Resource Pooling and Effective Bandwidths in CDMA Networks with Multiuser Receivers and Spatial Diversity

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***Abstract*—Much of the performance analysis on multiuser receivers for direct-sequence code-division multiple-access (CDMA) systems is focused on worst case near–far scenarios. The user capacity of power-controlled networks with multiuser receivers are less well-understood. In [1], it was shown that under some conditions, the user capacity of an uplink power-controlled CDMA cell for several important linear receivers can be very simply characterized via a notion of *effective bandwidth*. In the present paper, we show that these results extend to the case of antenna arrays. We consider a CDMA system consisting of users transmitting to an antenna array with a multiuser receiver, and obtain the limiting signal-to-interference (SIR) performance in a large system using random spreading sequences. Using this result, we show that the SIR requirements of all the users can be met if and only if the sum of the effective bandwidths of the users is less than the total number of degrees of freedom in the system. The effective bandwidth of a user depends only on its own requirement. Our results show that the total number of degrees of freedom of the whole system is the product of the spreading gain and the number of antennas. In the case when the fading distributions to the antennas are identical, we show that a curious phenomenon of "resource pooling" arises: the multiantenna system behaves like a system with only one antenna but with the processing gain the product of the processing gain of the original system and the number of antennas, and the received power of each user the sum of the received powers at the individual antennas.**

***Index Terms*—Antenna arrays, code-division multiple access (CDMA}, large system analysis, multiuser detection, random spreading, resource pooling.**

I. INTRODUCTION

IN recent years, there have been intense efforts in developing sophisticated multiuser techniques for wireless communications. A significant thrust of work has been on developing *multiuser* receiver structures which mitigate the interference between users in direct-sequence code-division multiple-access (DS-CDMA) systems. (See [2] for a comprehensive account of the state of the art.) Unlike the conventional matched filter receiver used in the IS-95 CDMA system, these techniques take into account the structure of the interference from other users when decoding a user. Another important line of work is the development of signal processing techniques in systems with antenna arrays [3]–[5]. While spread-spectrum techniques provide *frequency diversity* to the wireless system, antenna arrays provide *spatial diversity*. Both frequency and space provide *degrees of freedom* through which communication can take place.

Much work has already been undertaken on characterizing the performance of multiuser receivers, using measures such as asymptotic efficiency and near–far resistance [2]. These measures tend to be *user-centric*, focusing on the performance of a particular user being demodulated. Moreover, near–far resistance evaluates the worst case performance of a user in the face of arbitrary received powers of the interferers.

A different point of view can be taken from a networking perspective. Rather than focusing on the performance of individual users, we ask the following question: given desired levels of performance (quality of service, or QoS) for each of the users in the network, what is the number of users that can be accommodated? This leads to the *network-centric* performance measure of *user capacity*. In the case of a heterogeneous network with multiple class of users with different QoS, we are interested in the *user capacity region*, characterizing the tradeoff between the number of users in each class that can be simultaneously accommodated. Because of the need to meet the QoS of each of the users, power control is done in conjunction with multiuser reception. This necessitates a better understanding of the performance of multiuser receivers in a power-controlled environment rather than one with worst case interference.

A line of work toward a better understanding of these issues has recently been initiated in [1]. A network capacity analysis of linear multiuser receivers is done in the context of synchronous CDMA systems with *random* spreading sequences. The results are for large networks, asymptotic as both the number of users and processing gain grow. The QoS measure for each user is taken as the signal-to-interference ratio (SIR) achieved at the output of the multiuser receiver. Related results for the case when all users have the same SIR requirement are obtained independently in [6].

A concept of *effective bandwidth* has emerged from this work as a succinct measure of network capacity: given a set of SIR requirements for the users in an uplink power-controlled cell, they can all be met if and only if the sum of the effective bandwidths of the users is less than a certain invariant quantity, which depends only on the total degrees of freedom and the power constraints. Results are obtained for three linear receivers: the min-

Manuscript received May 1, 1999; revised May 5, 2000. The work of S. V. Hanly was supported by an Australia Research Council small grant. The work of D.N.C. Tse was supported in part by a National Science Foundation Early Faculty CAREER Award NCR-9734090.

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Communicated by M. L. Honig, Associate Editor for Communications.
Publisher Item Identifier S 0018-9448(01)02705-5.

imum mean-square error (MMSE) receiver [7]–[10], the decorrelator [11], [12], and the matched-filter receiver (as in IS-95 [13]). The effective bandwidths of a user with SIR requirement β under these three receivers are given by

$$e_{\rm mmse}(\beta) = \frac{\beta}{1+\beta} \qquad e_{dec}(\beta) = 1 \qquad e_{\rm mf}(\beta) = \beta.$$

The effective bandwidth concept is based on a more general notion of *effective interference*, which captures the effect of an interferer, received at arbitrary power, on the user to be demodulated. While the concept of effective bandwidth holds for a single power-controlled cell, the notion of effective interference can quantify the intercell interference effects as well.

In this paper, we extend the above concepts to DS-CDMA systems with spatial diversity. The spatial diversity can be obtained by multiple antenna elements at a single base station (microdiversity), or by combining of signals received at multiple base stations (macrodiversity). We show that the notion of effective bandwidths extends to both scenarios, again in the asymptotic regime of a system with large processing gain and many users but fixed number of antennas. The capacity region with and without power constraints is characterized, the latter we call the *interference-limited* capacity region of the network. In some cases, a curious phenomenon of "resource pooling" arises: the multiantenna system behaves like a system with only one antenna but with the processing gain the product of the processing gain of the original system and the number of antennas, and the received power of each user the sum of the received powers at the individual antennas. The focus of the analysis is on the linear MMSE receiver, which is the optimal linear receiver in terms of maximizing the SIR of each user. However, the performance of suboptimal receivers such as the matched filter and the decorrelator will also be presented for comparison. In contrast to the MMSE receiver, which requires knowledge of the received powers and signature sequences of all users, these suboptimal receivers require less information. We remark that the effective bandwidth result for the matched filter, macrodiversity antenna array was proven earlier in [14] in a similar model, but in the present paper we provide a more rigorous proof, using a more detailed model of the physical layer including flat fading.

The effective bandwidth results described above hold in a *large* system with *random* spreading sequences. A natural question is whether an effective bandwidth characterization exists in a *finite-sized* system with *arbitrary* spreading sequences. In Section VII, we present a characterization of the user capacity region for the MMSE receiver in terms of given arbitrary signature sequences, and show that under weak linear independence conditions on the sequences and the channel fading, the resulting *interference-limited* capacity region is identical to that under random sequences. These results provide insight as to why the effective bandwidth results emerge as they do in the limiting regime of random signature sequences.

Much work has already been undertaken on the signal processing aspects of CDMA antenna array systems, e.g., [15]–[17]. In contrast, this paper focuses on the issues of performance from the point of view of user capacity.

In this paper, random variables are denoted by capital letters X, Y, vectors by boldface letters $\boldsymbol{X}$, $\boldsymbol{Y}$, and matrices by calligraphic fonts $\mathcal{S}$, $\mathcal{D}$.

II. Model

A. Basic Multiantenna CDMA Model

In a DS-CDMA system, each of the user's information or coded symbols is spread onto a much larger bandwidth via modulation by its own *signature* or *spreading sequence*. We consider a sampled discrete-time baseband model for a symbol-synchronous multiaccess CDMA system with K users, L receive antennas, and processing gain N. The received signal at the lth antenna is given by

$$\boldsymbol{Y}(l) = \sum_{k=1}^{K} X_k \sqrt{T_k}\gamma_k(l)\boldsymbol{s}_k + \boldsymbol{W}(l) \tag{1}$$

where X_k is the symbol transmitted by user k at transmit power T_k, $\gamma_k(l)$ is the complex fading channel gain from user k to antenna l, $\boldsymbol{s}_k \in \mathbb{C}^N$ is the signature sequence of user k, $\boldsymbol{Y}(l) \in \mathbb{C}^N$, and $\boldsymbol{W}(l)$ is additive white Gaussian noise with variance σ^2, independent across l. The symbol energy $\mathbb{E}[X_i^2]$ is normalized to be 1. Here, we are assuming a flat-fading channel model. Moreover, the channel gains are assumed to be circular symmetric, as is typical for a baseband model.

Let

$$\begin{aligned}\boldsymbol{X} &:= (X_1, \ldots, X_K)^t\\ \boldsymbol{Y} &:= (\boldsymbol{Y}(1)^t, \ldots, \boldsymbol{Y}(L)^t)\\ \boldsymbol{W} &:= (\boldsymbol{W}(1)^t, \ldots, \boldsymbol{W}(L)^t)^t\end{aligned}$$

and

$$\begin{aligned}\mathcal{D} &:= \mathrm{diag}(T_1, \ldots, T_K)\\ \mathcal{S}(l) &:= [\gamma_1(l)\boldsymbol{s}_1, \ldots \gamma_K(l)\boldsymbol{s}_K]\\ \mathcal{S} &:= [\mathcal{S}(1)^t, \ldots, \mathcal{S}(L)^t]^t\end{aligned}$$

i.e., the $\mathcal{S}(l)$'s stacked one above the other. Also, let $\overline{\boldsymbol{s}}_k$ be the kth column of $\mathcal{S}$. Then we can write the overall channel as

$$\boldsymbol{Y} = \sum_{k=1}^{K} X_k \sqrt{T_k}\overline{\boldsymbol{s}}_k + \boldsymbol{W}. \tag{2}$$

In vector form, the channel can be written as

$$\boldsymbol{Y} = \mathcal{S}\mathcal{D}^{1/2}\boldsymbol{X} + \boldsymbol{W}.$$

We should point out that it is of interest to extend our results to the frequency-selective fading case, since that is often the relevant case for spread-spectrum systems, and that work in this direction is already progressing (see [18]). The applicability of our flat-fading model depends on how the spread bandwidth compares to the coherence bandwidth of the channel of interest. We focus on the flat-fading model in an effort to keep model complexity, and notation, to manageable levels.

B. MMSE Receiver

A *linear* receiver $\boldsymbol{c}_1$ for user 1 generates a soft decision $\hat{X}_1 = \boldsymbol{c}_1^H\boldsymbol{Y}$ for X_1 based on the entire observation $\boldsymbol{Y}$. The key performance measure for a linear receiver is the output *signal-to-interference* ratio (SIR), defined by

$$\mathrm{SIR}_1 = \frac{T_1|\boldsymbol{c}_1^H \overline{\boldsymbol{s}}_1|^2}{\sigma^2(\boldsymbol{c}_1^H \boldsymbol{c}_1) + \sum_{k=2}^{K} T_k |\boldsymbol{c}_1^H \overline{\boldsymbol{s}}_k|^2}. \tag{3}$$

Consider now coherent receivers that demodulate $\boldsymbol{X}$ from $\boldsymbol{Y}$ with perfect knowledge of the signature sequences as well as the channel gains and transmit powers of the users. Among these receivers, the MMSE receiver minimizes the mean-square error as well as maximizes the SIR for all the users, given the signature sequences, channel gains, and transmit powers.

The MMSE receiver for user 1 is given by

$$\begin{aligned} \boldsymbol{c}_1 &= (\mathbb{E}[\boldsymbol{Y}\boldsymbol{Y}^H])^{-1}\mathbb{E}[X_1^* \boldsymbol{Y}] \\ &= \sqrt{T_1}(\mathcal{S}\mathcal{D}\mathcal{S}^H + \sigma^2 I)^{-1}\overline{\boldsymbol{s}}_1 \\ &= \text{constant} \cdot (\mathcal{S}_1\mathcal{D}_1\mathcal{S}_1^H + \sigma^2 I)^{-1}\overline{\boldsymbol{s}}_1 \end{aligned} \tag{4}$$

where $\mathcal{S}_1$ is the matrix obtained by removing the column $\overline{\boldsymbol{s}}_1$ from $\mathcal{S}$, and $\mathcal{D}_1 = \mathrm{diag}(T_2, \ldots, T_K)$. The last step follows from the use of the matrix inversion lemma. The expectation is taken by averaging all quantities that are unknown to the receiver; in this case the unknown symbols of the users, and the background white noise.

The SIR of the MMSE receiver for user 1 is given by the expression

$$\mathrm{SIR}_1 = T_1 \overline{\boldsymbol{s}}_1^H (\mathcal{S}_1\mathcal{D}_1\mathcal{S}_1^H + \sigma^2 I)^{-1}\overline{\boldsymbol{s}}_1. \tag{5}$$

Observe that the SIRs achieved by each user are functions of the signature sequences, channel gains, and transmit powers of *all* users.

C. Random Signature Sequence Model

While (5) can be numerically computed given specific signature sequences, channel gains, and transmit powers, the qualitative dependence of performance on system parameters such as the number of users, the processing gain, the number of antennas or the received power profile is not clear. To obtain more insight, we will assume a *random* signature sequence model: the chip values of the sequences are independent and identically distributed (i.i.d.) circular symmetric complex Gaussian random variables with mean zero and variance $1/N$, and the sequences of different users are chosen independently.[1] The SIR depends on the realization of the random sequences as well as the channel gains, and is, therefore, also a random variable. It will be shown, however, that in the cases of interest, this random variable converges to a *deterministic* quantity in a large system, and thus provides a sequence-independent performance measure of the system. It should be re-emphasized that although the sequences are chosen randomly, knowledge of the sequences is assumed at the MMSE receiver so that interference suppression can be performed.

III. Microdiversity

In this section, we will focus on a fading model for microdiversity, where the receive antennas are assumed to be placed at the same base station. The diversity captured in this scenario is due to small-scale multipath fading effects. A reasonable model is to assume that the random channel gains $\gamma_k(l)$'s are independent for all users k and antennas l, and for any given user, the gains to all the antennas are identically distributed. The crucial assumption here is the symmetry of the channel fading statistics with respect to the antennas. However, the fading levels are not necessarily identically distributed across users. For example, some users may be close to the antenna array, and others far. We can think of the *distribution* of the $(\gamma_k(l))_{l=1}^L$ as being a function of the geographic location of user k with respect to the antenna array, on a *coarse* space scale, and the actual realizations of these random variables as a function of the user's random position as measured on a small space scale.

We will allow the transmit powers T_k's to depend on the magnitudes of the channel gains $\gamma_j(l)$ for all j and l, but independent of everything else. This models the use of power control. We will also assume that the transmit powers are a symmetrical function of the channel gains with respect to the different antennas. More precisely: if we denote $\Gamma(l) = (\gamma_1(l), \ldots, \gamma_K(l))$ be the vector of channel gains from all the users to antenna l, then for any permutation π on $\{1, \ldots, L\}$, for any channel gains $\Gamma(1), \ldots, \Gamma(L)$ and for any user k

$$T_k(\Gamma(\pi(1)), \ldots, \Gamma(\pi(L))) = T_k(\Gamma(1), \ldots, \Gamma(L)).$$

The reason for needing this assumption will be explained at the end of the next section. We believe that it holds for any sensible power control policy, and it certainly does for the power control policy we consider in Section IV-B. It makes precise the notion that with microdiversity, all antennas are identical and are treated in an identical manner by power control policies.

A. Resource Pooling

The following is the main result in this section, yielding the asymptotic performance of the MMSE receiver in a system with large processing gain and many users, but fixed number of antennas.

