{k=n}^{\infty}\int_0^{\infty}(1-\rho_1)P\{D(t) = k, q(t) = 0 | q(0) = 0\} \\ &\quad\cdot(\mu_2 - \lambda_2)e^{-(\mu_2-\lambda_2)t}dt.\end{aligned}$$

The remaining steps are identical to the lower bound proof in Theorem 2 from (27) to (28). □

V. Proof of Theorem 1

We will combine the results of the previous two sections and prove the main theorem. We wish to show that, without the loss of generality, when $\mu_1 - \lambda_1 \leq \mu_2 - \lambda_2$

$$\lim_{n\to\infty}\frac{1}{n}\log P\{q^r(t) \geq n\} = \max\left\{\log\frac{\lambda_2}{\lambda_2 + \mu_1 - \lambda_1}, \log\frac{4\lambda_1\mu_1}{(\lambda_1 + \mu_1 + \mu_2 - \lambda_2)^2}\right\}. \tag{45}$$

Proof: We need to find the asymptotic exponent for the two terms in (6), as n approaches infinity. The factors $P\{W_1(t) > W_2(t)\}$ and $P\{W_2(t) > W_1(t)\}$ are constants on $(0, 1)$, not dependent on n. For instance, $P\{W_1(t) > W_2(t)\}$ is given by (9). We will not carry these factors around in the subsequent analysis.

We will first consider the first term in (6). We wish to show

$$\lim_{n\to\infty}\frac{1}{n}\log P\{\hat{M}_2(t, W_*(t)) \geq n \mid W_1(t) > W_2(t)\} = \begin{cases}\log\frac{\lambda_2}{\lambda_2+\mu_1-\lambda_1}, & \text{if } \mu_2 - \lambda_2 \geq \mu_1 - \lambda_1 \\ \log\frac{4\lambda_2\mu_2}{(\lambda_2+\mu_2+\mu_1-\lambda_1)^2}, & \text{if } \mu_2 - \lambda_2 < \mu_1 - \lambda_1.\end{cases} \tag{46}$$

For the lower bound, we combine (7), (14), and (10), and get

$$\begin{aligned}P\{\hat{M}_2(t, W_*(t)) &\geq n \mid W_1(t) > W_2(t)\} \\ &= \int_{0^+}^{\infty}P\{\hat{M}_2(t, s) \geq n \mid W_2(t) < s\}f_{W_1|W_1>W_2}(s)ds \\ &\geq \int_{0^+}^{\infty}P\{\hat{M}_2(t, s) \geq n, q_2(t) = 0\}f_{W_1|W_1>W_2}(s)ds \\ &= \int_{0^+}^{\infty}P\{M_2(s) \geq n, q_2(s) = 0\}f_{W_1|W_1>W_2}(s)ds \\ &= K_1\int_0^{\infty}P\{M_2(s) \geq n, q_2(s) = 0\}e^{-(\mu_1-\lambda_1)s}ds \\ &\quad - K_2\int_0^{\infty}P\{M_2(s) \geq n, q_2(s) = 0\}e^{-(\mu_1-\lambda_1+\mu_2-\lambda_2)s}ds\end{aligned} \tag{47}$$

where $K_1 > 0$ and $K_2 > 0$ are constants given in (11) and (21). In the above, we have used the conditional density of W_1 given $\{W_1 > W_2\}$ from (10). By Lemma 4 with suitable substitution of variables, and since

$$\mu_2 - \lambda_2 < \mu_1 - \lambda_1 + \mu_2 - \lambda_2,$$

the second term in (47) has the following asymptotic exponent:

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log\int_0^{\infty}&P\{M_2(s) \geq n, q_2(s) = 0\} \\ &\cdot e^{-(\mu_1-\lambda_1+\mu_2-\lambda_2)s}ds \\ &= \log\frac{4\lambda_2\mu_2}{(\lambda_2 + \mu_2 + \mu_1 - \lambda_1 + \mu_2 - \lambda_2)^2} \\ &= \log\frac{4\lambda_2\mu_2}{(2\mu_2 + \mu_1 - \lambda_1)^2}.\end{aligned} \tag{48}$$

