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Optimum Asymptotic Multiuser Efficiency of Randomly Spread CDMA

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***Abstract*—This correspondence analyzes the high signal-to-noise ratio (SNR) performance of optimum multiuser detectors for synchronous direct-sequence spread spectrum with random spreading in an additive white Gaussian noise channel. Under very general conditions on the received powers, we show that the optimum asymptotic efficiency of a K-user system with spreading gain N converges to 1 almost surely as $K \to \infty$, and K/N is kept equal to an arbitrary nonzero constant. Therefore, the asymptotic behavior of the minimum bit error rate is equivalent to that of a single-user system.**

***Index Terms*—Asymptotic efficiency, CDMA, multiuser detection, optimum multiuser detection, spread spectrum.**

I. Introduction

This correspondence is concerned with the analysis of the capabilities of optimum multiuser detection for the basic synchronous CDMA multiple-access K-user channel [1]

$$y(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + \sigma n(t) \tag{1}$$

where $A_k \in (0, \infty)$, $b_k \in \{-1, 1\}$ and s_k are the received amplitudes, data, and unit-energy signature waveform of the kth user, respectively, and $n(t)$ is additive white Gaussian noise.

Uncoded bit error rate has received much attention as a performance measure of multiuser detectors. Of particular interest is the asymptotic multiuser efficiency which characterizes the performance loss (in effective signal-to-noise ratio (SNR)) as the background noise vanishes. If a particular receiver achieves bit-error rate $\mathsf{P}_k(\sigma)$ in the presence of multiple-access interference and additive white Gaussian noise with power spectral level equal to σ^2, then the asymptotic multiuser efficiency is given by [1]

$$\eta_k = \frac{2}{A_k^2} \lim_{\sigma \to 0} \sigma^2 \log 1/\mathsf{P}_k(\sigma). \tag{2}$$

The analysis of the error probability of optimum multiuser detectors for arbitrary signature waveforms and received powers goes back more than fifteen years [2]–[4]. Denote the diagonal matrix of received amplitudes $\boldsymbol{A} = \operatorname{diag}\{A_1, \ldots, A_K\}$, and the normalized crosscorrelation matrix by $\boldsymbol{R}$, with entries $\rho_{ij} = \langle s_i, s_j \rangle$. The optimum asymptotic multiuser efficiency is equal to [1]

$$\eta_k = \min_{\substack{\boldsymbol{v} \in \{-1,0,1\}^K \\ v_k = 1}} \frac{1}{A_k^2} \boldsymbol{v}^T \boldsymbol{ARAv}. \tag{3}$$

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In the worst case (over signature waveforms), solving for the minimum in (3) is an NP-hard combinatorial optimization problem in the number of users [5].

In direct-sequence spread-spectrum systems with spreading gain N, the signature waveform of user k is a linear combination of N orthogonal chip waveforms with energy $\|\psi_n\|^2 = 1/N$

$$s_k(t) = \sum_{n=1}^{N} s_{nk}\psi_n(t). \tag{4}$$

Then

$$\rho_{ij} = \frac{1}{N}\sum_{n=1}^{N} s_{ni}s_{nj}.$$

In the usual random spreading model,

$$\{s_{nk}, n = 1, \ldots N, K = 1, \ldots K\}$$

are drawn randomly and independently from a given symmetric distribution with zero mean and unit second moment [1]. This is a simple model which lends itself to analysis and is relevant to CDMA with long signature sequences. A number of recent works [1], [6]–[19] have analyzed the asymptotic performance of direct-sequence spread spectrum with random antipodal spreading in the *large-system limit*, when the number of users K grows without bound and the ratio of the number of users to the spreading gain N is kept fixed to

$$\frac{K}{N} = \beta. \tag{5}$$

More specifically, under that asymptotic regime, [13] has found the Shannon capacity achievable by the maximum-likelihood decoder, the single-user matched filter, the decorrelating detector, and the minimum mean square error (MMSE) multiuser detector, for systems with perfect power control; and [14] has investigated the signal-to-interference ratio of the single-user matched filter, the decorrelating detector, and the MMSE multiuser detector without assuming equal received powers.