Theorem 1: Let

$$P_k = T_k \sum_{l=1}^{L} |\gamma_k(l)|^2 \tag{6}$$

be the sum of the received powers of user k. Assume that almost surely the empirical distribution of $(P_1, \ldots, P_K)$ converges weakly to a limiting distribution F as K goes large, and that the P_k's are uniformly bounded for all k and N.[2] Each user selects a signature sequence randomly, as described in Section II-C. Then if $N, K \to \infty$ with $K/N \to \alpha$ but L fixed, SIR_1/P_1 converges in probability to a deterministic constant a, where a is the unique positive solution to the fixed-point equation

$$a = \frac{1}{\sigma^2 + \frac{\alpha}{L}\mathbb{E}\left[\frac{P}{1+Pa}\right]} \tag{7}$$

and P is a random variable having distribution F.

Proof: See Appendix-B. □

[1] We conjecture that all our results hold for general i.i.d. chip distribution.

[2] This latter assumption is a technicality to simplify the proofs, but we believe that it is not really necessary.

This result says that in a wide-band system with many users, the SIR of a user does not depend on the specific realization of the signature sequences, the channel gains, and the transmit powers. The SIR is a function of the user's own received powers at the antennas and depends on the the interferers' received powers only through the limiting empirical distribution of the P_k's. In a sense, there is an averaging of the effects across the large number of interferers. The convergence of the empirical distribution of the received power is a statistical regularity assumption and is necessary for such averaging to occur. It is satisfied, for example, when the transmit power T_k of any user k depends on the channel gains for that user only, which implies independence across users, and there is a bound on how big T_k can be. It also arises naturally when there are several classes of users with different SIR requirement, and power control is performed as a function of which class the user belongs to.

Theorem 1 is a natural extension of the single-antenna result [1, Theorem 3.1]. The rate of convergence in Theorem 1 requires further study, but we note that there are results for the single-antenna case—see [19].

When there is only one antenna element ($L = 1$), P_k is simply the received power of user k. Theorem 1, therefore, has the nice interpretation that for any fixed number of antennas L, the limiting performance of the MMSE receiver is the same as that for a system with a single antenna, having processing gain LN and with the received power of each user the sum of the received powers at the individual antennas. This is a form of *resource pooling*: all the degrees of freedom of the individual antennas are pooled together into a single resource and the system behaves like a single-antenna system.

The crucial condition for the resource pooling phenomenon to hold is that the limiting joint empirical distribution of received powers at the antennas is symmetrical with respect to the antennas; see the Proof of Theorem 1. This condition holds in the microdiversity environment under the assumption that the power control is a symmetrical function of the channel gains, an assumption we made just before the statement of Theorem 1, at the end of the preceding section. Without the symmetry condition, the SIR still converges but there is no resource pooling. The general case will be analyzed in Section IV.

B. Repeated Versus Completely Random Signature Sequences

One can consider the multiantenna spread-spectrum system as one with LN degrees of freedom, given by N units of processing gain per antenna. The "super" signature sequence of user k, of length LN, is then the signature sequence $\bar{\boldsymbol{s}}_k$, with the same signature sequence repeated L times and multiplied by the path gains $\gamma_k(l)$'s. One might imagine that this repetition would lead to some loss of degrees of freedom; the resource pooling interpretation of Theorem 1, however, seems to suggest that there is actually no loss. To substantiate this point, we consider the following alternative model.

The "Completely Random" Sequences Model: Instead of the basic model, suppose that the signature sequences received at each antenna, from the same user, are independently chosen sequences. Thus, for user k we generate L distinct randomly chosen sequences $\boldsymbol{s}_k^{(1)}, \boldsymbol{s}_k^{(2)}, \ldots, \boldsymbol{s}_k^{(L)}$, and set the received "super signature sequence" to be

$$\bar{\boldsymbol{s}}_k = (\gamma_k(1)\boldsymbol{s}_k^{(1)t}, \gamma_k(2)\boldsymbol{s}_k^{(2)t}, \ldots, \gamma_k(L)\boldsymbol{s}_k^{(L)t})^t.$$

The key difference is that now we do not have one sequence $\boldsymbol{s}_k$ repeated at each antenna, but rather a different sequence at each antenna. Somewhat surprisingly, we have the following result.

Theorem 2: In the completely random sequence model, SIR_1/P_1 converges to exactly the same limit as that given in Theorem 1.

We conclude from the above theorem that, asymptotically, there is no performance loss from sequence repetition. The proof of Theorem 1 shows that the uncorrelatedness of the channel gains $\gamma_k(l)$'s across antennas provides enough randomness to make the system behave as though the super sequences were fully random. Theorem 2 follows immediately from Corollary 4 to be presented in the next section.

To further reinforce this notion of no loss of degrees of freedom, we can consider the asymptotic efficiency of the receiver in the limiting system. By taking $\sigma^2 \to 0$, we see that $a\sigma^2 \to 1 - \alpha/L$ if $\alpha < L$. For a single-user system, the SIR is P_1/σ^2. Hence, the asymptotic efficiency of the MMSE is $1 - \alpha/L$; there are a total of LN degrees of freedom and at high signal-to-noise ratio (SNR), one interferer costs one degree of freedom.

The performance of the MMSE receiver depends crucially on the (colored) spectrum of the interference, and an important step in proving these results is to show that the empirical eigenvalue distribution of the interference covariance matrix $\mathcal{S}_1\mathcal{D}\mathcal{S}_1^H$ converges to a limit and to characterize the limit. While the limiting eigenvalue distribution for the special case of $L = 1$ in [1, Theorem 3.1] is known from existing random matrix results, new techniques have to be used to compute the limiting eigenvalue distribution for the general case, due to the more complex dependency in the elements of the random matrix $\mathcal{S}_1\mathcal{D}\mathcal{S}_1^H$ due to code repetition. The proofs are presented in Appendix-B.

While the completely random sequence model is not physically realizable, it is technically much easier to analyze than the realistic model with repeated signature sequences. In fact, our results for the more general macrodiversity scenario, to be presented in Section IV, are only proved for the completely random sequence model. Our belief that the results also hold for the original model with repeated sequences is supported by the asymptotic equivalence of the two models in the microdiversity scenario.

C. Effective Interference

Due to the resource pooling phenomenon, we can treat the system with microdiversity as one with a single-antenna element, to which we can immediately apply the interpretation of effective interference derived in [1]: in a large system, it follows from (7) that the SIR of user 1 approximately satisfies the fixed-point equation

$$\mathrm{SIR}_1 = \frac{P_1}{\sigma^2 + \frac{1}{LN}\sum_{k=2}^{K} I(P_k, P_1, \mathrm{SIR}_1)} \tag{8}$$

where $I(P_k, P_1, \beta) := P_1P_k/(P_1 + P_k\beta)$. Note that to determine the asymptotic performance we need to solve the fixed-point equation, which can be done numerically. However, to determine whether a desired SIR for user 1, β_1, is achievable, we can substitute β_1 into the right-hand side of (8), and check if

$$\frac{P_1}{\sigma^2 + \frac{1}{LN}\sum_{k=2}^{K} I(P_k, P_1, \beta_1)} \geq \beta_1.$$

(This is due to the fact that the right-hand side of (8) is a monotonic function of SIR_1. See [1] for details.) In doing so, we provide a way to decouple the interference effects of each user, and we can define the *effective interference* of an interferer of received power P_k on user 1 as $I(P_k, P_1, \mathrm{SIR}_1)$.

IV. Macrodiversity

The crucial assumption underlying our results in the microdiversity scenario is the symmetry of the fading distribution with respect to the receive antennas. This is justified by the closeness of the antennas. In a system where the antenna elements are widely separated (macrodiversity) such symmetry does not necessarily hold. For example, the antennas may be at two different base stations, and a user may be closer to one base station than the other. In this section, we will therefore relax the assumption that the channel gains $\gamma_k(1), \ldots, \gamma_k(L)$ to the different antennas are identically distributed for each user k. As in the case of microdiversity, we can think of the *distribution* of the $(\gamma_k(l))_{l=1}^{L}$ as being a function of coarse-scale propagation conditions for user k, which are assumed fixed, and the *realizations* as being functions of finer scale propagation conditions, which are assumed random. We will retain the assumption that the $\gamma_k(1), \ldots, \gamma_k(L)$ are *independent* random variables.

A. Limiting SIR Performance

The following result is the analog of Theorem 1 for the performance of the MMSE receiver in the macrodiversity scenario. At present, we are only able to prove the result for the (nonphysical) completely random signature sequence model described in Section III-A, but we conjecture that it also holds in the repeated signature sequence model of Section II-C. As discussed earlier, this conjecture is supported by the equivalence of the two models in the microdiversity case.

Theorem 3: Let $P_k(l) = T_k|\gamma_k(l)|^2$ be the received power of user k at antenna l. Assume that almost surely, the empirical joint distribution of

$$(P_1(1), \ldots, P_1(L)), (P_2(1), \ldots, P_2(L)), \ldots, (P_K(1), \ldots, P_K(L))$$

converges weakly to some limiting joint distribution G as $K \to \infty$. Each user selects L independent signature sequences randomly, as in the "completely random" model of Section III-A. Then, if $N, K \to \infty$ with $K/N \to \alpha$, SIR_1 converges in probability to $\sum_{l=1}^{L} P_1(l)a(l)$, where the constants $a(l)$'s are the unique positive solution to the system of fixed-point equations

$$a(l) = \frac{1}{\sigma^2 + \alpha \mathbb{E}\left[\frac{P(l)}{1+\sum_{n=1}^{L} P(n)a(n)}\right]}, \qquad l = 1, \ldots, L \quad (9)$$

and $(P(1), \ldots, P(L))$ are random variables having joint distribution G.

Proof: See Appendix-A. □

Thus, in the limit, the SIR of a user is a function of its own received powers at the antennas and system-wide constants $a(1), \ldots, a(L)$ which are user-independent.

In the case when the joint distribution G is exchangeable, i.e., for any permutation π of $\{1, \ldots, L\}$

$$G(x_1, \ldots, x_L) = G(x_{\pi(1)}, \ldots, x_{\pi(L)})$$

we have

$$\mathbb{E}\left[\frac{P(l)}{1+\sum_{n=1}^{L} P(n)a(n)}\right] = \frac{1}{L}\mathbb{E}\left[\frac{\sum_{n=1}^{L} P(n)}{1+\sum_{n=1}^{L} P(n)a(n)}\right]$$

the fixed point of (9) satisfies $a(l) = a$ for all l, and the following corollary holds.

Corollary 4: If the limiting distribution G is exchangeable, than SIR_1/P_1 converges to a constant a, where

$$P_1 = \sum_{l=1}^{L} T_1|\gamma_1(l)|^2$$

and a is the unique solution to the fixed-point equation

$$a = \frac{1}{\sigma^2 + \frac{\alpha}{L}\mathbb{E}\left[\frac{P}{1+Pa}\right]}$$

with $P = \sum_{l=1}^{L} P(l)$.

In this exchangeable case, the system of L equations becomes a single equation, and resource pooling occurs. In the microdiversity scenario considered in Section III, the channel gains of each user to all the antennas are i.i.d., and the transmit powers are symmetrical functions of the channel gains with respect to the different antennas. The exchangeability condition follows from these assumptions.

B. Interference-Limited Capacity and Effective Bandwidths

Let us now consider the interference-limited user capacity of the macrodiversity antenna array. In spite of the fact that one might expect the user capacity to depend on the user "geometry," i.e., the probability distribution of the fadings of the users to the antennas, this turns out not to be so, and the effective bandwidth results generalize from the single-antenna case [1] to the multiantenna scenario.

We consider the case in which users have particular SIR *targets*, i.e., user k has a target of β_k for its SIR. To achieve such a SIR target the user must control its transmit power, and we wish to find a necessary and sufficient condition on the targets β_k's for this to be possible, asymptotically, in a large system. This yields the interference-limited user capacity region of the system. Here, we allow the power control of a user to be possibly a function of the magnitudes of the channel gains of all users, but not of the signature sequences. However, we will see in Section VII that in a certain sense the interference-limited capacity does not increase even when the powers can depend on the signature sequences.

We make the following statistical assumptions on the targets β_k's.

1) Almost surely, the joint empirical distribution of

$$(\beta_1, \gamma_1(1), \ldots, \gamma_1(L)), \ldots, (\beta_K, \gamma_K(1), \ldots, \gamma_K(L))$$

converges to some limiting distribution H as K grows;

2) The limiting distribution H satisfies

$$H(\beta, \gamma(1), \ldots, \gamma(L)) = H_1(\beta)H_2(\gamma(1), \ldots, \gamma(L)).$$

Assumptions 1) and 2) would hold, for example, when there are a finite number of classes of users and the fraction of users in each class approaches a limit, and the fading gains of the users are independent of each other and independent of which class the user belongs to.

Using Theorem 3, it can be seen that to meet the target SIRs, asymptotically, the transmit power T_k of user k should satisfy

$$T_k \sum_{l=1}^{L} |\gamma_k(l)|^2 a(l) = \beta_k \tag{10}$$

where $a(l)$'s satisfy the fixed-point equations (9), and $(P(1), \ldots, P(L))$ in (9) follows the limiting empirical distribution of the received powers

$$(P_1(1), \ldots, P_1(L)), \ldots, (P_K(1), \ldots, P_K(L)).$$

This limiting distribution can, in turn, be calculated from (10) and Assumptions 1) and 2). Let us denote the vector $(a(1), a(2), \ldots, a(L))$ by $\boldsymbol{a}$. We note that Theorem 3 implies that $\boldsymbol{a}$ is strictly positive. But since

$$P_k(l) = T_k|\gamma_k(l)|^2 = \frac{|\gamma_k(l)|^2\beta_k}{\sum_{n=1}^{L} |\gamma_k(n)|^2 a(n)}, \quad l = 1, \ldots, L, \ k = 1, \ldots, K$$

and

$$\sum_{n=1}^{L} P_k(n)a(n) = \beta_k$$

it follows that

$$\mathbb{E}\left[\frac{P(l)}{1 + \sum_{n=1}^{L} P(n)a(n)}\right] = \mathbb{E}\left[\frac{\beta}{1+\beta}\right] \mathbb{E}\left[\frac{|\gamma(l)|^2}{\sum_{n=1}^{L} |\gamma(n)|^2 a(n)}\right]$$

where $(\gamma(l))_{l=1}^{L}$ has the limiting distribution H_2 of the channel gains of the users, and β has the limiting distribution H_1 of the SIR targets of the users. Define

$$B_{\text{mmse}} := \mathbb{E}\left[\frac{\beta}{1+\beta}\right] = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \frac{\beta_k}{1+\beta_k}.$$

It follows from the above calculations that a necessary condition that the given SIR targets can be met asymptotically is that the fixed point equations

$$a(l) = \frac{1}{\sigma^2 + \alpha B_{\text{mmse}} \mathbb{E}\left[\frac{|\gamma(l)|^2}{\sum_{n=1}^{L} |\gamma(n)|^2 a(n)}\right]}, \qquad l = 1, \ldots L \tag{11}$$

have a strictly positive solution for $(a(1), \ldots, a(L))$.