By Lemma 4 with suitable substitution of variables, the first term in (47) has the following asymptotic exponent:

$$\lim_{n\to\infty}\frac{1}{n}\log\int_0^{\infty}P\{M_2(s) \geq n, q_2(s) = 0\}e^{-(\mu_1-\lambda_1)s}ds = \begin{cases}\log\frac{\lambda_2}{\lambda_2+\mu_1-\lambda_1}, & \text{if } \mu_2 - \lambda_2 \geq \mu_1 - \lambda_1 \\ \log\frac{4\lambda_2\mu_2}{(\lambda_2+\mu_2+\mu_1-\lambda_1)^2}, & \text{if } \mu_2 - \lambda_2 < \mu_1 - \lambda_1.\end{cases} \tag{49}$$

Now

$$\frac{\lambda_2}{\lambda_2+\mu_1-\lambda_1} = \frac{4\lambda_2\mu_2}{4\lambda_2\mu_2+4\mu_1\mu_2-4\lambda_1\mu_2}$$
$$\frac{4\lambda_2\mu_2}{(2\mu_2+\mu_1-\lambda_1)^2} = \frac{4\lambda_2\mu_2}{4\mu_2^2+4\mu_1\mu_2-4\lambda_1\mu_2+(\mu_1-\lambda_1)^2}.$$

Hence, we have

$$\frac{\lambda_2}{\lambda_2+\mu_1-\lambda_1} > \frac{4\lambda_2\mu_2}{(2\mu_2+\mu_1-\lambda_1)^2}.$$

Also, because $\lambda_2 < \mu_2$, we have

$$\frac{4\lambda_2\mu_2}{(\lambda_2+\mu_2+\mu_1-\lambda_1)^2} > \frac{4\lambda_2\mu_2}{(2\mu_2+\mu_1-\lambda_1)^2}.$$

Therefore, we can ignore the contribution from (48) when considering the lower bound of the left-hand side in (46). Then, (49) gives the lower bound.

For the upper bound of the left-hand side in (46), we combine (7) and (13), and get

$$P\{\hat{M}_2(t,W_*(t)) \ge n \mid W_1(t) > W_2(t)\} \le \frac{1}{1-\rho_2}\int_{0^+}^{\infty} P\{\hat{M}_2(t,s)\ge n\} f_{W_1|W_1>W_2}(s)ds. \quad (50)$$

Using a similar argument as in the derivation of the lower bound, but with Theorem 2 substituting the role of Lemma 4, we get

$$\lim_{n\to\infty}\frac{1}{n}\log\int_{0^+}^{\infty} P\{\hat{M}_2(t,s)\ge n\} f_{W_1|W_1>W_2}(s)ds = \begin{cases}\log\frac{\lambda_2}{\lambda_2+\mu_1-\lambda_1}, & \text{if } \mu_2-\lambda_2 \ge \mu_1-\lambda_1\\ \log\frac{4\lambda_2\mu_2}{(\lambda_2+\mu_2+\mu_1-\lambda_1)^2}, & \text{if } \mu_2-\lambda_2 < \mu_1-\lambda_1.\end{cases} \quad (51)$$

Since the upper and lower bounds agree with each other, we get (46).

To determine $P\{q^r(t)\ge n\}$ for large n, we also need to consider the second term in (6), $P\{\hat{M}_1(t,W_*(t))\ge n \mid W_2(t) > W_1(t)\}$. By symmetry

$$\lim_{n\to\infty}\frac{1}{n}\log P\{\hat{M}_1(t,W_*(t))\ge n \mid W_2(t) > W_1(t)\} = \begin{cases}\log\frac{\lambda_1}{\lambda_1+\mu_2-\lambda_2}, & \text{if } \mu_1-\lambda_1 \ge \mu_2-\lambda_2\\ \log\frac{4\lambda_1\mu_1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^2}, & \text{if } \mu_1-\lambda_1 < \mu_2-\lambda_2.\end{cases} \quad (52)$$

When $\mu_1-\lambda_1 \le \mu_2-\lambda_2$, combining (6), (46) and (52), we get (45). □

VI. Conclusion

To conclude, we discuss the implications of Theorem 1. When, $\lambda_1=\lambda_2$ and $\mu_1=\mu_2$

$$\lim_{n\to\infty}\frac{1}{n}\log P\{q^r(t)\ge n\} = \log\rho_1.$$

When $\mu_1-\lambda_1=\mu_2-\lambda_2$

$$\lim_{n\to\infty}\frac{1}{n}\log P\{q^r(t)\ge n\} = \max\{\log\rho_1,\log\rho_2\}.$$