Under the foregoing random spreading model, the following behavior is known for the asymptotic efficiency of the single-user matched filter η^c, decorrelator η^d, and linear MMSE detector η^m, as $K \to \infty$ and $K/N = \beta$

$$\eta^c \to 0 \tag{6}$$
$$\eta^d \to [1-\beta]^+ \tag{7}$$
$$\eta^m \to [1-\beta]^+ \tag{8}$$

where (6), (7), and (8) can be found in [1, Problem 3.39, Problem 5.27, and eq. (6.62), respectively]. Thus for all linear multiuser detectors, the asymptotic efficiency decreases as the system load β increases and vanishes for $\beta > 1$.

In this correspondence, we study the asymptotic efficiency of the *optimum* detector in the same large-system limit. In sharp contrast to the performance of the linear detectors, we have the following result.

Theorem I.1: Assume

1) $E[|s_{nk}|^3] < \infty, n = 1, \ldots, N, k = 1, \ldots, K$;
2) there exist positive constants θ and γ such that for all $k = 2, 3, \ldots$

$$\theta < \frac{A_k}{A_1} < \gamma \tag{9}$$

but $A_1, A_2, \ldots$ are otherwise arbitrary constants.

Then, for any $\beta > 0$, as $N, K \to \infty$ and $K/N \to \beta$, the optimum asymptotic multiuser efficiency for user 1 converges to 1 almost surely.

Moreover, in the case when $\beta < 1$, the same result is obtained without the condition on the uniform boundedness of A_k/A_1 from above.

It is worthwhile highlighting that the large-system analysis of asymptotic efficiency of the various receivers above involves taking two limits to the log bit-error rate (cf. (2)) first with respect to σ and then with respect to K. The question is whether the limits commute.

Suppose that, for fixed σ, a certain multiuser detector achieves error probability $\mathsf{P}_k(\sigma)$, then its multiuser efficiency is defined by [1, p. 121][1]

$$\eta_k(\sigma) = \frac{\sigma^2}{A_k^2}(Q^{-1}(\mathsf{P}_k(\sigma)))^2. \tag{10}$$

It is known that the decorrelator achieves [1]

$$\lim_{K\to\infty} \eta_k^d(\sigma) = [1-\beta]^+ \tag{11}$$

regardless of σ and the received amplitudes.

To state the corresponding results for the single-user matched filter and the MMSE linear filter, it is convenient to denote a random variable A whose distribution is the limit of the empirical distribution of $\{A_1, \ldots A_K\}$. From the results of [14], [13], and [20] it easily follows that

$$\lim_{K\to\infty} \eta_k^c(\sigma) = \left(1 + \beta\frac{E[A^2]}{\sigma^2}\right)^{-1} \tag{12}$$

and that

$$\lim_{K\to\infty} \eta_k^m(\sigma) = x_k(\sigma) \tag{13}$$

where $x_k(\sigma)$ is the positive solution to

$$x + \beta E\left[\frac{A^2 x}{A^2 x + \sigma^2}\right] = 1. \tag{14}$$

It is easy to recover the results in (6) and (8) letting $\sigma \to 0$ in (12) and (14), respectively.

Thus the limits with respect to $\sigma \to 0$ and $K \to \infty$ commute for all the three linear receivers.

At present, we do not have any corresponding results on the large-system limit of the *optimum* multiuser efficiency for a fixed background noise σ, and hence we do not make any claims regarding the commutativity of the limits in Theorem I.1.