Conversely, suppose that (11) has a fixed point $\boldsymbol{a}^*$ that is strictly positive. Let us now choose the transmit power for user k to be

$$T_k^* = \frac{\beta_k}{\sum_{l=1}^{L} |\gamma_k(l)|^2 a^*(l)}.$$

With this choice of transmit powers and the fact that $(a^*(1), \ldots, a^*(L))$ satisfies (11), it follows that $(a^*(1), \ldots, a^*(L))$ also satisfies (9), with the expectation over the limiting empirical distribution of $(P(1), \ldots, P(L))$ induced by the choice of transmit powers $\{T_k^*\}$. It follows from Theorem 3 that, due to the uniqueness of the solution to (9), the limiting SIR of user k is precisely β_k. Furthermore, the uniqueness of the fixed point in (9) guarantees that (11) can have at most one strictly positive fixed point.

Next, we investigate the condition on the SIR targets under which (11) has a strictly positive solution. Note that such a condition can only depend on the SIR targets through B_{mmse}. To derive a necessary condition, write (11) as

$$\mathbb{E}\left[\frac{|\gamma(l)|^2 a(l)}{\sum_{n=1}^{L} |\gamma(n)|^2 a(n)}\right] \alpha B_{\text{mmse}} = 1 - a(l)\sigma^2$$

and adding up these L equations, we obtain

$$\alpha B_{\text{mmse}} = L - \sigma^2 \sum_{l=1}^{L} a(l). \tag{12}$$

Thus, a necessary condition for there being a positive solution is that $\alpha B_{\text{mmse}} < L$. The following proposition shows that this condition is also sufficient.

Proposition 5: If $\alpha B_{\text{mmse}} < L$, then the system of equations (11) has a unique strictly positive solution.

Proof: Assume $\alpha B_{\text{mmse}} < L$, and define the mapping

$$F : \Re_+^L \to \Re_+^L$$

by

$$F_l(\boldsymbol{a}) = \frac{1}{\sigma^2 + \alpha B_{\text{mmse}} \mathbb{E}\left[\frac{|\gamma(l)|^2}{\sum_{n=1}^{L} |\gamma(n)|^2 a(n)}\right]}, \quad l = 1, 2, \ldots, L.$$

We will establish that F has a fixed point by generating a sequence of vectors $\boldsymbol{a}^{(0)}, \boldsymbol{a}^{(1)}, \ldots$ which provably converges to such a fixed point.

Note that if $\boldsymbol{a}^{(i)}$ is defined inductively via

$$\boldsymbol{a}^{(i)} := F(\boldsymbol{a}^{(i-1)})$$

and if $\mathbf{0} \le \boldsymbol{a}^{(i)} \le \boldsymbol{a}^{(i-1)}$, then by the monotonicity of the mapping F (i.e., $F(\boldsymbol{x}) \le F(\boldsymbol{y})$ if $\boldsymbol{x} \le \boldsymbol{y}$ component-wise), $(\boldsymbol{a}^{(i)})_{i=0}^{\infty}$ is a decreasing sequence. Note also that in this case

$$\sigma^2 a^{(i)}(l) + \alpha B_{\text{mmse}} \mathbb{E}\left[\frac{|\gamma(l)|^2 a^{(i)}(l)}{\sum_{n=1}^{L} |\gamma(n)|^2 a^{(i-1)}(n)}\right] = 1$$

and adding up these L equations, we obtain

$$\sigma^2 \sum_{l=1}^{L} a^{(i)}(l) + \alpha B_{\text{mmse}} \mathbb{E}\left[\frac{\sum_{n=1}^{L} |\gamma(n)|^2 a^{(i)}(n)}{\sum_{n=1}^{L} |\gamma(n)|^2 a^{(i-1)}(n)}\right] = L. \tag{13}$$

Let us set $a^{(0)}(l) = 1/\sigma^2 \ l = 1, 2, \ldots L$. Then the inductive condition is satisfied and $(\boldsymbol{a}^{(i)})_{i=0}^{\infty}$ is a decreasing sequence of positive vectors. Now the assumption of the theorem is that there exists a positive $\epsilon > 0$, such that $\alpha B_{\rm mmse} < L - \epsilon$, and thus, it follows that

$$\limsup_{i\to\infty} \alpha B_{\rm mmse}\mathbb{E}\left[\frac{\sum_{n=1}^{L}|\gamma(n)|^2 a^{(i)}(n)}{\sum_{n=1}^{L}|\gamma(n)|^2 a^{(i-1)}(n)}\right] \le L - \epsilon.$$

We conclude from (13) that $\sum_{l=1}^{L} a^{(i)}(l)$ is bounded away from 0, and hence that it is not possible for $\boldsymbol{a}^{(i)}$ to converge to $\mathbf{0}$. We conclude that $\boldsymbol{a}^{(i)}$ converges to a nonzero, nonnegative solution $\boldsymbol{a}^*$ to (11).

Equations (11) admit the solution $\boldsymbol{a} = \mathbf{0}$. However, we have established that $\boldsymbol{a}^*$ is not this solution. It is also immediate from (11) that there can be no nonnegative solution that is a fixed point, with some components zero, apart from the solution $\boldsymbol{a} = \mathbf{0}$. Thus, $\boldsymbol{a}^*$ must be strictly positive. Uniqueness follows from the fact that (11) can have at most one strictly positive fixed point. $\square$

We can summarize the above development in the following theorem.

Theorem 6: Let $(\beta_k)_{k=1}^{\infty}$ be the SIR targets of the users, and suppose Assumptions 1) and 2) hold. Define

$$B_{\rm mmse} := \lim_{k\to\infty} \frac{1}{K}\sum_{k=1}^{K}\frac{\beta_k}{1+\beta_k}.$$

Then

1) if $\alpha B_{\rm mmse} > L$ then there is no way to assign transmit powers $(T_k)_{k=1}^{\infty}$ such that the users asymptotically achieve SIR targets β_k, $k = 1, 2, \ldots$;
2) if $\alpha B_{\rm mmse} < L$, then (11) has a unique positive solution $(a(1), \ldots, a(L))$ and users' target SIRs can be asymptotically met, with transmit powers given by

$$T_k = \beta_k\left(\sum_{l=1}^{L}|\gamma_k(l)|^2 a(l)\right)^{-1}. \tag{14}$$

Thus, the condition $\alpha B_{\rm mmse} < L$ characterizes the interference-limited user capacity of the system. The above theorem also provides us with a notion of *effective bandwidth*. There are a total of LN degrees of freedom provided by spreading and the antenna array in the limit as $N \to \infty$. If user k achieves its target SIR of β_k, then it occupies an effective bandwidth of $\frac{\beta_k}{1+\beta_k}$ degrees of freedom, and to achieve the SIR targets, the sum of effective bandwidths must be less than the total number of degrees of freedom.

C. Capacity Under Power Constraints

Theorem 6 characterizes the interference-limited user capacity of the system, when no power constraints are imposed on the users. To exploit the total degrees of freedom, however, an enormous amount of transmit power may be required, as in the case when some antennas are very far away from the users. What do our results say about power consumption? Recall that the transmit power of user k to achieve target β_k is given by

$$T_k = \frac{\beta_k}{\sum_{l=1}^{L}|\gamma_k(l)|^2 a(l)} \tag{15}$$

where the $a(n)$'s satisfy the fixed-point equations (11). The transmit power of user k certainly depends on its own target SIR and channel gains, but the main observation is that it depends on the effects of other users in the system only through the system constants $a(1), \ldots, a(L)$, which in turn depends only on the empirical mean of their effective bandwidths, $\alpha B_{\rm mmse}$. Thus one can think of $\alpha B_{\rm mmse}$ as a measure of *congestion* in the system. We will speculate this can be used as a basis of real-time admissions control. We will now show that this can be used to define a precise power-limited capacity region in the case when there are only a finite number of classes of users.

Suppose there are J classes of users, and all users in class j having SIR requirement β_j. In this case, we assume that in the limit, a proportion $\frac{\alpha_j}{\alpha}$ of users is of class j. In this case

$$B_{\rm mmse} = \sum_{j=1}^{J}\alpha_j\beta_j/(1+\beta_j). \tag{16}$$

We also assume that users in class j have a power constraint $\overline{p}_j$. Outage is said to occur for a user when its required transmit power exceeds its power constraint. The capacity region is for a particular level of outage probability, which we denote by θ^*. Let us denote the worst outage probability, among all users, by $\theta(\alpha B_{\rm mmse})$, where we note explicitly the functional dependence of outage probability on $\alpha B_{\rm mmse}$. Thus, $\theta(\alpha B_{\rm mmse})$ increases monotonically with its argument, and there is a unique C, $C < L$, such that $\theta(C) = \theta^*$. Hence, to satisfy the outage probability constraint, $\alpha B_{\rm mmse} < C$. Substituting (16), we conclude that the capacity region is described by the effective bandwidth constraint

$$\left\{(\alpha_1, \ldots, \alpha_J)\colon \sum_{j=1}^{J}\alpha_j\frac{\beta_j}{1+\beta_j} < C\right\}. \tag{17}$$

Note that C increases as the tolerable outage probability θ^* increases, but that it can never exceed L. On the other hand, no matter how small θ^*, C can be made arbitrarily close to L if users can tolerate a large enough maximum transmit power constraint.

Let us specialize the result to the microdiversity scenario and with the channel gains $\gamma_k(l)$'s identically distributed for all k and l. In this case, when $\alpha B_{\rm mmse} < L$, the unique solution $\boldsymbol{a}^*$ of (11) satisfies $a^*(l) = a^*$ for all l, and

$$a^* = \frac{1}{\sigma^2}\left(1 - \frac{\alpha B_{\rm mmse}}{L}\right).$$

The transmit power of user k required to maintain an SIR of β_j in class j is

$$T_k = \frac{\beta_j\sigma^2}{(1-\alpha B_{\rm mmse}/L)\sum_{l=1}^{L}|\gamma_k(l)|^2}.$$

Hence, given the outage constraint for every user k in class j is

$$\mathbb{P}(T_k > \overline{p}_j) \leq \theta^*.$$

This translates to a power-limited user capacity region given by

$$\left\{(\alpha_1, \ldots, \alpha_J)\colon \sum_{j=1}^{J} \alpha_j \frac{\beta_j}{1+\beta_j} < L\left(1 - \frac{\sigma^2}{R^{-1}(1-\theta^*)} \max_{1\leq j\leq J} \frac{\beta_j}{\overline{p}_j}\right)\right\}$$

where R is the cumulative distribution function (cdf) of the random variable $\sum_{l=1}^{L} |\gamma_k(l)|^2$ (same for all k). So in this case, the user capacity region (17) can be computed explicitly. The observation is that in this case, the class with the highest value of $\beta_j/\overline{p}_j$ has the highest outage probability, and limits the user capacity of the system. As the power constraints are relaxed, this power-limited region approaches the interference-limited region.

D. Effective Interference

A notion of "effective interference" was defined in [1] for the single-antenna case, and we showed in Section III that this notion extends to the microdiversity antenna array. This notion pertains to the interference created by an individual interferer on the desired user, and enables the system to be decoupled into a sum of interference effects from all the users in the system. In the macrodiversity array, it is not possible to decouple interference effects in this way. The reason for this is that there are now L constants $a(1), \ldots, a(L)$ which depend on the interaction of all users in the system. In the microdiversity case, there is only one constant a that is equivalent to an SIR, and this can be replaced by a target SIR requirement for the desired user, providing a way to decouple the SIR equations of the users in the system. However, in the macrodiversity case, this would require knowing not just the target SIR requirement of the desired user, which is independent of the other users, but also the target effective SIRs at the separate antenna elements, which does depend on a tight coupling between the users.

In [1], the notion of effective bandwidth was derived by building on the concept of effective interference. It is interesting to note that in the macrodiversity scenario, one can still define a meaningful notion of effective bandwidth even without the existence of effective interference.

V. Suboptimal Receivers

The MMSE receiver maximizes the SIR for user 1, given knowledge of the signature sequences, channel gains, and transmit powers of *all* users. Suboptimal linear receivers can be defined which do not require full knowledge of all the attributes of the interferers. In this section, we consider two such receivers, and their performance will be contrasted with the optimal MMSE receiver. The analysis will be in the general macrodiversity environment with the repeated signature sequence model.

A. The Matched-Filter Receiver

Consider the situation when the demodulator for user 1 has knowledge of the signature sequence, channel gains, and transmit power of user 1, but has no knowledge of those of the interferers other than their statistics. In such a scenario, we can consider the receiver for user 1 which minimizes the mean-square error $\mathbb{E}[(\hat{X}_1 - X_1)^2]$, with the averaging over the signature sequences, transmit powers, and channel gains of all interferers in addition to the transmitted symbols and white noise. Following (4) for the MMSE receiver under perfect knowledge, this present receiver can be derived in the same way but with expectations taken over all the additional quantities assumed to be unknown to the receiver

$$\boldsymbol{c}_1 = (\mathbb{E}[\boldsymbol{Y}\boldsymbol{Y}^H])^{-1}\mathbb{E}[X_1^*\boldsymbol{Y}] \tag{18}$$

which is proportional to the vector

$$\left[\frac{\gamma_1^*(1)}{\sigma^2 + \frac{1}{N}\mathbb{E}\left[\sum_{k=2}^{K} T_k|\gamma_k(1)|^2\right]}\boldsymbol{s}_1^t, \ldots, \frac{\gamma_1^*(L)}{\sigma^2 + \frac{1}{N}\mathbb{E}\left[\sum_{k=2}^{K} T_k|\gamma_k(L)|^2\right]}\boldsymbol{s}_1^t\right]^t.$$

Thus, this receiver operates by despreading the received signal at each antenna using $\boldsymbol{s}_1$, and then performing a maximal ratio combining of the despread signals according to the average SIR at each antenna. We shall call this the *matched-filter receiver.* We observe that this is effectively the receiver implemented in the "softer handoff" mode of the IS-95 standard [13], where signals received in different sectors are combined. It is also the receiver considered in various works on CDMA with antenna arrays [15], [14], [4].

The optimal MMSE receiver had perfect knowledge of the signature sequences and fading levels of all users, including interferers, and such knowledge in practice must come from measurements. The matched filter, however, explicitly assumes the interferers' parameters are unknown, and this accounts for the expectations in the definition of the matched filter. Measurements will be required, of course, to determine the *statistics* of the fading levels of the interferers, which *are* assumed known. The matched filter should also be able to incorporate measurements of *realized* total interference levels $(\sum_{k=1}^{K} T_k|\gamma_k(l)|^2)_{l=1}^{L}$, if these are available. This can be accommodated in our definition if we interpret the expectations in (18) as *conditional* expectations, conditioned on the measurements. Any remaining randomness then comes from measurement error.

The matched-filter receiver is much simpler than the MMSE receiver considered in the previous subsections, but it entails a loss in capacity. Using effective bandwidths, we will quantify precisely this loss in performance. We will also find that there is a striking synergy between the results for the MMSE and for the matched-filter receiver.

We first evaluate the limiting SIR of user 1 under this matched-filter receiver. Although the receiver was derived assuming no knowledge of the sequences and channel gains of the other users, we shall prove a stronger result that the SIR under the matched filter, viewed as a function of the signature sequences, channel gains and transmit powers of all users as in definition (3), in fact converges to a quantity which depends only on the received powers of user 1. This result is analogous to Theorem 3, but holds in the repeated signature sequence model.