Like all GI/GI/1 queues, the RSQ size depends on the arrival and departure rates through a dimensionless parameter. This implies that the RSQ size does not change with the link speed of the network, if all links involved scale their bandwidth by the same factor and if the traffic characteristics are not altered by the technology change. This is in contrast with the models from our previous paper [2], where the improvement of network speed worsens the packet resequencing problem in terms of both the queue size and the delay. In the current model, there can be many ways to produce the large RSQ size, which, in general, depends on parameters for both queues in the DN. According to Theorem 1, for $\mu_1-\lambda_1\le\mu_2-\lambda_2$ and for large n

$$P\{q^r(t)\ge n\} \approx \max\left\{\left(\frac{\lambda_2}{\lambda_2+\mu_1-\lambda_1}\right)^n, \left(\frac{4\lambda_1\mu_1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^2}\right)^n\right\}.$$

In the first term above, $\frac{\lambda_2}{\lambda_2+\mu_1-\lambda_1}\approx 1$ if $\mu_1-\lambda_1 \ll \lambda_2$. In the second term above, $\frac{4\lambda_1\mu_1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^2}\approx 1$ if $\mu_2-\lambda_2 \ll \lambda_1+\mu_1$ and $\lambda_1\approx\mu_1$. This implies that, for the second term to decay slowly, we need to have

$$\mu_1-\lambda_1 \le \mu_2-\lambda_2 \ll 2\mu_1 \approx 2\lambda_1.$$

An interesting observation is that it can be large even when the queue sizes in the DN are both small. This occurs when the two disordering queues are "mismatched," that is, when one of the disordering queues is much faster than the other in terms of both the arrival rate and the service rate. For example, suppose $\mu_i = 2\lambda_i$ for $i=1$ and 2. Hence, $\rho_1=\rho_2=1/2$, and for $i=1$ and 2

$$\lim_{n\to\infty}\frac{1}{n}\log P\{q_i(t)\ge n\} = \log\frac{1}{2}.$$

Suppose, in the DN, queue 2 is ten times faster than queue 1, i.e., $\lambda_2=10\lambda_1$ and $\mu_2=10\mu_1$. Then

$$\lim_{n\to\infty}\frac{1}{n}\log P\{q^r(t)\ge n\} = \log\frac{10}{11}.$$

Intuitively, the queue size of the RSQ can be large because many later packets can go through queue 2 in the DN and end up waiting in the RSQ, while some earlier packets are waiting in queue 1.

In Fig. 2, we show the simulation results for $P\{q^r = n\}$ and compare them with the analytical results in Theorem 1. In Fig. 2(a), the parameters are chosen so that

$$\frac{\lambda_2}{\lambda_2+\mu_1-\lambda_1} = \frac{10}{11} = 0.9091.$$

In Fig. 2(b), the parameters are chosen so that

$$\frac{\lambda_2}{\lambda_2+\mu_1-\lambda_1} = \frac{1}{21} = 0.0476.$$

Loosely speaking, Theorem 1 says, for $\mu_1-\lambda_1\le\mu_2-\lambda_2$ and for large n

$$P\{q^r(t)\ge n\} = e^{-\delta n + o(n)} \quad (53)$$

where $o(n)$ is a function that grows more slowly than n, i.e., $o(n)/n\to 0$ as n tends to infinity. The large deviations analysis of this paper is able to give an expression for the parameter δ

$$\delta = -\max\left\{\log\frac{\lambda_2}{\lambda_2+\mu_1-\lambda_1}, \log\frac{4\lambda_1\mu_1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^2}\right\}$$