II. Proof

Focusing on user 1, the optimum asymptotic efficiency is given as in (3):

$$\eta_1 = \min_{\substack{\boldsymbol{v}\in\{-1,0,1\}^K \\ v_1=1}} \frac{1}{A_1^2}\boldsymbol{v}^T\boldsymbol{ARAv}. \tag{15}$$

To simplify notation, we shall consider without loss of generality the case $A_1 = 1$, since both (15) and the conditions on the amplitudes are invariant to scaling of all amplitudes by A_1.

Let C_K be the set of all $\boldsymbol{v} \in \{-1, 0, 1\}^K$ with $v_1 = 1$ and $v_j \neq 0$ for some $j > 1$. Let E_K be the event that $\boldsymbol{v}^T\boldsymbol{ARAv} < 1$ for some $\boldsymbol{v} \in C_K$. In what follows, we shall show that

$$\mathrm{Prob}[E_K] \to 0 \tag{16}$$

[1] $Q(\cdot)$ is the complementary cumulative distribution function (cdf) of the $N(0,1)$ random variable; $x = Q^{-1}(z)$ if $z = \int_x^\infty (\exp[-t^2/2]/\sqrt{2\pi})\, dt$.

exponentially in N as $N \to \infty$. This together with an application of the Borel–Cantelli Lemma yields the desired conclusion that η_1 converges to 1 almost surely.

By the union bound

$$\begin{aligned}\mathrm{Prob}[E_K] &\le \sum_{\boldsymbol{v}\in C_K} \mathrm{Prob}[\boldsymbol{v}^T\boldsymbol{ARAv} < 1] \\ &= \sum_{\boldsymbol{v}\in C_K, w(\boldsymbol{v})\le M_0} \mathrm{Prob}[\boldsymbol{v}^T\boldsymbol{ARAv} < 1] \\ &\quad + \sum_{\boldsymbol{v}\in C_K, w(\boldsymbol{v})> M_0} \mathrm{Prob}[\boldsymbol{v}^T\boldsymbol{ARAv} < 1] \qquad (17)\end{aligned}$$

where $M_0 \ge 2$ is a constant and the number of nonzero components in $\boldsymbol{v} \in \{-1, 0, 1\}^K$ is denoted by

$$\begin{aligned} w(\boldsymbol{v}) &\stackrel{\mathrm{def}}{=} \sum_{k=1}^{K} |v_k| \\ &= \|\boldsymbol{v}\|^2. \end{aligned}$$

We will show below that for any choice of the constant M_0, the first sum in the right side of (17) goes to zero exponentially in N. Moreover, for sufficiently large (but fixed) M_0, the second sum in the right side of (17) will also be shown to go to zero exponentially in N.

A. Step 1: First Sum

Fix $\boldsymbol{v} \in C_K$ and define

$$Y_n \stackrel{\mathrm{def}}{=} \left[s_{n1} + \sum_{k=2}^{K} v_k A_k s_{nk} \right]^2.$$

Note that Y_n are independent and identically distributed (i.i.d.) random variables with expectation

$$E[Y_n] = 1 + \sum_{k=2}^{K} |v_k| A_k^2. \qquad (18)$$

According to (9), for every $\boldsymbol{v} \in C_K$ with $w(\boldsymbol{v}) \le M_0$

$$1 + \theta^2 < E[Y_n] < 1 + M_0 \gamma^2. \qquad (19)$$

The reason we are interested in Y_n is that

$$\frac{1}{N} \sum_{n=1}^{N} Y_n = \boldsymbol{v}^T \boldsymbol{ARAv} \qquad (20)$$

and

$$\begin{aligned}\mathrm{Prob}[\boldsymbol{v}^T\boldsymbol{ARAv} < 1] &= \mathrm{Prob}\left[\frac{1}{N}\sum_{n=1}^{N} Y_n < 1\right] \\ &\le \exp[-N \Lambda_{\boldsymbol{v}}^*(1)] \qquad (21)\end{aligned}$$

where

$$\Lambda_{\boldsymbol{v}}^*(\mu) \stackrel{\mathrm{def}}{=} \sup_{r<0} [\mu r - \log E(\exp(rY_n))]$$

is the large deviations rate function for the sum of i.i.d. random variables Y_n's. Inequality (21) follows from the Chernoff bound.