Theorem 7: Let $P_k(l) = T_k|\gamma_k(l)|^2$ be the received power of user k at antenna l. Assume that almost surely, the empirical joint distribution of

$$(P_1(1), \ldots, P_1(L)), (P_2(1), \ldots P_2(L)), \ldots, (P_K(1), \ldots, P_K(L))$$

converges weakly to some limiting joint distribution G as $K \to \infty$, and that the received powers $P_k(l)$ are uniformly bounded over all k and l. Then, if $N, K \to \infty$ with $K/N \to \alpha$ and L fixed, SIR_1 of the single-user matched filter converges in probability to $\sum_{l=1}^{L} P_1(l)a(l)$, where

$$a(l) = \frac{1}{\sigma^2 + \alpha\mathbb{E}[P(l)]}, \qquad l = 1, \ldots, L \tag{19}$$

and $(P(1), \ldots, P(L))$ are random variables having joint distribution G.

Proof: See Appendix-C. □

Note that $\alpha\mathbb{E}[P(l)]$ is the power of the limiting multiaccess interference at the output of the matched filter receiver for antenna l, and $P_1(l)a(l)$ the limiting SIR at the output of the same filter. Also, the matched-filter receiver asymptotically approaches

$$\left[\frac{\gamma_1^*(1)}{\sigma^2 + \alpha\mathbb{E}[P(1)]}\,\bar{\boldsymbol{s}}_1^t, \ldots, \frac{\gamma_1^*(L)}{\sigma^2 + \alpha\mathbb{E}[P(l)]}\,\bar{\boldsymbol{s}}_1^t\right]^t.$$

The proof of the theorem reveals that the overall SIR is the sum of the SIRs at the individual antennas because of the asymptotic uncorrelatedness of the multiaccess interference at the despread outputs at the different antenna, for almost all choice of signature sequences, channel gains, and transmit powers.

We have just given a straightforward interpretation for the $a(n)$'s in the matched-filter receiver, and it is tempting to give a similar interpretation to the $a(n)$'s for the MMSE receiver in Theorem 3; however, at the present time we have no proof of this. We will discuss this further in Section VIII.

In the special case of microdiversity, all the $a(l)$'s are the same, and in this case resource pooling occurs, and we can think of the array as a single antenna with the effective received power of a user at the single resource being the sum of the received powers of the user at the separate antennas. In this case, the notion of the effective interference of an interferer on a desired user can be defined.

Let us now consider the issue of user capacity and effective bandwidths. Again, consider a set of SIR targets for the users: $(\beta_k)_{k=1}^{\infty}$. Proceeding exactly as in the derivation of (11), but replacing (9) by (19), $\mathbb{E}[\frac{\beta}{1+\beta}]$ by $\mathbb{E}[\beta]$, and B_{mmse} by B_{mf}, where

$$B_{\mathrm{mf}} := \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \beta_k$$

we obtain the fixed-point equation

$$a(l) = \frac{1}{\sigma^2 + \alpha B_{\mathrm{mf}}\mathbb{E}\left[\frac{|\gamma(l)|^2}{\sum_{n=1}^{L}|\gamma(n)|^2 a(n)}\right]}, \qquad l = 1, \ldots, L. \tag{20}$$

The existence of a strictly positive solution $\boldsymbol{a}^*$ to (20) is again a necessary and sufficient condition for users being able to achieve their SIR targets. The necessity of the condition $\alpha B_{\mathrm{mf}} < L$ follows directly from

$$\alpha B_{\mathrm{mf}} = L - \sigma^2 \sum_{l=1}^{L} a(l) \tag{21}$$

which itself can be derived in the same way as (12). Sufficiency, and uniqueness, follow from Theorem 8 below, which provides a remarkable synergy with Theorem 6.

Theorem 8: Suppose Assumptions 1) and 2) from Section IV-B hold for the limiting empirical distributions of the SIR requirements $(\beta_k)_{k=1}^{\infty}$ and channel gains. Define

$$B_{\mathrm{mf}} := \lim_{k\to\infty} \frac{1}{K} \sum_{k=1}^{K} \beta_k.$$

Then we have the following.

1) If $\alpha B_{\mathrm{mf}} > L$ then there is no way to assign transmit powers $(T_k)_{k=1}^{\infty}$ in such a way that the users asymptotically achieve SIR targets β_k, $k = 1, 2, \ldots$.
2) If $\alpha B_{\mathrm{mf}} < L$ then one can assign transmit powers $(T_k)_{k=1}^{\infty}$ such that the users asymptotically achieve SIR targets β_k, $k = 1, 2, \ldots$. This power control is of the form

$$T_k = \beta_k \left(\sum_{n=1}^{L} |\gamma_k(n)|^2 a(n)\right)^{-1} \tag{22}$$

where $(a(1), \ldots, a(L))$ is the unique solution to the fixed-point equations (20).

Proof: Analogous to the proof of Theorem 6. □

As in the MMSE receiver case, the $a(l)$'s depend on the target SIRs only through a scalar B_{mf}. Moreover, the fixed-point equations (20) are identical to the corresponding equations (11) for the MMSE receiver, except for B_{mf} replacing B_{mmse}. This implies that in the case when there is a finite number of classes, if the power-limited capacity region under the MMSE receiver is given by

$$\left\{(\alpha_1, \ldots, \alpha_J)\colon \sum_{j=1}^{J} \alpha_j \frac{\beta_j}{1+\beta_j} < C\right\}$$

then the user capacity region under the matched filter for the same power constraints and outage probability is given by

$$\left\{(\alpha_1, \ldots, \alpha_J): \sum_{j=1}^{J} \alpha_j \beta_j < C\right\}$$

for the same constant C. Perhaps the synergy here relies on the fact that the matched filter is actually an MMSE receiver, albeit one based on less information. The fact that the effective bandwidths are larger for the matched-filter receiver is due to the lack of knowledge about the signature sequences of the interferers, and hence the inability to suppress them.

We remark that the effective bandwidth result in Theorem 8 was obtained earlier in [14], but for a slightly different model. In that paper, the underlying physical layer (complex fadings and signature sequences) were not explicitly modeled, although in the present version of the result, such complex fadings are needed to obtain the asymptotic independence of the interference at each antenna. In [14], independence was taken as an assumption, with a heuristic justification based on the chip asynchrony that results from realistic propagation delays, but no rigorous justification was provided. While that approach might be made rigorous, it is much stronger to prove the result in the chip-synchronous case, and make use of the random phases that arise from multipath fading; chip asynchrony can also be modeled, but we do not attempt this extension in the present paper.

B. The Decorrelator Receiver

The matched-filter receiver has no knowledge of the signature sequences of the interferers, but does make use of known statistics of the total interference levels of the interferers at each antenna. Let us consider the other way we could lose information at the receiver: assume the receiver knows the signature sequences of the interferers, but absolutely nothing about their interference levels. In a system where the signature sequence of an interferer is repeated on a symbol-by-symbol basis but the fading is fast, it may be easier to keep track of the sequences than the interference levels, so this is a plausible assumption to make. As for the matched-filter receiver, we assume perfect knowledge of the gains and transmit power of the user to be demodulated, to focus on the interference suppression capability of the receiver.

In the single-antenna case, an interferer lies in a single-dimensional subspace in $\mathbb{C}^N$, which is known to the receiver even when the receiver does not know the interferer's channel gain. The decorrelator [11] operates by projecting the received signal onto the subspace orthogonal to all interferers. In the case of L antennas, each interferer is only known to lie in an L-dimensional subspace when its signature sequence is known but the channel gains are not. For interferer k, this L-dimensional subspace is spanned by the vectors

$$\boldsymbol{v}_k(1) := [\boldsymbol{s}_k^H, 0, \ldots, 0]^H, \ldots \boldsymbol{v}_k(L) := [0, \ldots, 0, \boldsymbol{s}_k^H]^H.$$

A natural generalization of the decorrelator for user 1 is one which projects the received signal onto the subspace orthogonal to the vectors $\boldsymbol{v}_k(l)$, $k = 2, \ldots, K$, $l = 1, \ldots, L$, while maximizing the SIR of user 1. This receiver is given by the first row of the matrix $(\mathcal{V}^H\mathcal{V})^{-1}\mathcal{V}^H$ where

$$\mathcal{V} = \left[\sqrt{T_1}\bar{\boldsymbol{s}}_1, \boldsymbol{v}_2(1), \ldots, \boldsymbol{v}_K(1), \ldots, \boldsymbol{v}_2(L), \ldots, \boldsymbol{v}_K(L)\right]$$

and we have assumed $\mathcal{V}^H\mathcal{V}$ is invertible. The SIR for user 1 under the decorrelator is given by the expression

$$\text{SIR}_1 = \frac{1}{\sigma^2(\mathcal{V}^H\mathcal{V})^{-1}_{11}}. \tag{23}$$

Let

$$\mathcal{U} := [\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_K]$$
$$\mathcal{U}_1 := [\boldsymbol{s}_2, \ldots, \boldsymbol{s}_K]$$

and consider the eigendecomposition $\mathcal{U}_1 = O_1\Lambda_1 O_1^H$. Let $\mathcal{F}_1$ be the diagonal matrix such that the (i, i)th entry is 1 if $\Lambda_{ii} = 0$ and is 0 if $\Lambda_{ii} > 0$, and define $\mathcal{H}_1 := O_1\mathcal{F}_1 O_1^H$. Using [18, Lemma B.4] we have

$$\frac{1}{(\mathcal{U}^H\mathcal{U})^{-1}_{11}} = \boldsymbol{s}_1^H \mathcal{H}_1 \boldsymbol{s}_1$$

and

$$\frac{1}{(\mathcal{V}^H\mathcal{V})^{-1}_{11}} = T_1 \bar{\boldsymbol{s}}_1^H \operatorname{diag}(\mathcal{H}_1, \ldots, \mathcal{H}_1)\bar{\boldsymbol{s}}_1.$$

Substituting this in (23) yields

$$\text{SIR}_1 = \frac{T_1}{\sigma^2} \sum_{l=1}^{L} |\gamma_1(l)|^2 / (\mathcal{U}^H\mathcal{U})^{-1}_{11}.$$

The quantity

$$\frac{T_1|\gamma_1(l)|^2}{\sigma^2(\mathcal{U}^H\mathcal{U})^{-1}_{11}}$$

is the SIR achieved at output of the decorrelator for user 1 at antenna 1. Thus, the overall decorrelator can be re-interpreted as applying a decorrelator at each antenna and then performing a maximal-ratio combining of the individual antenna's outputs. The noise at the outputs is independent since it contains only the background noise and not the interference from any other users.

Applying [1, Theorem 7.2] now yields that as $K, N \to \infty$, $K/N \to \alpha < 1$

$$\frac{1}{(\mathcal{U}^H\mathcal{U})^{-1}_{11}} \xrightarrow{\mathcal{P}} 1 - \alpha.$$

Hence, it follows that under the decorrelator, for fixed L

$$\text{SIR}_1 \xrightarrow{\mathcal{P}} (1-\alpha)\frac{T_1}{\sigma^2} \sum_{l=1}^{L} |\gamma_1(l)|^2.$$

Hence, the system under the decorrelator is now equivalent to one with K users, N degrees of freedom, and received power of each user the coherent sum of the received powers at the individual antennas. Compared to the single-antenna setting, we see that having more antennas provides more diversity to the demodulated user through the pooling of received powers.

This can be viewed as a "single-user" benefit, which exists even when there are no interferers. On the other hand, there is no improvement in the multiuser interference suppression ability in the sense that the reduction in SIR is proportional to $1-\alpha$ for any L, and that no more than N interferers can be nulled out. This is because each interferer essentially occupies L degrees of freedom under the decorrelator. Thus, compared to the perfect knowledge MMSE receiver, there is pooling of received powers but no pooling of degrees of freedom for the decorrelator. Thus, we can conclude that under the decorrelator, the effective bandwidth of each user is L in a system with LN degrees of freedom. One can also compare the asymptotic efficiencies of the decorrelator with that of the MMSE receiver. The former is $1-\alpha$ and the latter is $1-\frac{\alpha}{L}$.

VI. EXAMPLES

In this section, we will present a few simple examples to give some insight into our effective bandwidth results, focusing on the MMSE receiver (Theorem 6) and the matched filter (Theorem 8). Both of these results boil down to solving the fixed-point equations

$$a(l)=\frac{1}{\sigma^2+\alpha B\mathbb{E}\left[\frac{|\gamma(l)|^2}{\sum_{n=1}^{L}|\gamma(n)|^2a(n)}\right]}, \qquad l=1,\ldots,L \tag{24}$$

where we use $B:=B_{\rm mmse}$ in the MMSE receiver case, and $B:=B_{\rm mf}$ in the matched-filter receiver case.

A. Capacity Gain of Macrodiversity Combining

Consider the two antenna scenario depicted in Fig. 1. Users are in one of two possible locations, and in each location, the magnitudes of the gains to the antennas are deterministic. For location 1, we set $|\gamma(1)|^2=1$, $|\gamma(2)|^2=f$, and for location 2, we set $|\gamma(1)|^2=f$, $|\gamma(2)|^2=1$. Here, typically we would assume that $f<1$.

Consider first the case where an equal number of users are in the two locations, so that there are $\frac{\alpha N}{2}$ users at each location. Note that the limiting empirical distribution H_2 for the magnitudes of the channel gains is given by

$$\mathbb{P}(|\gamma(1)|^2=1,\,|\gamma(2)|^2=f)=0.5$$
$$\mathbb{P}(|\gamma(1)|^2=f,\,|\gamma(2)|^2=1)=0.5.$$

Since this joint distribution is exchangeable, it follows that $a(1)=a(2):=a$, where

$$a=\frac{1}{\sigma^2}\left(1-\frac{1}{2}\alpha B\right) \tag{25}$$

so that the transmit power of user k with SIR requirement β_k is given by

$$T_k=\frac{\beta_k\sigma^2}{(1-\alpha B/2)(1+f)}.$$

Thus, as αB increases up to 2, a decreases down to zero and the transmit power required goes to infinity. As expected, αB cannot go beyond 2, the user capacity of the system.

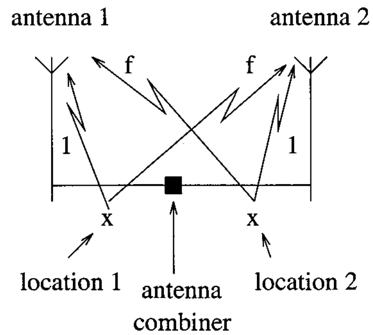

Fig. 1. Two-antenna macrodiversity.

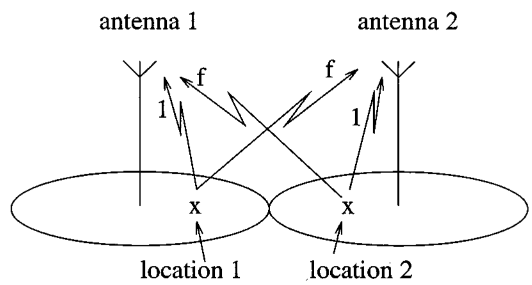

Fig. 2. Two cells without macrodiversity.

It is interesting to contrast this capacity result with the user capacity for the two-cell scenario depicted in Fig. 2 in which macrodiversity is not used. We do this first for the MMSE receiver, and then for the matched-filter receiver.