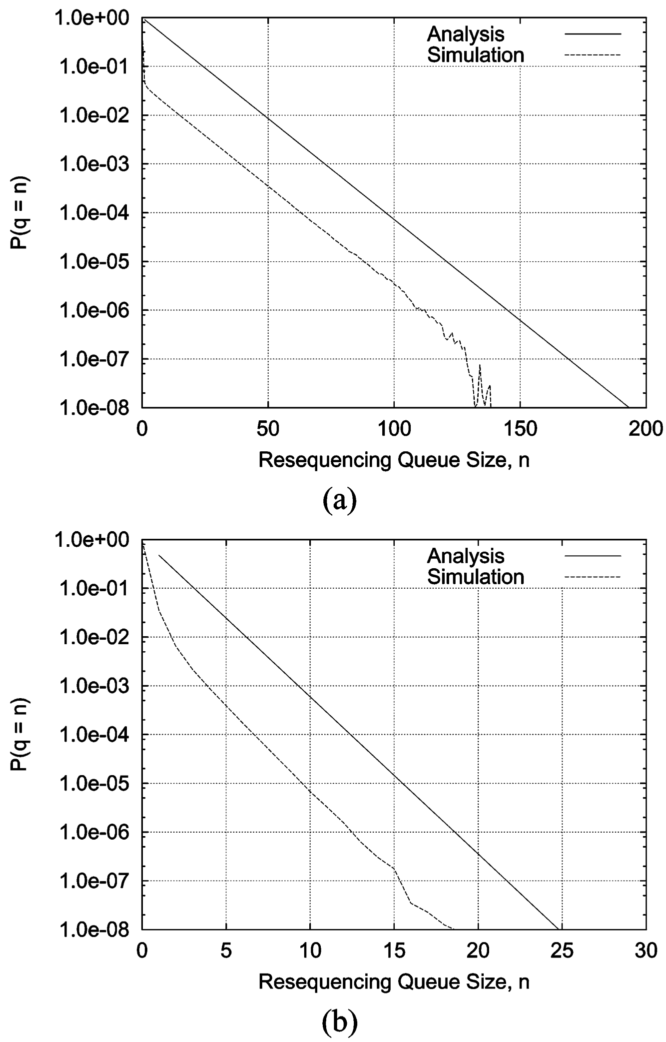

Fig. 2. $P\{q^r = n\}$: Simulation results. (a) $\lambda_1 = 1, \mu_1 = 2, \lambda_2 = 10, \mu_2 = 20$; (b) $\lambda_1 = 10, \mu_1 = 20, \lambda_2 = 1, \mu_2 = 12$.

but cannot capture the nature of $o(n)$. In each plot of Fig. 2, the gap between the two curves shows the "imprecision" of the large deviations result. That is, it shows how much the large deviations result misses the actual tail probability of the queue size.

From the modeling point of view, compared with those in [2], the model in this paper allows non-i.i.d. packet delays in the DN and it specifically models packet disordering caused by routing on different paths. As for generalization, our preliminary work shows that there are similar large deviations results for more complex arrival and service processes for the queues in the DN, even for the case of non-i.i.d. arrival processes. However, in order to generalize, we must rely on more generalizable arguments than many probabilistic arguments used in this paper, which specifically depend on the underlying probability distributions. In another direction of generalization, we can consider a DN with k parallel M/M/1 queues, where $k \geq 3$. Preliminary investigation seems to show that there is no conceptual hurdle in that direction but careful bookkeeping is required. Finally, one weakness of these models is that they do not allow situations that yield heavy tailed distributions for the RSQ.

Appendix
Alternative Proofs for the Upper Bound in the Proof of Theorem 2: Case of $\mu_1 - \lambda_1 < \mu_2 - \lambda_2$

We will work directly with the departure process. Following the approach in [27, Ch. 2,Sec. 4], let

$$H_{ij}(t) = P\{q(t) = i, D(t) = j | q(0) = 0\} \tag{54}$$

where $D(t)$ is the number of departures on the interval $[0, t]$. We will show the following.

Lemma 5:

$$H_{ij}(t) = \frac{\rho_1^{i+j} e^{-(\lambda_1+\mu_1)t} (\mu_1 T)^{2j+i}}{j!} \sum_{l=0}^{\infty} \frac{(i+l+1)(\mu_1 t)^l}{(j+i+l+1)!}. \tag{55}$$

Therefore

$$\begin{aligned} P\{M(t) = j | q(0) = 0\} &= \sum_{i=0}^{\infty} H_{ij}(t) \\ &= \frac{\rho_1^j e^{-(\lambda_1+\mu_1)t} (\mu_1 t)^{2j}}{j!} \sum_{i=0}^{\infty} \rho_1^i (\mu_1 t)^i \sum_{l=0}^{\infty} \frac{(i+l+1)(\mu_1 t)^l}{(j+i+l+1)!}. \end{aligned} \tag{56}$$