For a fixed M_0, the number of terms in the first sum of (17) grows polynomially in N. Hence, the first sum decays exponentially in N if there is a positive lower bound on $\Lambda_{\boldsymbol{v}}^*(1)$ uniform for all $\boldsymbol{v} \in C_K$ and $w(\boldsymbol{v}) \le M_0$, for all A_k's satisfying (9) and for all K. To that end, define the random variable

$$Z^{(m)} \stackrel{\mathrm{def}}{=} \left(X_1 + \sum_{k=2}^{m} A_k X_k \right)^2$$

where X_k's are i.i.d. random variables having the same distribution as s_{11}. Denote

$$\Lambda_{m,\vec{A}}^*(\mu) \stackrel{\mathrm{def}}{=} \sup_{r<0} \left[\mu r - \log E\left(\exp\left(r Z^{(m)}\right)\right)\right] \qquad (22)$$

where $\vec{A} = (A_1, \ldots, A_m)$. We now find a lower bound on $\Lambda_{m,\vec{A}}^*(1)$ uniform for all A_k's such that

$$\sum_{k=2}^{m} A_k^2 \ge \theta^2.$$

We may assume this condition as a result of the assumption on the uniform lower bound in (9).

We observe that

$$E\left(Z^{(m)}\right) = 1 + \sum_{k=2}^{m} A_k^2 > 1 \qquad (23)$$

so by the strict concavity of the rate function, it follows that $\Lambda_{m,\vec{A}}^*(1) > 0$. Furthermore, $\Lambda_{m,\vec{A}}^*(1)$ is a continuous function of $\vec{A}$. One can, therefore, choose $\varepsilon_m > 0$, such that $\Lambda_{m,\vec{A}}^*(1) > \varepsilon_m$, uniformly in $\vec{A}$ within the compact set with all A_k's bounded by 2 and $\sum_{k=2}^{m} A_k^2 \ge \theta^2$. If there is $A_k > 2$, then let

$$x \stackrel{\mathrm{def}}{=} \max_{2\le k\le m} A_k$$

which satisfies $2 < x \le \gamma$. Now we can lower-bound (22) by

$$\begin{aligned}\Lambda_{m,\vec{A}}^*(1) &= \sup_{r<0} \left\{ r - \log E\left\{ \exp\left[r x^2 \left(\sum_{k=1}^{m} \frac{A_k}{x} X_k \right)^2 \right] \right\} \right\} \\ &= \Lambda_{m,\frac{\vec{A}}{x}}^*\left(\frac{1}{x^2}\right) \\ &\ge \Lambda_{m,\frac{\vec{A}}{x}}^*\left(\frac{1}{4}\right)\end{aligned}$$

where we have used the fact that the function in (22) is decreasing for $\mu < 1$. Since $\vec{A}/x$ now lies in a compact set, by the above argument, there exists $\delta_m > 0$ such that $\Lambda_{m,(\vec{A}/x)}^*(1/4) > \delta_m$ for all $\vec{A}$. Thus for all $\vec{A}$

$$\Lambda_{m,\vec{A}}^*(1) > \min\{\varepsilon_m, \delta_m\} > 0.$$

Hence there exists

$$\varepsilon(M_0) \stackrel{\mathrm{def}}{=} \min_{2\le m\le M_0} \min\{\varepsilon_m, \delta_m\} > 0$$

for all $\boldsymbol{v} \in C_K$ and $w(\boldsymbol{v}) \le M_0$

$$\Lambda_{\boldsymbol{v}}^*(1) > \varepsilon(M_0).$$

Hence the first term in (17) decays exponentially in N for any fixed M_0.