By symmetry, we can obtain the capacity of this system by focusing on cell 1 and using the single cell results of [1]. For simplicity, let us assume that all users have the same SIR target β, which implies that $B=\frac{\beta}{1+\beta}$. We can think of the user in cell 2, as being in cell 1 but with SIR target $f\beta$, since this is the SIR it will get there if it attains a target of β in its own cell. Therefore, from [1, Sec. 6], we obtain the required common received power of all users in cell 1 (excluding the users in cell 2)

$$P_{\rm mmse}=\frac{\beta\sigma^2}{1-\frac{1}{2}\,\alpha\frac{\beta}{1+\beta}\,-\frac{1}{2}\,\alpha\frac{f\beta}{1+f\beta}}. \tag{26}$$

It follows that without macrodiversity, the capacity constraint is

$$\alpha\left(B+\frac{f\beta}{1+f\beta}\right)<2.$$

We see that the loss in capacity is due to the interference in cell 1 created by users in cell 2, and vice versa. We note that even if f is very large, the other-cell interference effect is bounded, as we would expect since the MMSE receiver is near–far resistant. Nevertheless, the example shows that other-cell interference can still substantially reduce capacity, depending on the value of f.

Now let us consider the matched-filter receiver, which has $B=\beta$ and

$$P_{\rm mf}=\frac{\beta\sigma^2}{1-\frac{1}{2}\alpha\beta-\frac{1}{2}\alpha f\beta}. \tag{27}$$

In this case, the capacity without macrodiversity is given by

$$\alpha(1+f)B < 2$$

and the penalty term $\alpha f B$ is unbounded as f increases, which is a consequence of the fact that the matched filter is not near–far resistant.

It is interesting to note that the macrodiversity capacities in our examples (MMSE and matched filter) are independent of f. Intuitively, as cells get closer together (f increases) a user experiences interference from a larger number of users, but at the same time it gets more benefit from macrodiversity combining. We observe that these two effects cancel each other out exactly [14].

B. Nonuniform Traffic

To get a feel for nonuniform traffic, let us return to the macrodiversity model of Fig. 1, but consider the scenario in which all users are located in location 1. Then we have

$$a(1) = \frac{1}{\sigma^2 + \alpha B \frac{1}{a(1)+a(2)f}} \tag{28}$$

$$a(2) = \frac{1}{\sigma^2 + \alpha B \frac{f}{a(1)+a(2)f}}. \tag{29}$$

Set $x := \frac{1}{a(1)+a(2)f}$. Then adding the above equations we get

$$\frac{1}{x} = \frac{1}{\sigma^2 + \alpha B x} + \frac{f}{\sigma^2 + \alpha B f x}$$

which gives a quadratic equation in x

$$(2-\alpha B)\alpha B f x^2 - (1+f)\sigma^2(\alpha B - 1)x - \sigma^4 = 0.$$

If $f<1$ and $1<\alpha B<2$, or $f>1$ and $\alpha B<1$, then the positive solution x is given by

$$\sigma^2(\alpha B-1)\frac{1+f+(1-f)\left(1+\frac{4f}{(1-f)^2(\alpha B-1)^2}\right)^{1/2}}{(2-\alpha B)2\alpha B f}.$$

In the other cases, $f<1$ and $\alpha B<1$, or $f>1$ and $1<\alpha B<2$, the positive solution x is given by

$$\sigma^2(\alpha B-1)\frac{1+f-(1-f)\left(1+\frac{4f}{(1-f)^2(\alpha B-1)^2}\right)^{1/2}}{(2-\alpha B)2\alpha B f}.$$

In all cases, the condition for a positive solution is that $\alpha B < 2$. Thus, we have numerically verified the capacity constraint given in Theorem 6, for this scenario.

Note that we can substitute x back into (28) and (29) to obtain numerical values for $a(1)$ and $a(2)$. To gain some insight, let us focus on the case in which the two antennas are far apart, i.e., the case of small f. We set $f := \epsilon$, where ϵ is a parameter that we take small, and obtain an approximation for $a(1)$ and $a(2)$ up to $O(\epsilon)$ accuracy. We assume that $1 < \alpha B < 2$. Then its simple to derive that

$$\begin{aligned} a(1) &= \frac{2-\alpha B}{\alpha B - 1}\epsilon + o(\epsilon) \\ a(2) &= 2 - \alpha B + O(\epsilon) \end{aligned}$$

as $\epsilon \downarrow 0$. But for any user k, (14) gives us that

$$\begin{aligned} T_k &= \beta_k \left(\frac{2-\alpha B}{\alpha B - 1}\epsilon + \epsilon(2-\alpha B) + o(\epsilon)\right)^{-1} \\ &= O(\epsilon^{-1}). \end{aligned}$$

Thus, transmit powers increase dramatically as ϵ decreases. The fact that $a(1)$ is small means that users need to transmit with enough power to get received at antenna 2, but since this antenna is far away, transmit powers are very large.

C. Rayleigh Fading

A simple way to extend our example to incorporate Rayleigh fading is to specify any random variable $(\gamma_k(1), \gamma_k(2))$ associated with location 1 as being a pair of independent, zero-mean, complex Gaussian random variables, the first with variance 1, and the second with variance f. In this model, the mean of $|\gamma_k(l)|^2$ is a function of large-scale geographic effects, while its fluctuations are due to multipath fading.

Let us return to our example in which the proportion of users at each location are the same, but under the present Rayleigh fading model. In this case, the limiting empirical distribution of the channel gains can be described as first a random selection of location 1 or location 2 with equal probability, followed by a conditional selection of channel gains. If location 1 is selected then the gains are independent Rayleigh with parameters 1 and f, respectively, and if location 2 is selected the gains are independent Rayleigh with parameters f and 1, respectively. Thus, the limiting empirical distribution of the gains is exchangeable with respect to the antennas. It follows from (24) that $a(1)$ and $a(2)$ are the same, and that the common value a is as given in (25). We have, therefore, verified for this example that the capacity constraint is still $\alpha B < 2$. The transmit powers are given by

$$T_k = \frac{\beta_k \sigma^2}{(1-\alpha B/2)(|\gamma_k(1)|^2 + |\gamma_k(2)|^2)}$$

and these tend to infinity as capacity is approached.

To get a better picture of the impact of macrodiversity, we now present a *numerical* study of macrodiversity capacity, and contrast it with the single-cell case (without macrodiversity combining, and treating out-of-cell signals as interference), both for the MMSE receiver and for the matched-filter receiver. As before, we do the MMSE case first, followed by the matched filter.

In the MMSE case, the interference-limited macrodiversity capacity is $2\frac{1+\beta}{\beta}$ users per degree of freedom of spreading, which we note to be independent of f. This is plotted as curve "mmse-macro" in Fig. 3 for the case $\beta = 10$ dB. In the single-cell case, the capacity depends on f, and (26) for the received power in either cell becomes

$$P_{\text{mmse}} = \frac{\beta\sigma^2}{1 - \frac{1}{2}\frac{\beta}{1+\beta}\alpha - \frac{1}{2}\alpha\mathbb{E}\left[\frac{|\gamma(1)|^2\beta}{|\gamma(1)|^2\beta + |\gamma(2)|^2}\right]}$$

where the expectation is taken with respect to the fading from location 2. The expectation in the denominator

$$\mathbb{E}\left[\frac{|\gamma(1)|^2\beta}{|\gamma(1)|^2\beta + |\gamma(2)|^2}\right]$$

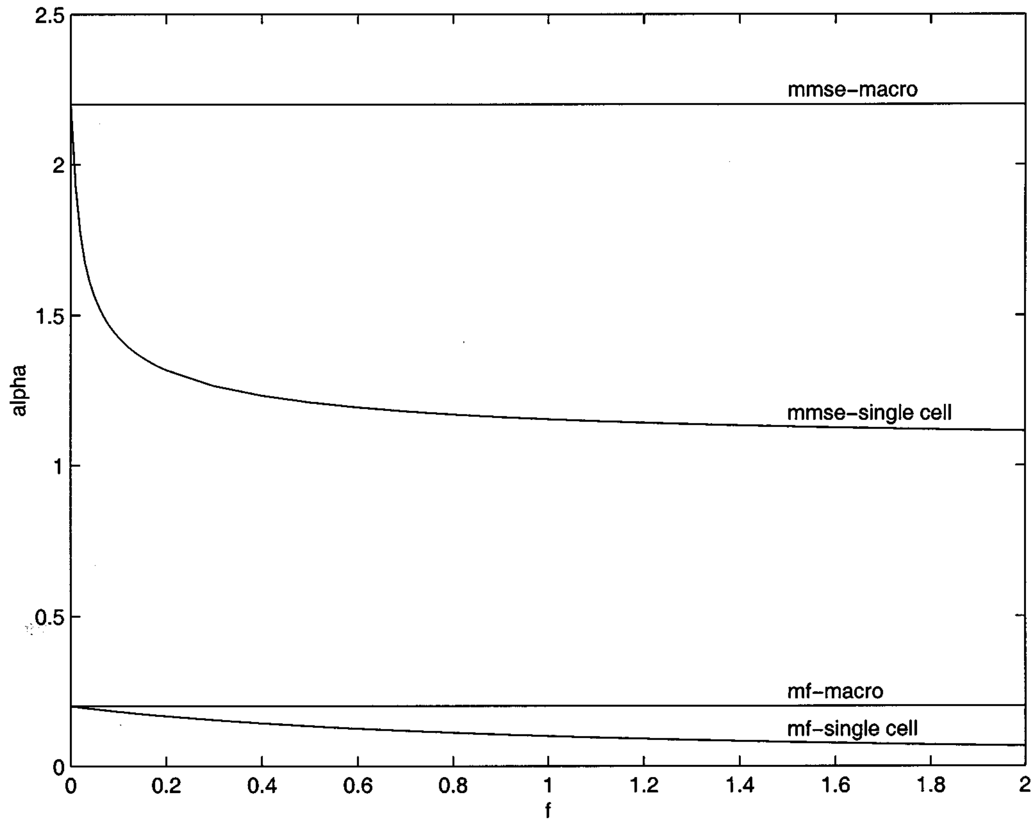

Fig. 3. User capacity per unit processing gain versus f, for two antennas and Rayleigh fading. SIR requirement $\beta = 10$ dB.

can be evaluated as the double integral

$$\frac{1}{f}\int_0^\infty \int_0^\infty \frac{r_1\beta}{r_1\beta + r_2} \exp(-r_2 - r_1/f)\, dr_1\, dr_2.$$

By evaluating this integral for different values of f, and for $\beta = 10$ dB, we obtain the "single-cell (mmse)" curve in Fig. 3. Note that the capacity does not decrease to zero, as the mean other-cell interference f tends to infinity, but instead approaches a nonzero limiting value. The value of this limit can easily be calculated numerically to be $22/21$. It can also be deduced from single-antenna results [1], by noting that for large f, each user in location 2 effectively takes up an effective bandwidth of 1 degree of freedom in cell 1. In terms of single-antenna theory [1], cell 1 consists of $(\alpha N)/2$ users per degree of freedom each taking an effective bandwidth of $\frac{\beta}{1+\beta} = 10/11$, and $(\alpha N)/2$ users each taking an effective bandwidth of 1. Thus, the constraint on α is $(21/22)\alpha < 1$.

In our example, we observe that the MMSE single-cell capacity drops away very quickly from the macrodiversity capacity as f increases. The gain from macrodiversity is due to the fact that "other-cell interference" does not exist with this receiver, whereas other-cell interference increases in the single-cell receiver case as f increases. In the single-cell case, it is interesting to see if the other-cell interference effect can be overcome by bandwidth partitioning. We note that the single-cell MMSE receiver can achieve a capacity of exactly 1.1 users per degree of spreading (i.e., 0.55 users per degree of freedom (d.f.) per cell) if we split the bandwidth between the cells, giving $N/2$ degrees of freedom to cell 1, and $N/2$ degrees of freedom to cell 2. This is *less* efficient than full frequency reuse, except for very large values of f. Full frequency reuse seems optimal for "sensible" values of f in our example (say, $f < 1$), but we observe that bandwidth partitioning is superior as $f \uparrow \infty$, where the limiting value was calculated above to be approximately 1.05 users per degree of spreading. This example, therefore, illustrates the fact that full frequency reuse is not inherently optimal for spread-spectrum systems with linear receivers (see [20] for more general information-theoretic results about this issue) but we remark that it is likely to provide a very good solution for real systems, or for models with more realistic fading parameters. This is especially so for the MMSE receiver, which has the ability to null out a strong interferer in an adjacent cell.

In the matched-filter case, the macrodiversity capacity is also independent of f, and is $2/\beta$ users per degree of spreading. This is depicted in Fig. 3, for the case $\beta = 10$ dB. In the single-cell case, the capacity depends on f, and (27) for the received power in either cell remains valid for our fading example. Using this equation, we plot the "mf-single cell" curve in Fig. 3, again for $\beta = 10$ dB. Note the matched filter, no-macrodiversity capacity goes to zero as f increases, precisely because of the lack of "near–far resistance" in the matched-filter receiver.

The present model of Rayleigh fading is still rather simplistic. For example, we have assumed that the mean strengths of all users are deterministic, and the users split into two groups, each with identical mean strengths to the two antennas. A more realistic example would allow each user to have distinct mean strengths, and a reasonable model would be to select the mean strengths from a log-normal distribution to model shadow fading effects. Another aspect of realistic fading in spread-spectrum systems is that it exhibits frequency selectivity i.e., multiple complex taps are required to model the channel of a user. We will not attempt to extend our theory, or our examples, to more realistic fading models in the present paper, but note that other results in this direction have been obtained in [18].

VII. A Deterministic Formulation

In the previous sections, we have focused on an asymptotic model where the signature sequences are randomly chosen and the number of users and the processing gain are both large. Moreover, the power control was allowed to be based on only the channel gains but not the specific signature sequences. It is usually assumed that it is not appropriate for power control in CDMA to be functionally dependent on the signature sequences of users. In multiuser detection, however, it is required for the receiver to know the structure in the interference, as determined by the signature sequences, and it may therefore be reasonable to allow power control at the transmitter to depend on the choice of signature sequences as well. This seems plausible, especially if signature sequences are repeated on a symbol-by-symbol basis.

In the present section, we consider a *finite-sized* system with *arbitrary* signature sequences, and look at the capacity under the MMSE receiver. We show that the asymptotic capacity results we derived in the preceding sections, actually hold in the finite case, provided we can control transmit powers as a function of the given signature sequences. Our results will also provide insight into the structure of the interference-limited capacity regions in the asymptotic regime in which signature sequences are chosen randomly.

Given a set of signature sequences $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_K$, channel gains $\gamma_k(l)$'s, and transmit powers T_k's, the SIR for user 1 is given by (5), with the matrix $\mathcal{S}_1\mathcal{D}_1$ defined there. We assume that user k has a required SIR of β_k.The *power control problem* is to find transmit powers $T_1, \ldots, T_k$ for the users in such a way that each user attains its required SIR. We find necessary and sufficient conditions for guaranteeing that the power control problem is feasible, and in this way obtain *capacity constraints* that depend on the signature sequences and SIR requirements of the users. However, we show that under a weak assumption on the signature sequences, the constraints reduce to a single constraint that only depends on the SIR requirements of the users.