Proof: We will start with the integral transform of $H_{ij}(t)$. Define

$$H^*(p, q, s) := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p^i q^j \int_0^{\infty} e^{-s\tau} H_{ij}(\tau/\mu_1) d\tau \tag{57}$$

where $|p| < 1$, $|q| < 1$, and $\mathrm{Re}\ s > 0$. It is shown in [27, p. 198] that

$$H^*(p, q, s) = \frac{(q-p)x_2(q) - (q - x_2(q))p}{(q - x_2(q))(\rho_1 p^2 - (1 + \rho_1 + s)p + q)} \tag{58}$$

where $x_2(q)$ is one solution to the equation

$$\rho_1 p^2 - (1 + \rho_1 + s)p + q = 0.$$

The two solutions for p are

$$x_1(q) = \frac{1 + \rho_1 + s + \sqrt{(1+\rho_1+s)^2 - 4\rho_1 q}}{2\rho_1}$$
$$x_2(q) = \frac{1 + \rho_1 + s - \sqrt{(1+\rho_1+s)^2 - 4\rho_1 q}}{2\rho_1}.$$

It can be shown that for $\mathrm{Re}\ s > 0$ and $|q| \leq 1$

$$|x_1(q)| > 1, \quad |x_2(q)| < 1.$$

We will use the fact that, for $n = 1, 2, \ldots$

$$\begin{aligned} x_2^n(q) &= \int_0^{\infty} e^{-(1+\rho_1+s)\tau} \frac{n q^n}{\tau (\rho_1 q)^{\frac{n}{2}}} I_n(2\tau\sqrt{\rho_1 q}) d\tau \\ &= \sum_{m=0}^{\infty} q^{n+m} \int_0^{\infty} e^{-(1+\rho_1+s)\tau} \frac{n \rho_1^m}{m!(m+n)!} \tau^{2m+n-1} d\tau \end{aligned} \tag{59}$$

where $I_n(x)$ is the modified Bessel function of the first kind with series expansion

$$I_n(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^{n+2m}}{m!(m+n)!}, \quad n = 0, 1, \ldots.$$

To find an expression for $H_{ij}(t)$, we will expand $H^*(p,q,s)$ into power series of p. Starting with (58)

$$\begin{aligned} H^*(p,q,s) &= \frac{q(x_2(q)-p)}{\rho_1(q-x_2(q))(p-x_1(q))(p-x_2(q))} \\ &= \frac{-q}{\rho_1(q-x_2(q))(p-x_1(q))} \\ &= \frac{q}{\rho_1(q-x_2(q))}\frac{1}{x_1(q)}\sum_{i=0}^{\infty}\left(\frac{p}{x_1(q)}\right)^i. \end{aligned} \tag{60}$$

In the above, the series expansion is valid when $|p| < |x_1(q)|$, which is satisfied when $|p|$ is small enough. By (60) and by the definition of $H^*(p,q,s)$ in (57), the coefficient for p^i in the power series expansion of $H^*(p,q,s)$ with respect to p is

$$\sum_{j=0}^{\infty} q^j \int_0^{\infty} e^{-s\tau} H_{ij}(\tau/\mu_1)d\tau = \frac{q}{\rho_1(q-x_2(q))}\frac{1}{x_1^{i+1}(q)}. \tag{61}$$

Next, we use the fact

$$x_1(q)x_2(q) = \frac{q}{\rho_1}$$

and we express $\frac{q}{q-x_2(q)}$ in power series of $\frac{x_2(q)}{q}$. We get

$$\sum_{j=0}^{\infty} q^j \int_0^{\infty} e^{-s\tau} H_{ij}(\tau/\mu_1)d\tau = \rho_1^i \sum_{l=0}^{\infty}\frac{x_2^{i+l+1}(q)}{q^{i+l+1}}. \tag{62}$$

The above series expansion is true when $|x_2(q)| < |q|$, which can be satisfied if $|s|$ is large enough. By (59)

$$\begin{aligned} &\sum_{j=0}^{\infty} q^j \int_0^{\infty} e^{-s\tau} H_{ij}(\tau/\mu_1)d\tau \\ &= \rho_1^i \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} q^j \int_0^{\infty} e^{-(1+\rho_1+s)\tau}\frac{(i+l+1)\rho_1^j}{j!(j+i+l+1)!}\tau^{2j+i+l}d\tau. \end{aligned} \tag{63}$$