B. Step 2: Second Sum

We now bound the second sum in (17), corresponding to those terms in which the number of nonzero v_k's is "large." For these terms, we approximate the distribution of Y_n by that of a χ-squared random variable. We first need to invoke the Berry–Esseen refinement of the Central Limit Theorem, which gives a uniform upper bound on the error of the Gaussian approximation to the cdf of a sum of independent but not necessarily identical random variables.

Lemma 1 [21, Theorem 7.4.1]: Let $W_1, \ldots, W_n$ be independent zero-mean random variables, and let $\sigma_i^2 = E(W_i^2), \nu_i = E(|W_i|^3)$. Let

$$S \stackrel{\text{def}}{=} \frac{1}{\sqrt{\sum_{i=1}^n \sigma_i^2}} \sum_{i=1}^n W_i$$

and F_S be the cdf of S. Then

$$\sup_x |F_S(x) - \Phi(x)| < B_0 \frac{\sum_{i=1}^n \nu_i}{\left\{\sum_{i=1}^n \sigma_i^2\right\}^{\frac{3}{2}}}$$

where Φ is the unit normal cdf and B_0 is a universal constant.

Now, let $X_1, \ldots, X_K$ be i.i.d. random variables having the same distribution as s_{11}. For a fixed $\boldsymbol{v}$ and $r < 0$, define

$$\begin{aligned} S &= \|\boldsymbol{A}v\|^{-1}\left(X_1 + \sum_{k=2}^K v_k A_k X_k\right) \\ Y &= \left(X_1 + \sum_{k=2}^K v_k A_k X_k\right)^2 = \|\boldsymbol{A}v\|^2 S^2 \\ U &= \exp(rY). \end{aligned}$$

Now, let F_S and F_U be the cdf's of S and U, respectively.

$$\begin{aligned} E[\exp(rY)] &= \int_0^1 [1 - F_U(u)]\, du \\ &= \int_0^1 \left[1 - 2F_S\left(\frac{\log u}{r\|\boldsymbol{A}v\|}\right)\right] du \\ &= \int_0^1 \left[1 - 2\Phi\left(\frac{\log u}{r\|\boldsymbol{A}v\|}\right)\right] du \\ &\quad + 2\int_0^1 \left[\Phi\left(\frac{\log u}{r\|\boldsymbol{A}v\|}\right) - F_S\left(\frac{\log u}{r\|\boldsymbol{A}v\|}\right)\right] du. \end{aligned} \tag{24}$$

If we let $\tilde{Y}$ be a χ^2 random variable with the same mean as Y (i.e., $\tilde{Y} = Y$ if s_{11} is Gaussian), then (24) results in

$$\begin{aligned} &|E[\exp(rY)] - E[\exp(r\tilde{Y})]| \\ &\quad = \left|2\int_0^1 \left[\Phi\left(\frac{\log u}{r\|\boldsymbol{A}v\|}\right) - F_S\left(\frac{\log u}{r\|\boldsymbol{A}v\|}\right)\right] du\right| \\ &\quad \le 2B_0 \frac{E[|s_{11}|^3]\left(1 + \sum_{k=2}^K |v_k|^3 A_k^3\right)}{\left\{1 + \sum_{k=2}^K v_k^2 A_k^2\right\}^{\frac{3}{2}}} && (25) \\ &\quad \le 2B_0 \frac{E[|s_{11}|^3]\gamma^3 \|\boldsymbol{v}\|^2}{\theta^3 \|\boldsymbol{v}\|^3} && (26) \\ &\quad = \frac{B_1}{\sqrt{w(\boldsymbol{v})}} && (27) \end{aligned}$$

for some constant B_1, where (25) follows from Lemma 1, and (26) follows from (9). Now we can use (27) in conjunction with the Chernoff bound, for all $r < 0$