The following theorem gives us the effective bandwidth characterization of the capacity region of the finite system under discussion. We remark that the result, and the argument we use to prove it, are both very similar to Theorem 6. The difference, however, is that in Theorem 6, either the system is feasible for all users, or it is infeasible for all users; if it is infeasible then all users' SIRs remain bounded away from their targets. In the present case, it is possible to have a subset of users with bounded transmit powers and with SIRs converging to their targets, but with the complementary set of users remaining infeasible, with SIRs bounded away from their targets. Thus, a subset of users is "feasible," whereas the remainder is "infeasible." This is possible precisely because the feasible users have enough degrees of freedom to "null out" the infeasible users. This observation accounts for the reason why we have to consider constraints on subsets of users, and not just the whole set of users, in the following theorem. In what follows, let SIR $(\boldsymbol{T})$ denote the vector of attained SIRs when the transmit power vector is $\boldsymbol{T} = (T_1, \ldots, T_K)^t$. For a subset of users $U \subseteq \{1, 2, \ldots, K\}$, let $\mathcal{S}(U)$ be the $LN \times |U|$ matrix consisting of the columns of $\mathcal{S}$ corresponding to users in U, and $\mathcal{D}(U)$ be the diagonal matrix of transmit powers of users in U. First we present the following key lemma, proven in [21].

Lemma 9: Let $\lambda_1, \ldots, \lambda_{LN}$ be the eigenvalues of the matrix $\mathcal{S}(U)\mathcal{D}(U)\mathcal{S}(U)^H$, and SIR_k^U be the SIR of user k under the MMSE receiver when only users in the subset U are present in the system. Then

$$\sum_{k \in U} \frac{\mathrm{SIR}_k^U}{1+\mathrm{SIR}_k^U} = \sum_{j=1}^{LN} \frac{\lambda_j}{\lambda_j + \sigma^2}. \tag{30}$$

Theorem 10: The constraints

$$\sum_{k \in U} \frac{\beta_k}{1+\beta_k} < \mathrm{rank}(\mathcal{S}(U)) \qquad \forall\, U \subseteq \{1, 2, \ldots, K\} \tag{31}$$

are necessary and sufficient for there to exist a positive transmit power vector $\boldsymbol{T}^*$ that solves the SIR equations

$$\mathrm{SIR}_k(\boldsymbol{T}^*) = \beta_k \qquad k = 1, 2, \ldots, K. \tag{32}$$

If a solution exists then it is unique.

Proof: See Appendix-D. □

Corollary 11: Assume that $K \geq LN$. Suppose that the matrix $\mathcal{S}$ has the property

$$\text{every subset of } LN \text{ columns are linearly independent.} \tag{33}$$

Then a necessary and sufficient condition for there to be a positive transmit power vector $\boldsymbol{T}^*$ that solves the SIR equations in (32) is that

$$\sum_{i=1}^{K} \frac{\beta_i}{1+\beta_i} < LN. \tag{34}$$

Proof: If U is a subset of users with $|U| < LN$ then $\mathrm{rank}\,(\mathcal{S}(U)) = |U|$, but $\forall\, i \in U$, $\frac{\beta_i}{1+\beta_i} < 1$. Therefore, (31) is satisfied. If $|U| \geq LN$ then $\mathrm{rank}\,(\mathcal{S}(U)) = LN$ and hence (31) is just $\sum_{i \in U} \frac{\beta_i}{1+\beta_i} < LN$. The tightest of all constraints of this form is

$$\sum_{i=1}^{K} \frac{\beta_i}{1+\beta_i} < LN. \qquad \square$$

Under what conditions on the original signature sequences $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_K$ does $\mathcal{S}$ have the property (33)? Let $\mathcal{T} = [\boldsymbol{s}_1, \ldots, \boldsymbol{s}_K]$ and for any subset of users U, $\mathcal{T}(U)$ be the $N \times |U|$ matrix the columns of which are the signature sequences of users in U (in increasing order, say).

Proposition 12: Assume $K \geq LN$. If every set of users of size LN can be partitioned into L disjoint subsets $U_1, \ldots, U_L$, each of size N, such that $\mathcal{T}(U_1), \ldots, \mathcal{T}(U_L)$ are all full-rank, then property (33) is satisfied almost surely in the (random) channel gains $\gamma_k(l)$.

Proof: Suppose first that $K = LN$. Then $\mathcal{S}$ is an LN by LN matrix. Let $\mathbb{P} = \{U_1, \ldots, U_L\}$ be any partition of the users into L disjoint subsets each of size N. By the Cauchy–Binet formula [22, p. 22]

$$\det \mathcal{S} = \sum_{\mathbb{P}} \left[\prod_{l=1}^{L} \det \mathcal{T}(U_l) \right] \left[\prod_{l=1}^{L} \prod_{k \in U_l} \gamma_k(l) \right]$$

where the sum is over all partitions $\mathbb{P}$. In this representation, $\det \mathcal{S}$ is a polynomial in the channel gains. By the assumed property of the signature sequences, there exist at least one nonzero coefficient in this polynomial and the polynomial is not identical to zero. Since the channel gains are circular symmetric (hence continuous phases) and the phases are independent, this implies that almost surely the determinant of $\mathcal{S}$ is nonzero and the rank property (33) holds. For the general case when $K > LN$, an identical argument holds for any subset of columns of size LN. □

It is worth noting that if the elements of the signature sequences are drawn randomly and independently from the same *continuous* distribution then the assumed condition in Proposition 12 will hold with probability 1. Moreover, even when the elements are discrete, say taking on values $+1$ or -1, it can be shown that the condition will hold almost surely in a large system as $N, K \to \infty, K/N \to \alpha$.

It is interesting to observe that the feasibility condition (34) is similar to the interference-limited capacity constraint for random sequences in a large system, as in Theorem 6. However, using random sequences, the power control to achieve the same capacity region is much simpler than for arbitrary sequences: the transmit power of a user can be chosen as a function of only its SIR requirement and its own channel gains. Moreover, the effective bandwidth characterization extends to power-limited capacity regions for random sequences, and this does not hold for arbitrary sequences. These results are consequences of deeper random matrix properties.

During the completion of this paper, a recent Ph.D. dissertation [23] has been brought to our attention, which investigates the issue of effective bandwidths for finite systems with CDMA and antenna arrays. In this work, one of our necessary conditions is derived, namely, the overall constraint in our equation (31). Simulation evidence is given that this constraint seems to describe capacity very well, and our Corollary 11 can be viewed as a substantiation of this claim.

VIII. Power Control: Macroscopic Versus Microscopic Scale

The proof of Theorem 10, given in Appendix-D, relies on an iterative procedure in which a solution $\boldsymbol{T}^*$ to (32) is shown to exist by constructing a sequence $\boldsymbol{T}^{(n)}$, $n = 1, 2, \ldots$. The basic idea of the proof is that given a set of transmit powers $\boldsymbol{T}^{(n)}$, this specifies an interference covariance matrix $\Sigma^{(n)} := \mathcal{S}\mathcal{D}^{(n)}\mathcal{S}^H$, and a corresponding set of eigenvalues $(\lambda_i^{(n)})_{i=1}^{LN}$. These in turn specify a new set of transmit powers $\boldsymbol{T}^{(n+1)}$, where the transmit power $T_i^{(n+1)}$ of user i is chosen to satisfy user i's SIR requirement assuming all the other user's transmit powers are fixed at $T_j^{(n)}$, $j \neq i$. This can be thought of as a power control algorithm, for potential practical implementation, and in the present section we wish to derive an explicit expression for $T_1^{(n+1)}$, the transmit power of user 1, at step $n+1$ of the algorithm.

The interference covariance matrix for user 1 is given by $\Sigma^{(1,n)} := \mathcal{S}_1\mathcal{D}_1^{(n)}\mathcal{S}_1^H$, which is the matrix in which the effect of user 1 has been removed. Let $(\lambda_j^{(1,n)})_{j=1}^{LN}$ be the eigenvalues of this matrix, and let $\boldsymbol{u}^{(1,n)}$ be the coordinates of the signature sequence of user 1 in the basis of eigenvectors. Then

$$T_1^{(n+1)} := \beta_k \left(\sum_{j=1}^{LN} \frac{u_j^{(1,n)}}{\lambda_j^{(1,n)} + \sigma^2} \right)^{-1} \tag{35}$$

is an explicit representation of the power control algorithm. We can interpret $\lambda_j^{(1,n)}$ as the level of interference experienced by user 1 in the direction j of the orthogonal basis of eigenvectors. Equation (35) specifies a power control algorithm, in which powers are selected as a function of interference measurements, and the user's own energies, in the different directions.

In the random sequence model, the proof of Theorem 6 can also be viewed as providing a power control algorithm to satisfy the SIR requirements as expressed in (10). The proof explicitly identifies a sequence $\boldsymbol{a}^{(n)}$ for which $\boldsymbol{a}^{(n)} \downarrow \boldsymbol{a}^*$, as $n \to \infty$, where $\boldsymbol{a}^*$ is the solution to (11). But if we set

$$T_1^{(n+1)} := \beta_1 \left(\sum_{l=1}^{L} |\gamma_1(l)|^2 a^{(n)}(l) \right)^{-1} \tag{36}$$

then this provides an analogous power control algorithm to (35). The analogy with (35) is even stronger if we define a new set of variables

$$q(l) := \frac{1}{a(l)} - \sigma^2, \qquad l = 1, 2, \ldots, L \tag{37}$$

after which (36) becomes

$$T_1^{(n+1)} := \beta_1 \left(\sum_{l=1}^{L} \frac{|\gamma_1(l)|^2}{q^{(n)}(l) + \sigma^2} \right)^{-1}. \tag{38}$$

This is a power control algorithm based on the "macroscopic" variables $(q_1, \ldots, q_L)$, as opposed to "microscopic" variables $(\lambda_j)_{j=1}^{LN}$.

The reduction in dimensions from LN variables to L variables is a consequence of the law of large numbers embodied in Theorem 3. This law of large numbers is very important from a practical point of view: in a large system, implementing an algorithm of the form (35) involves measuring a large number of "microscopic" variables, the λ_j's and u_j's, which are fluctuating on the fast time scale of burstiness and fading. Theorem 3 states that if the underlying probability distributions are fixed, then the limiting $a(n)$'s are constants, and in a finite, but large system, the fluctuations will be small. Moreover, Theorem 3 shows that the constants are the same for all users. A power control algorithm such as (36) can, therefore, be implemented, in which the number of variables to be measured and fed back to the users is much smaller, and which change much more slowly, as the underlying probability distributions change.

Both (35) and (36) fall into a general class of power control algorithms studied in [24] and also surveyed in [25]. That these algorithms converge, when a solution exists, is due to an underlying monotonicity inherent to these problems [26], [24], [14], and indeed central to the proofs in the present paper as well. In the present paper, we have also exploited fundamental conservation laws ((30) in the deterministic signature sequence model, and (12) in the asymptotic random sequences model) in order to show existence of solutions to these power control problems. See [25] for further discussions on conservation laws. Other relevant works on power control for multiuser CDMA receivers and for antenna arrays include [27] and [28].

The matched-filter results in the random sequences model are surprisingly similar to the MMSE results; all one does is replaces B_{mmse} by B_{mf}. Thus, a power control algorithm for the matched-filter receiver is also given by

$$T_k^{(n+1)} := \beta_k \left(\sum_{l=1}^{L} |\gamma_k(l)|^2 a^{(n)}(l) \right)^{-1}, \quad k = 1, 2, \ldots, K.$$

We remark that the convergence properties of this algorithm were obtained earlier in [14]. It is also interesting to observe that in the matched-filter case, the change of variables (37) has a direct interpretation: $q(l)$ is the interference level at the output of the matched-filter receiver at antenna l, and this follows from the proof of Theorem 7. In the MMSE case, we conjecture that the $q(l)$s have a similar interpretation as despread interference levels at the antennas, where we assume the global MMSE receiver is partitioned among the antennas to get intermediate outputs before combining.

IX. Conclusion

Our main results are the limiting SIR performance and the effective bandwidth characterization of capacity for a DS-CDMA system with random spreading, spatial diversity, and MMSE multiuser receiver. We also provide analogous results for the decorrelator and matched-filter receiver, of interest in their own right, but particularly to provide a contrast with the performance of the optimal MMSE receiver. The asymptotics are for large processing gain N and large number of users K, but fixed number of antennas L. We consider both microdiversity, in which the channel gains to the antenna elements are i.i.d. for each user, and macrodiversity, in which the channel gains are independent but not identically distributed.

In the case of microdiversity, a curious "resource pooling" effect is observed. We have shown that one can consider the microdiversity array as a single antenna with processing gain LN, i.e., a pooling of degrees of freedom provided by spreading and antennas. In the single-antenna system, the equivalent amount of received power is obtained by adding up the received power at each of the individual antenna elements in the multiantenna system. This resource pooling result is surprising in several ways. When one considers the pooling of degrees of freedom, one notes that the signature sequences are repeated at each antenna, so it is far from clear that one has LN "independent" dimensions, yet this is what our result shows. As to the pooling of received power, this may appear to be a fairly standard result for antenna arrays, where this effect is observed to occur for maximal ratio combining of antenna elements, but the underlying assumption there is of independent interference at each antenna element (see [29, Sec. 7.4]). However, in our spread-spectrum model, the interference at each antenna is *not* independent; in fact, it is precisely the dependence of interference over chips and antennas that the MMSE receiver exploits to gain its advantage. Therefore, it is quite surprising to see the same additivity of power that occurs in the independent interference case arising here as well. Moreover, our results show that not only is the received power pooled for the user to be demodulated, it is also pooled for the *interferers*. This certainly goes beyond the standard maximal ratio combining result.

Generalizing the results in [1], we provided an effective bandwidth characterization of the user capacity region of a system with antenna diversity. As a function of the target SIR β, the effective bandwidths for each user under the MMSE, decorrelator, and matched filter are

$$e_{\mathrm{mmse}}(\beta) = \frac{\beta}{1+\beta} \qquad e_{dec}(\beta) = L \qquad e_{\mathrm{mf}}(\beta) = \beta$$

with the interference-limited capacity region asymptotically given by

$$\sum_{k=1}^{K} e(\beta) < LN.$$

Thus, in a system with processing gain N, and L antennas, the overall number of degrees of freedom is LN. Analogous results are obtained for power-limited user capacity regions where users have transmit power constraints.

We observe that the matched filter gains capacity whenever we increase L, but it is still susceptible to the *near–far* problem; the fact that the signature sequences of the interferers are not used means that the receiver cannot null them out. Thus, the N degrees of freedom from spreading are not used to their full potential. The decorrelator does not gain user capacity when we increase L, because the effective bandwidth occupied by each user increases linearly with L. Thus, the L degrees of freedom provided by the antennas are not used to their full potential. However, it does use the N degrees of freedom provided by spreading effectively; since it knows the signature sequences of the interferers it can null out their interference.

Somewhat surprisingly, the interference-limited capacity regions do not depend on the statistics of the channel gains, and are the same in both the macrodiversity and microdiversity models. For the MMSE receiver, we gave an explanation of the interference-limited capacity region by means of an underlying deterministic conservation law governing the tradeoff between the mean-square errors of the different users.

Our results are obtained in a synchronous CDMA model with flat fading. Extensions to asynchronous systems with frequency-selective fading can perhaps be obtained by combining the ideas of this paper with those in [30] and [18], which address similar questions in the asynchronous and multipath settings, respectively, for a single-antenna CDMA system.

Appendix

A. Proof of Theorem 3

We need the following result, [31, Corollary 10.1.2].