Matching both sides term-by-term, we get

$$H_{ij}(\tau/\mu_1) = \rho_1^i \sum_{l=0}^{\infty}\frac{(i+l+1)\rho_1^j}{j!(j+i+l+1)!}e^{-(1+\rho_1)\tau}\tau^{2j+i+l}. \tag{64}$$

Replacing τ/μ_1 by t, we get (55). □

We will now show the upper bound. First, we notice that for all integer $i \geq 0$

$$P\{M(t)=j|q(0)=i\} \leq P\{M(t)=j|q(0)=0\}.$$

This is because

$$\begin{aligned} &P\{M(t)=j|q(0)=i\} \\ &\quad = P\{\text{At least } j \text{ customers arrived} \\ &\qquad\quad \text{on } [0,t] \text{ the } i \text{ customers} \\ &\qquad\quad \text{in the queue at time } 0 \text{ and the first} \\ &\qquad\quad j \text{ customers who} \\ &\qquad\quad \text{arrived on } [0,t] \text{ are served by time } t\} \end{aligned}$$

and this probability should be monotonically decreasing in i. Hence

$$\begin{aligned} &P\{M(T)=n\} \\ &\quad = \int_0^{\infty} P\{M(t)=n\}(\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \\ &\quad = \int_0^{\infty}\sum_{i=0}^{\infty} P\{M(t)=n|q(0)=i\}P\{q(0)=i\} \\ &\qquad \cdot (\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \\ &\quad \leq \int_0^{\infty}\sum_{i=0}^{\infty} P\{M(t)=n|q(0)=0\}P\{q(0)=i\} \\ &\qquad \cdot (\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \\ &\quad = \int_0^{\infty} P\{M(t)=n|q(0)=0\}(\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt. \end{aligned}$$

By (56)

$$\begin{aligned} &P\{M(T)=n\} \\ &\quad \leq \int_0^{\infty}\frac{\rho_1^n e^{-(\lambda_1+\mu_1)t}(\mu_1 t)^{2n}}{n!}\sum_{i=0}^{\infty}\rho_1^i(\mu_1 t)^i \\ &\qquad \cdot \sum_{l=0}^{\infty}\frac{(i+l+1)(\mu_1 t)^l}{(n+i+l+1)!}(\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt. \end{aligned}$$

Because $n+i+l+1 \geq i+l+1$ for all $n \geq 0$

$$\begin{aligned} &P\{M(T)=n\} \\ &\quad \leq \int_0^{\infty}\frac{\rho_1^n e^{-(\lambda_1+\mu_1)t}(\mu_1 t)^{2n}}{n!}\sum_{i=0}^{\infty}\rho_1^i(\mu_1 t)^i \\ &\qquad \cdot \sum_{l=0}^{\infty}\frac{(\mu_1 t)^l}{(n+i+l)!}(\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \\ &\quad = (\mu_2-\lambda_2)\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{\rho_1^{n+i}(2n+i+l)!}{n!(n+i+l)!} \\ &\qquad \cdot \int_0^{\infty}\frac{e^{-(\lambda_1+\mu_1+\mu_2-\lambda_2)t}(\mu_1 t)^{2n+i+l}}{(2n+i+l)!}dt \\ &\quad = \frac{\mu_2-\lambda_2}{\lambda_1+\mu_1+\mu_2-\lambda_2} \\ &\qquad \cdot \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{\rho_1^{n+i}(2n+i+l)!}{n!(n+i+l)!}\left(\frac{\mu_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^{2n+i+l}. \end{aligned}$$

In deriving the last step, (16) has been used. Note that

$$\begin{aligned} \frac{(2n+i+l)!}{n!(n+i+l)!} &= \frac{(2n)!}{n!n!}\frac{(2n+1)(2n+2)\dots(2n+i+l)}{(n+1)(n+2)\dots(n+i+l)} \\ &\leq \frac{(2n)!}{n!n!}2^{i+l}. \end{aligned}$$