$$\begin{aligned} &\operatorname{Prob}[\boldsymbol{v}^T \boldsymbol{ARAv} < 1] \\ &\quad = \operatorname{Prob}\left[\frac{1}{N}\sum_{n=1}^N Y_n < 1\right] \\ &\quad \le \exp[-N\{r - \log E[\exp(rY)]\}] \\ &\quad \le \exp\left[-N\left\{r - \log\left(E[\exp(r\tilde{Y})] + \frac{B_1}{\sqrt{w(\boldsymbol{v})}}\right)\right\}\right] \end{aligned} \tag{28}$$

where the last inequality follows from (27). By direct evaluation of the moment-generating function

$$\begin{aligned} E[\exp(r\tilde{Y})] &= \frac{1}{\sqrt{1 - 2r\|\boldsymbol{A}v\|^2}} \\ &\le \frac{1}{\sqrt{1 - 2r\theta^2 w(\boldsymbol{v})}} \end{aligned} \tag{29}$$

where (29) follows from (9) and the fact that $r < 0$. Substituting (29) into (28) and letting $r = -1$, we get

$$\begin{aligned} &\operatorname{Prob}[\boldsymbol{v}^T \boldsymbol{ARAv} < 1] \\ &\quad \le \exp\left[-N\left\{-1 - \log\left(\frac{1}{\sqrt{1 + 2\theta^2 w(\boldsymbol{v})}} + \frac{B_1}{\sqrt{w(\boldsymbol{v})}}\right)\right\}\right] \\ &\quad \le \exp\left[-N\left\{-1 + \log\left(\frac{\sqrt{w(\boldsymbol{v})}}{B_2}\right)\right\}\right] \end{aligned} \tag{30}$$

for some universal constant B_2. Hence, the second sum in (17) is bounded by

$$\begin{aligned} &\sum_{\boldsymbol{v} \in C_K, w(\boldsymbol{v}) > M_0} \operatorname{Prob}[\boldsymbol{v}^T \boldsymbol{ARAv} < 1] \\ &\quad \le \sum_{\boldsymbol{v} \in C_K, w(\boldsymbol{v}) > M_0} \exp\left[-N\left\{-1 + \log\left(\frac{\sqrt{w(\boldsymbol{v})}}{B_2}\right)\right\}\right] \\ &\quad < |C_K| \exp\left[-N\left\{-1 + \log\left(\frac{\sqrt{M_0}}{B_2}\right)\right\}\right] \\ &\quad < 3^{\beta N} \exp\left[-N\left\{-1 + \log\left(\frac{\sqrt{M_0}}{B_2}\right)\right\}\right] \\ &\quad = \exp\left[-N\left\{-1 - \beta\log 3 + \log\left(\frac{\sqrt{M_0}}{B_2}\right)\right\}\right]. \end{aligned}$$

Hence, for sufficiently large M_0, this sum goes to zero exponentially in N. Combining this with Step 1 completes the proof of the main theorem.

C. Special Case of $\beta < 1$

For the case of $\beta < 1$, we can slightly strengthen the above result by doing away with the uniform upper bound condition on the amplitudes A_k's. We observe that in the above proof, we only use the upper bound condition in Step 2, while in Step 1 we only need the uniform lower bound. Thus we can relax the upper bound condition if we can show that for $\beta < 1$, there exists an M_0 such that for all N, $\boldsymbol{v}^*$ (the argument that minimizes (15)) satisfies $w(\boldsymbol{v}^*) < M_0$.