Theorem 13: Let $\mathcal{A}$ be a $\lfloor cn \rfloor \times \lfloor dn \rfloor$ random matrix with independent entries which are zero-mean and satisfy the condition

$$n \operatorname{Var}(\mathcal{A}_{ij}) < B$$

for some uniform bound $B < \infty$. Moreover, suppose that if we define for each n a function v_n: $[0, c] \times [0, d] \to \Re$ by

$$v_n(x, y) = n\operatorname{Var}(\mathcal{A}_{ij}), \text{ for } i, j \text{ satisfying:} \\ \frac{i}{n} \le x \le \frac{(i+1)}{n} \qquad \frac{j}{n} \le y \le \frac{(j+1)}{n}$$

and that v_n converges uniformly to a limiting bounded function v.[3] Then for each $a, b \in [0, c]$, $a < b$, and $z \in \mathbb{C}^+$

$$\frac{1}{n} \sum_{i=\lfloor an \rfloor}^{\lfloor bn \rfloor} (\mathcal{A}\mathcal{A}^H - zI)_{ii}^{-1} \xrightarrow{\mathcal{P}} \int_a^b u(x, z)\, dx$$

where $u(x, z)$ satisfies the equation

$$u(x, z) = \frac{1}{-z + \int_0^d \frac{v(x, y)\, dy}{1 + \int_0^c u(w, z) v(w, y)\, dw}} \tag{39}$$

for every $x \in [0, c]$. The solution of (39) exists and is unique in the class of functions $u(x, z) \ge 0$, analytic on $z \in \mathbb{C}^+$ and continuous on $x \in [0, c]$.

Moreover, almost surely, the empirical eigenvalue distribution of $\mathcal{A}\mathcal{A}^H$ converges weakly to a limiting distribution G^*, whose Stieltjes transform

$$m(z) := \int_0^\infty 1/(\lambda - z)\, dG^*(\lambda)$$

is given by $\int_0^1 u(x, z)\, dx$.

The following lemma is a modification of a corresponding lemma of [32] and the proof proceeds along similar lines.

Lemma 14: Suppose $\nu_1, \ldots, \nu_{NL}$ are uncorrelated random variables, each zero mean and variance $1/N$ with finite fourth moment. Let $\mathcal{B}$ be an LN by LN constant Hermitian matrix. Define the vector

$$\boldsymbol{s} := \left[d_1\nu_1, \ldots, d_1\nu_N, d_2\nu_{N+1}, \ldots, d_L\nu_{N(L-1)+1}, \ldots, d_L\nu_{LN} \right]^t$$

where $d_1, \ldots, d_L \in \mathbb{C}$. Then

$$\mathbb{E}\boldsymbol{s}^H B\boldsymbol{s} = \frac{1}{N} \sum_{l=1}^{L} |d_l|^2 \left(\sum_{i=(l-1)N+1}^{lN} \mathcal{B}_{ii} \right).$$

If furthermore the ν_i's and i.i.d., then

$$\operatorname{Var}[\boldsymbol{s}^H \mathcal{B}\boldsymbol{s}] \le \frac{C_1}{N} \lambda_{\max}^2(\mathcal{B})$$

for some constant C_1 which depends only on the fourth moment of ν_1 and the d_l's. Here, $\lambda_{\max}(\mathcal{B})$ is the largest eigenvalue of $\mathcal{B}$.

[3]We note that here the x-axis is identified with the rows of the matrix, and the y-axis to the columns.

Proof of Theorem 3: In the completely random model, the SIR for user 1 is given by

$$\mathrm{SIR}_1 = T_1 r_1^H (\mathcal{R}_1 \mathcal{D}_1 \mathcal{R}_1^H + \sigma^2 I)^{-1} r_1$$

where T_k is the transmit power of user k

$$\boldsymbol{r}_i = \left[\gamma_i(1)\nu_{1,i}, \ldots, \gamma_i(1)\nu_N, \gamma_i(2)\nu_{N+1,i}, \ldots, \gamma_i(L)\nu_{N(L-1)+1,i}, \ldots, \gamma_i(L)\nu_{LN,i} \right]^t$$

$\mathcal{R}_1 = [\boldsymbol{r}_2, \ldots, \boldsymbol{r}_K]$, and $\mathcal{D}_1 = \operatorname{diag}(T_2, \ldots, T_K)$. Here, ν_{ij} are all i.i.d. random variables with mean zero and variance $1/N$. Recall that $P_k(m) = T_k|\gamma_k(m)|^2$. For simplicity, assume that the power vectors $(P_k(1), \ldots, P_k(L))$ take on only a finite set of J possible values, say $(p_{1j}, \ldots, p_{Lj})$, $j = 1, \ldots, J$, and let the limiting empirical distribution of $(P_k(1), \ldots, P_k(L))$, $k = 1, \ldots, K$ be equal to $(p_{1j}, \ldots, p_{Lj})$ with probability q_j. With this assumption, we can identify each user with the class j its vector of received powers belongs to. The same result can be proved for the continuous case using an approximation argument. (See [30, proof of Theorem 4.1] for a similar argument.)

Consider now the random matrix $\mathcal{A} := \mathcal{R}_1 \mathcal{D}_1^{1/2}$. Conditional on the P_k's and $\gamma_k(l)$'s, the entries of $\mathcal{A}$ are independent. Each column of $\mathcal{A}$ is associated with a user. By reordering the columns so that the users are in increasing order of the their class indexes j, the function v_n defined in the statement of Theorem 13 for $\mathcal{A}$ can be seen to converge to v almost surely as $N, K \to \infty$, $K/N \to \alpha$, where

$$v(x, y) = p_{lj}, \\ \text{if } \alpha \sum_{i<j} q_i \le y < \alpha \sum_{i \le j} q_i \text{ and } l - 1 \le x < l. \tag{40}$$

Then, for each $a, b \in [0, L]$, $a < b$

$$\frac{1}{n} \sum_{i=\lfloor an \rfloor}^{\lfloor bn \rfloor} (\mathcal{A}\mathcal{A}^H + \sigma^2 I)_{ii}^{-1} \xrightarrow{\mathcal{P}} \int_a^b u(x)\, dx \tag{41}$$

where u satisfies the equation

$$u(x) = \frac{1}{\sigma^2 + \int_0^\alpha \frac{v(x, y)\, dy}{1 + \int_0^L u(w) v(w, y)\, dw}} \tag{42}$$

for all $x \in [0, L]$. It can be seen that since v is constant as a function of x in intervals $[l - 1, l)$, $l = 1, \ldots L$, u is also constant in those intervals. Let $u(x) = a(l)$ for $x \in [l - 1, l)$. Using (40), (42) is now reduced to the following system of L simultaneous equations:

$$a(l) = \frac{1}{\sigma^2 + \mathbb{E}\left[\frac{P(l)}{1 + \sum_{m=1}^{L} a(m) P(m)} \right]}$$

where $(P(1), \ldots, P(L)) = (p_{1j}, \ldots, p_{Lj})$ with probability q_j, $j = 1, \ldots, J$. Applying Lemma 14 and (41) now yields the theorem, since $\lambda_{\max}((\mathcal{A}\mathcal{A}^H + \sigma^2 I)^{-1}) \le 1/\sigma^2$.

B. Proof of Theorem 1

The focus of the analysis is on the quantity

$$\mathrm{SIR}_1 = T_1 \bar{\boldsymbol{s}}_1^H (\mathcal{S}_1 \mathcal{D}_1 \mathcal{S}_1^H + \sigma^2 I)^{-1} \bar{\boldsymbol{s}}_1 \tag{43}$$

where the various variables are defined in Section II.

The asymptotic behavior of the SIR is quite complicated to analyze due to the dependence of the elements of the super signature sequences of the users. As a result, we cannot directly use existing random matrix results as in the proof of Theorem 3. Essentially, what we want to show is that the asymptotic behavior of SIR_1 is the same as if the super signature sequences were generated in the completely random sequence model. There are two main parts to the proof.

1) We show that the limiting spectrum of $\mathcal{S}\mathcal{D}\mathcal{S}^H$ is the same as that of $\mathcal{R}\mathcal{D}\mathcal{R}^H$, where $\mathcal{R}$ is the analogous matrix in the completely random sequence model. This is done by bootstraping from the special case where the received powers at all the antennas from all the users are the same, a result already proved in [18] using techniques from free probability theory. The more general result follows from a combinatorial argument.
2) We show that $T_1\bar{\boldsymbol{s}}_1^H(\mathcal{S}_1\mathcal{D}_1\mathcal{S}_1^H+\sigma^2 I)^{-1}\bar{\boldsymbol{s}}_1$ is asymptotically
$$T_1\sum_{m=1}^{L}|\gamma_1(m)|^2\cdot\frac{1}{N}\,\mathbf{Tr}\,(\mathcal{S}_1\mathcal{D}_1\mathcal{S}_1+\sigma^2 I)^{-1}.$$
This would have followed from a simple mean and variance calculation if $\bar{\boldsymbol{s}}_1$ were generated from the completely random sequence model (as in Lemma 14). However, the dependence in entries of $\bar{\boldsymbol{s}}_1$ makes the proof much more complicated. The analysis of this step is done in the main body of the proof.

First Step:

Lemma 15: Assume that $T_k|\gamma_k(l)|^2=1$ for all k and l. Then the moments of the expected empirical eigenvalue distributions of $\mathcal{S}\mathcal{S}^H$ and $\mathcal{R}\mathcal{R}^H$ both converge to the same limits as $N,\,K\to\infty$ and $K/N\to\alpha$.

We now want to generalize the above result to the case of arbitrary but bounded $T_k|\gamma_k(l)|^2$'s. We need another random matrix result.

Let $\mathcal{Z}$ be an n by $\lfloor cn\rfloor$ random matrix with independent zero-mean entries and such that $\mathbb{E}|\mathcal{Z}_{ij}|^{2+\delta}<B$ for some $\delta>0$ and $B>0$, and let G_n be the empirical eigenvalue distribution of $1/n\mathcal{Z}\mathcal{Z}^H$. The rth moment of G_n, $\int x^r\,dG_n(x)$, is
$$\frac{1}{n}\,\mathbf{Tr}\left[\left(\frac{1}{n}\,\mathcal{Z}\mathcal{Z}^H\right)^r\right] = \frac{1}{n^{r+1}}\sum \mathcal{Z}_{s_1s_2}\mathcal{Z}^*_{s_3s_2}\cdots\mathcal{Z}_{s_{2r-1}s_{2r}}\mathcal{Z}^*_{s_1s_{2r}}$$
and the summation is over all indexes $s_1,\,\ldots,\,s_{2r}$, with the odd indexes ranging from 1 to n and the even indexes from 1 to $\lfloor cn\rfloor$. Consider now the following sum of a subset of the terms:
$$H(n,\,r):=\frac{1}{n^{r+1}}\sum_{\mathcal{L}}\mathcal{Z}_{s_1s_2}\mathcal{Z}^*_{s_3s_2}\cdots\mathcal{Z}_{s_{2r-1}s_{2r}}\mathcal{Z}^*_{s_1s_{2r}} \tag{44}$$
where each summand in $\mathcal{L}$ satisfies the further constraints that the number of distinct odd-indexed s_{2i-1} plus the number of distinct even-indexed s_{2i} equals $r+1$ and that there is a one-to-one pairing of the unconjugated and conjugated terms in the product. Hence, $H(n,\,r)$ can also be written as
$$H(n,\,r)=\sum|\mathcal{Z}_{t_1t_2}|^2\cdots|\mathcal{Z}_{t_{2r-1}t_{2r}}|^2.$$
It should also be noted that there is no further repeat of terms in each of the products, because of the first constraint. The following result is due to Wachter [33, Theorem 1.1.4].

Theorem 16: Almost surely the empirical distribution G_n converges to a limit G^* if and only if for each k, $\mathbb{E}[H(n,\,r)]$ converges as $n\to\infty$. Moreover, if the limit G^* exists , then
$$\int x^r\,dG^*(x)=\lim_{n\to\infty}\mathbb{E}\left[\int x^r\,dG_n(x)\right] = \lim_{n\to\infty}\mathbb{E}[H(n,\,r)].$$

Theorem 17: Assume there exists a B such that $P_k(l)<B$ for all k and l. Then the moments of the expected empirical eigenvalue distributions of $\mathcal{S}\mathcal{D}\mathcal{S}^H$ and $\mathcal{R}\mathcal{D}\mathcal{R}^H$ both converge to the same limits as $N,\,K\to\infty$ and $K/N\to\alpha$. Here $\mathcal{D}=\mathrm{diag}\,(T_1,\,\ldots,\,T_k)$.

Proof: First we condition on the $P_k(l)$'s. Define
$$\mathcal{Z}=\sqrt{N}\mathcal{R}\mathcal{D}^{1/2}$$
and let $n=LN$. Let G_n be the empirical eigenvalue distribution of RDR^H. We have for each r
$$\begin{aligned}\int x^r\,dG_n(x)&=\frac{1}{n}\,\mathbf{Tr}\left[\left(\frac{1}{n}\,\mathcal{Z}\mathcal{Z}^H\right)^r\right]\\&=\frac{1}{n^{r+1}}\sum\mathcal{Z}_{s_1s_2}\mathcal{Z}^*_{s_3s_2}\cdots\mathcal{Z}_{s_{2r-1}s_{2r}}\mathcal{Z}^*_{s_1s_{2r}}\\&=H(n,\,r)+G(n,\,r)\end{aligned}$$
where $H(n,\,r)$ is as defined in (44) and $G(n,\,r)$ consists of the remaining terms in the sum. Let $H_0(n,\,r)$ and $G_0(n,\,r)$ be the corresponding term when $P_k(l)=1$ for all $l,\,k$ in the repeated sequence model; $H^c(n,\,r)$ and $G^c(n,\,r)$ be the corresponding terms in the completely random sequence model, for general $P_k(l)$'s; and $H_0^c(n,\,r)$ and $G_0^c(n,\,r)$ be the corresponding terms in the completely random model with $P_k(l)=1$ for all $l,\,k$. We observe that

1) $\mathbb{E}[H(n,\,r)]=\mathbb{E}[H^c(n,\,r)], \mathbb{E}[H_0(n,\,r)]=\mathbb{E}[H_0^c(n,\,r)]$;
2) $\lim_{n\to\infty}\mathbb{E}[G_0^c(n,\,r)]=0$;
3) $\lim_{n\to\infty}\mathbb{E}[G_0(n,\,r)]=0$;
4) $\lim_{n\to\infty}\mathbb{E}[G^c(n,\,r)]=\lim_{n\to\infty}\mathbb{E}[G(n,\,r)]=0$.

The first statement follows from the fact that $\mathbb{E}[H(n,\,r)]$ and $\mathbb{E}[H^c(n,\,r)]$ depend only on the variances of the individual entries of $\mathcal{S}\mathcal{D}\mathcal{S}^H$ and $\mathcal{R}\mathcal{D}\mathcal{R}^H$, respectively, which are identical. The second fact follows from Theorem 16 applied to the completely random model with $P_k(l)=1$. The third fact follows from Lemma 15 that the completely random and repeated sequence models have the same limiting moments. The fourth statement follows from the fact that there is a one-to-one correspondence of the summands in $G^c(n,\,r)$ and $G_0^c(n,\,r)$ with nonzero expectation, each of which are positive and that the expectation of each such term in $G^c(n,\,r)$ can be at most a factor of B^r larger than that of the corresponding term in $G^c(n,\,r)$. Analogous statement holds for $G(n,\,r)$ and $G_0(n,\,r)$.