Hence

$$\begin{aligned} &P\{M(T)=n\} \\ &\quad \leq \frac{\mu_2-\lambda_2}{\lambda_1+\mu_1+\mu_2-\lambda_2}\frac{(2n)!}{n!n!}\rho_1^n\left(\frac{\mu_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^{2n} \\ &\qquad \cdot \sum_{i=0}^{\infty}2^i\rho_1^i\left(\frac{\mu_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^i \end{aligned}$$

$$\cdot \sum_{l=0}^{\infty} 2^l \left(\frac{\mu_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^l$$
$$= \frac{\mu_2-\lambda_2}{\lambda_1+\mu_1+\mu_2-\lambda_2} \frac{(2n)!}{n!n!} \left(\frac{\lambda_1\mu_1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^2}\right)^n$$
$$\cdot \sum_{i=0}^{\infty} \left(\frac{2\lambda_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^i \sum_{l=0}^{\infty} \left(\frac{2\mu_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^l. \tag{65}$$

Since $\mu_1 - \lambda_1 < \mu_2 - \lambda_2$ and $\lambda_1 < \mu_1$, it follows that

$$\frac{2\lambda_1}{\lambda_1+\mu_1+\mu_2-\lambda_2} < 1 \tag{66}$$
$$\frac{2\mu_1}{\lambda_1+\mu_1+\mu_2-\lambda_2} < 1. \tag{67}$$

Hence, the two infinite sums in (65) are both finite. Furthermore, by Stirling's approximation

$$\frac{(2n)!}{n!n!} = \frac{\sqrt{4\pi n}(2n)^{2n}e^{-2n}(1+O(1/n))}{(\sqrt{2\pi n}(n)^n e^{-n}(1+O(1/n)))^2} \leq \frac{C_2}{\sqrt{n}}4^n \tag{68}$$

for some constant $C_2 > 0$. Therefore

$$P\{M(T)=n\} \leq \frac{C_3}{\sqrt{n}} \left(\frac{4\lambda_1\mu_1}{(\lambda_1+\mu_1+\mu_2-\lambda_2)^2}\right)^n \tag{69}$$

for some constant $C_3 > 0$. We are done with the proof for the upper bound.

Another perhaps shorter proof for the upper bound starts with (30). By writing

$$P\left\{\sum_{i=1}^{n} X_i \leq t\right\} = \sum_{l=n}^{\infty} \frac{e^{-\mu_1 t}(\mu_1 t)^l}{l!}$$

we have

$$\begin{aligned}
&P\{M(T) \geq n\} \\
&\leq \int_0^\infty \sum_{i=n}^{\infty} \frac{e^{-\lambda_1 t}(\lambda_1 t)^i}{i!} \sum_{l=n}^{\infty} \frac{e^{-\mu_1 t}(\mu_1 t)^l}{l!} \\
&\quad \cdot (\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \\
&= \int_0^\infty \sum_{i=0}^{\infty} \frac{e^{-\lambda_1 t}(\lambda_1 t)^{n+i}}{(n+i)!} \sum_{l=0}^{\infty} \frac{e^{-\mu_1 t}(\mu_1 t)^{n+l}}{(n+l)!} \\
&\quad \cdot (\mu_2-\lambda_2)e^{-(\mu_2-\lambda_2)t}dt \\
&= (\mu_2-\lambda_2)\sum_{i=n}^{\infty}\sum_{l=n}^{\infty} \frac{\rho_1^{n+i}(2n+i+l)!}{(n+i)!(n+l)!} \\
&\quad \cdot \int_0^\infty \frac{e^{-(\lambda_1+\mu_1+\mu_2-\lambda_2)t}(\mu_1 t)^{2n+i+l}}{(2n+i+l)!}dt \\
&= \frac{\mu_2-\lambda_2}{\lambda_1+\mu_1+\mu_2-\lambda_2} \\
&\quad \cdot \sum_{i=n}^{\infty}\sum_{l=n}^{\infty} \frac{\rho_1^{n+i}(2n+i+l)!}{(n+i)!(n+l)!} \left(\frac{\mu_1}{\lambda_1+\mu_1+\mu_2-\lambda_2}\right)^{2n+i+l}.
\end{aligned} \tag{70}$$

Next

$$\begin{aligned}
\frac{(2n+i+l)!}{(n+i)!(n+l)!} &= \frac{(2n)!}{n!n!} \frac{(2n+1)(2n+2)\dots(2n+i+l)}{(n+1)\dots(n+i)(n+1)\dots(n+l)} \\
&\leq \frac{(2n)!}{n!n!}2^{i+l}.
\end{aligned} \tag{71}$$

The remaining steps are exactly the same as those starting at (65).

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