To that end, let λ_N (≥ 0) be the smallest (random) eigenvalue of the crosscorrelation matrix $\boldsymbol{R}$. Then for any vector $\boldsymbol{v}$

$$\boldsymbol{v}^T \boldsymbol{ARAv} \ge \lambda_N \|\boldsymbol{Av}\|^2 \ge \lambda_N \theta^2 w(\boldsymbol{v}) \tag{31}$$

where the last inequality follows from the lower bound condition on the amplitudes. By definition of the asymptotic efficiency, the minimizing $\boldsymbol{v}_N^*$ in (3) must satisfy

$$\boldsymbol{v}_N^{*T} \boldsymbol{ARAv}_N^* \le 1.$$

Combining this with inequality (31), we get for every N

$$\lambda_N \theta^2 w(\boldsymbol{v}_N^*) \le 1.$$

Now, it is proved in [22] that almost surely

$$\lambda_N \to (1 - \sqrt{\beta})^2.$$

Thus almost surely

$$\limsup_{N\to\infty} w(\boldsymbol{v}_N^*) \le \frac{1}{\theta^2(1-\sqrt{\beta})^2}. \tag{32}$$

Setting M_0 to be the right-hand side constant yields the desired conclusion.

III. Conclusion

The proof of convergence to unit asymptotic efficiency has relied on an application of the Berry–Esseen refinement of the Central Limit Theorem, in addition to the Chernoff large deviations bounds.

The sufficient condition on the nature of the direct-sequence spread-spectrum format, namely, that chips are modulated by a random variable whose absolute third moment exists, encompasses all cases of practical interest. If the amplitudes are not subject to any restrictions, then optimum asymptotic efficiency does not converge to unity. This is because the worst case optimum asymptotic efficiency taken over all amplitudes (allowing them to depend on the choice of signature waveforms) is the optimum near–far resistance, and this is the same as the asymptotic efficiency of the decorrelator, which approaches $[1-\beta]^+$ as $N \to \infty$. Theorem I.1 states that, provided huge disparities on the amplitudes are avoided and the amplitudes are not chosen as a function of the signature waveforms, the optimum asymptotic efficiency converges to 1. For example, the following cases satisfy (9):

- $A_1 = \ldots A_K = A(K)$, i.e., equal received amplitudes which may depend on the number of users.
- There exists a positive constant μ such that $A_i/A_k < \mu$ for all i and k.

Theorem I.1 shows the surprising result that even if the number of users is a large multiple of the spreading gain, the odds are that the nearest neighbor to each of the transmitted signals is at the same distance as if no interferers were present in the channel. Thus in the asymptotic setting of large K and high SNR, the typical way for an error to occur in maximum-likelihood multiuser detection is for all users but one to be detected correctly. Naturally, as the number of users per degree of freedom β increases we expect that the number of nearby neighbors increases, and accordingly, the asymptotic efficiency is representative of the low-noise behavior only for increasingly higher SNR.

Unlike uncoded bit-error rate, the highest achievable rate using error control codes is indeed a function of β. For example, in the special case of equal received amplitudes $A_1 = \ldots A_K = A$, the asymptotic capacity (per degree of freedom) with average power constrained transmitters is equal to [13, eq.(9)]

$$\frac{\beta}{2}\log\left(1+\mathsf{SNR}-\frac{1}{4}\mathcal{F}(\mathsf{SNR},\beta)\right) + \frac{1}{2}\log\left(1+\mathsf{SNR}\beta-\frac{1}{4}\mathcal{F}(\mathsf{SNR},\beta)\right) - \frac{\log e}{8\mathsf{SNR}}\mathcal{F}(\mathsf{SNR},\beta) \tag{33}$$

where $\mathsf{SNR} = A^2/\sigma^2$ and

$$\mathcal{F}(x,z) \stackrel{\text{def}}{=} \left(\sqrt{x(1+\sqrt{z})^2+1} - \sqrt{x(1-\sqrt{z})^2+1}\right)^2.$$

The result of this correspondence applies to a receiver which takes into account all active users. If the structure of the multiaccess interference caused by some users (e.g., out-of-cell interferers) is neglected, then the effective background noise level is increased. Note that in some models this could introduce a coupling between K and σ.

It is an open problem whether or not the limiting behavior of minimum error probability under high SNR and large number of users commutes. A more ambitious and interesting open problem is the behavior of the bit-error rate for fixed SNR as K and N grow.

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