Now, in the completely random sequence model, the empirical eigenvalue distribution converges almosst surely, from Theorem 3. Therefore, by Theorem 16, for each r,

$\lim_{n\to\infty} E[H^c(n, r)]$ exists and is the limit of the rth moment of the expected empirical eigenvalue distribution of $\mathcal{SDS}^H$. Applying facts (1) and (4) to (45) allows us to conclude that the corresponding moments for the repeated sequence model converge to those same limits. Moreover, this limit is the same almost surely in the received powers $P_k(l)$'s. An application of the Dominated Convergence Theorem allows us to conclude that the same conclusions holds for expectations over $P_k(l)$'s as well. □

Second Step: We need one more lemma.

Lemma 18: Let $\mathcal{A}$ be a deterministic $N \times N$ complex matrix with uniformly bounded spectral radius for all N. Let $\boldsymbol{q} = \frac{1}{\sqrt{N}}[q_1, \ldots, q_N]^H$, where q_i's are i.i.d. zero-mean unit variance random variables with finite eighth moment, Let $\boldsymbol{r}$ be identically distributed and independent of q. Then

$$\mathbb{E}[|\boldsymbol{q}^H \mathcal{A} \boldsymbol{r}|^4] \leq \frac{B}{N^2}$$

for some constant B that does not depend on $\mathcal{A}$ or N.

Proof: The proof proceeds along the same lines as the proof of [34, Lemma 2.7]. □

We can now give a proof of Theorem 1.

Proof of Theorem 1: Define

$$\begin{aligned}
\mathcal{S}_i &:= [\overline{\boldsymbol{s}}_1, \ldots, \overline{\boldsymbol{s}}_{i-1}, \overline{\boldsymbol{s}}_{i+1}, \ldots, \overline{\boldsymbol{s}}_K] \\
\mathcal{D}_i &:= \operatorname{diag}(T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_K) \\
\mathcal{S} &:= [\overline{\boldsymbol{s}}_1, \ldots, \overline{\boldsymbol{s}}_K] \\
\mathcal{D} &:= \operatorname{diag}(T_1, \ldots, T_K)
\end{aligned}$$

where $\overline{\boldsymbol{s}}_i$ is defined in Section II-A, and let

$$\beta_i^N := T_i \overline{\boldsymbol{s}}_i^H (\mathcal{S}_i \mathcal{D}_i \mathcal{S}_i^H + \sigma^2 I)^{-1} \overline{\boldsymbol{s}}_i \tag{45}$$

which is the SIR achieved by user i under the MMSE receiver. In [21, eq. (12)], a key equation relating the β_i^N's and the trace of $(\mathcal{SDS}^H + \sigma^2 I)^{-1}$ was derived

$$\frac{1}{LN} \sum_{i=1}^{K} \frac{\beta_i^N}{1+\beta_i^N} = 1 - \frac{\sigma^2}{LN} \mathbf{Tr}(\mathcal{SDS}^H + \sigma^2 I)^{-1}.$$

Rearranging terms and then taking expectation with respect to the random sequences, channel gains, and transmit powers, we get

$$\frac{1}{LN} \mathbb{E}\left[\sum_{i=1}^{K} \frac{1}{1+\beta_i^N}\right] = \frac{K}{LN} - 1 + \mathbb{E}\left[\frac{\sigma^2}{LN} \sum_{i=1}^{LN} \frac{1}{\lambda_i^N + \sigma^2}\right] \tag{46}$$

where $\lambda_1^N, \ldots, \lambda_{LN}^N$ are the eigenvalues of $\mathcal{SDS}^H$. Let us investigate what happens as $N \to \infty$. Applying Lemma 15, we have

$$\mathbb{E}\left[\frac{1}{LN} \sum_{i=1}^{LN} \frac{1}{\lambda_i^N + \sigma^2}\right] = \int_0^\infty \frac{1}{\lambda + \sigma^2} \, dF_{\mathcal{SDS}^H}(\lambda)$$

and

$$\int_0^\infty \frac{1}{\lambda + \sigma^2} \, dF_{\mathcal{SDS}^H}(\lambda) \to \int_0^\infty \frac{1}{\lambda + \sigma^2} \, dG^*(\lambda) \tag{47}$$

as $N \to \infty$, where G^* is the limiting expected empirical eigenvalue distribution of $\mathcal{SDS}^H$. This convergence holds since $f(\lambda) = 1/(\lambda + \sigma^2)$ is a bounded continuous function and $F_{\mathcal{SDS}^H}$ converges weakly to G^*. Let us define

$$\beta^* := \int_0^\infty \frac{1}{\lambda + \sigma^2} \, dG^*(\lambda).$$

We now analyze what happens asymptotically to the left-hand side of (46). Recalling that $P_i = T_i \sum_{l=1}^{L} |\gamma_i(l)|^2$, we first observe that

$$\mathbb{E}\left[\frac{\beta_i^N}{P_i}\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{\beta_i^N}{P_i} \,|T_i, |\gamma_i(1)|, \ldots, |\gamma_i(L)|\right]\right]$$

which, by Lemma 14, implies that

$$\mathbb{E}\left[\frac{\beta_i^N}{P_i}\right] = \frac{1}{N} \mathbb{E}\left[\frac{1}{\sum_{l=1}^{L} |\gamma_i(l)|^2} \sum_{l=1}^{L} |\gamma_i(l)|^2 \cdot \left(\sum_{j=(l-1)N+1}^{lN} (\mathcal{S}_i \mathcal{D}_i \mathcal{S}_i^H + \sigma^2 I)_{jj}^{-1}\right)\right] \tag{48}$$

since the circular symmetry and independence of the γ_i's imply that the entries of $\overline{\boldsymbol{s}}_i$ are uncorrelated.

In the microdiversity model, the $\gamma_k(l)$'s are identically distributed and independent for each fixed k. By permuting the rows of $(\mathcal{S}_i \mathcal{D}_i \mathcal{S}_i^H + \sigma^2 I)^{-1}$, we see that the random variables

$$U_l := \sum_{j=(l-1)N+1}^{lN} (\mathcal{S}_i \mathcal{D}_i \mathcal{S}_i^H + \sigma^2 I)_{jj}^{-1}, \qquad l = 1, \ldots, L \tag{49}$$

are identically distributed. So we can rewrite (48) as

$$\mathbb{E}\left[\frac{\beta_i^N}{P_i}\right] = \frac{1}{LN} \mathbb{E}\left[\mathbf{Tr}(\mathcal{S}_i \mathcal{D}_i \mathcal{S}_i^H + \sigma^2 I)^{-1}\right] \to \beta^* \tag{50}$$

as $N \to \infty$. Our goal now is to show from (46) that β_i/P_i in fact converges to β^* in probability for every i. To this end, let

$$\frac{\beta_i^N}{P_i} = \beta^* + \Delta_i^N.$$

Then, expanding about β^*

$$\frac{1}{1+\beta_i^N} = \frac{1}{1+P_i\beta^*} - \frac{P_i}{(1+P_i\beta^*)^2} \Delta_i^N + \frac{P_i^2}{(1+P_i\xi_i^N)^3} (\Delta_i^N)^2$$

for some ξ_i^N in between β_i^N/P_i and β^*. Substituting into the left-hand side of (46), we get

$$\begin{aligned}
\frac{1}{LN} \mathbb{E}\left[\sum_{i=1}^{K} \frac{1}{1+\beta_i^N}\right] = &\, \mathbb{E}\left[\frac{1}{LN} \sum_{i=1}^{K} \frac{1}{1+P_i\beta^*}\right] \\
&- \mathbb{E}\left[\frac{1}{LN} \sum_{i=1}^{K} \frac{P_i}{(1+P_i\beta^*)^2} \Delta_i^N\right] \\
&+ \mathbb{E}\left[\frac{1}{LN} \sum_{i=1}^{K} \frac{P_i^2}{(1+P_i\xi_i^N)^3} (\Delta_i^N)^2\right].
\end{aligned} \tag{51}$$

As $N,\, K \to \infty$ and $K/N \to \alpha$, the first term approaches

$$\mathbb{E}\left[\frac{1}{LN}\sum_{i=1}^{K}\frac{1}{1+P_i\beta^*}\right] \to \frac{\alpha}{L}\,\mathbb{E}\left[\frac{1}{1+P\beta^*}\right] \tag{52}$$

where the random variable P has the limiting empirical distribution F of the P_i's. This follows from the almost sure convergence of the empirical distribution together with an application of the Dominated Convergence Theorem.

Consider now the second expectation in (51). From (50)

$$\mathbb{E}[\Delta_i^N] = \frac{1}{LN}\,\mathbb{E}[\mathbf{Tr}\,(\mathcal{S}_i\mathcal{D}_i\mathcal{S}_i^H + \sigma^2 I)^{-1}] - \beta^*.$$

If we let

$$\mathcal{C}_i := (\mathcal{S}_i\mathcal{D}_i\mathcal{S}_i^H + \sigma^2 I)^{-1}$$

and

$$\mathcal{C} := (\mathcal{S}\mathcal{D}\mathcal{S}^H + \sigma^2 I)^{-1}$$

application of the matrix inversion lemma yields

$$\mathcal{C}_i = \mathcal{C} + \frac{T_i\mathcal{C}_i\bar{\boldsymbol{s}}_i s_i^H\mathcal{C}_i}{1+T_i\bar{\boldsymbol{s}}_i^H\mathcal{C}_i s_i}$$

so

$$\mathbb{E}[\Delta_i^N] = \frac{1}{LN}\,\mathbb{E}[\mathbf{Tr}\,\mathcal{C}] + \frac{1}{LN}\,\mathbb{E}\left[\frac{T_i\bar{\boldsymbol{s}}_i^H\mathcal{C}_i\bar{\boldsymbol{s}}_i}{1+T_i\bar{\boldsymbol{s}}_i^H\mathcal{C}_i\bar{\boldsymbol{s}}_i}\right] - \beta^*.$$

Note that

$$0 < \frac{1}{LN}\,\mathbb{E}\left[\frac{T_i\bar{\boldsymbol{s}}_i^H\mathcal{C}_i^2\bar{\boldsymbol{s}}_i}{1+T_i\bar{\boldsymbol{s}}_i^H\mathcal{C}_i\bar{\boldsymbol{s}}_i}\right] < \frac{1}{\sigma^4 LN}\,\mathbb{E}[T_i\|\bar{\boldsymbol{s}}_i\|^2]$$

and

$$\frac{1}{\sigma^4 LN}\,\mathbb{E}[T_i\|\bar{\boldsymbol{s}}_i\|^2] = \frac{\sigma^4}{N}\,\mathbb{E}\left[T_i\sum_{l=1}^{L}|\gamma_i(l)|^2\right] < \frac{\sigma^4 B}{N}$$

since the total received powers are assumed to be uniformly bounded from above by B.

Substituting this into the second term of (51) yields

$$\begin{aligned}\mathbb{E}&\left[\frac{1}{LN}\sum_{i=1}^{K}\frac{P_i}{(1+P_i\beta^*)^2}\,\Delta_i^N\right]\\ &\geq \mathbb{E}\left[\left(\frac{1}{LN}\,\mathbf{Tr}\,\mathcal{C} - \beta^*\right)\cdot\frac{1}{LN}\sum_{i=1}^{K}\frac{P_i}{(1+P_i\beta^*)^2}\right]\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}&\left[\frac{1}{LN}\sum_{i=1}^{K}\frac{P_i}{(1+P_i\beta^*)^2}\,\Delta_i^N\right]\\ &\leq \mathbb{E}\left[\left(\frac{1}{LN}\,\mathbf{Tr}\,\mathcal{C} - \beta^*\right)\cdot\frac{1}{LN}\sum_{i=1}^{K}\frac{P_i}{(1+P_i\beta^*)^2}\right]\\ &\quad + \frac{\sigma^4 B}{N}\,\mathbb{E}\left[\frac{1}{LN}\sum_{i=1}^{K}\frac{P_i}{(1+P_i\beta^*)^2}\right].\end{aligned}$$

The P_i's are assumed to be uniformly bounded from above so that the terms $\frac{P_i^2}{1+P_i\beta^*}$ are uniformly bounded from above for all i and N. Also, $\frac{1}{LN}\,\mathbb{E}[\mathbf{Tr}\,\mathcal{C}] \to \beta^*$. We can now conclude that

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{LN}\sum_{i=1}^{K}\frac{P_i}{(1+P_i\beta^*)^2}\,\Delta_i^N\right] \to 0. \tag{53}$$

Substituting (47), (52), and (53) into (46) and taking limit as $N \to \infty$, we get that

$$\text{LHS} = \frac{\alpha}{L} - 1 + \sigma^2\beta^* \tag{54}$$

where

$$\begin{aligned}\text{LHS} &= \frac{\alpha}{L}\,\mathbb{E}\left[\frac{1}{1+P\beta^*}\right]\\ &\quad + \lim_{N\to\infty}\mathbb{E}\left[\frac{1}{LN}\sum_{i=1}^{K}\frac{P_i^2}{(1+P_i\xi_i^N)^3}(\Delta_i^N)^2\right].\end{aligned}$$

Recall that

$$\beta^* = \int 1/(\lambda+\sigma^2)\,dG^*(\lambda)$$

where G^* is the limiting expected eigenvalue distribution of $\mathcal{S}\mathcal{D}\mathcal{S}^H$. However, we have shown in Theorem 17 that the limiting distribution is the same in the repeated sequence and completely random sequence models. This equality allows us to apply Theorem 13, applicable for the completely random model, to compute β^*. Because of the i.i.d. nature of the fading gains in the special case of microdiversity, the system of fixed-point equations (39) collapses to a single fixed-point equation, to which β^* is a unique solution

$$\beta^* = \frac{1}{\sigma^2 + \frac{\alpha}{L}\,\mathbb{E}\left[\frac{P}{1+P\beta^*}\right]}.$$

Some simple algebra translates this into

$$\frac{\alpha}{L}\,\mathbb{E}\left[\frac{1}{1+P\beta^*}\right] = \frac{\alpha}{L} - 1 + \sigma^2\beta^*.$$

Comparing this with (54) yields

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{LN}\sum_{i=1}^{K}\frac{P_i^2}{(1+P_i\xi_i^N)^3}(\Delta_i^N)^2\right] = 0. \tag{55}$$

The remaining step of the proof is to show that this implies that $\lim_{N\to\infty}\mathbb{E}[(\Delta_1^N)^2] = 0$.

Using the matrix inversion lemma, for any $i,\, j$

$$\bar{\boldsymbol{s}}_i^H\mathcal{C}_{(ij)}\bar{\boldsymbol{s}}_i = \bar{\boldsymbol{s}}_i^H\mathcal{C}_i\bar{\boldsymbol{s}}_i + \frac{T_j\left(\bar{\boldsymbol{s}}_i^H\mathcal{C}_{(ij)}\bar{\boldsymbol{s}}_j\right)^2}{1+T_j\bar{\boldsymbol{s}}_j^H\mathcal{C}_{(ij)}\bar{\boldsymbol{s}}_j} \tag{56}$$

where

$$\mathcal{C}_{(ij)} := (\mathcal{S}_i\mathcal{D}_i\mathcal{S}_i^H - P_j\bar{\boldsymbol{s}}_j\bar{\boldsymbol{s}}_j^H + \sigma^2 I)^{-1}.$$

Now, by definition

$$\Delta_i^N = \frac{\beta_i^N}{P_i} - \beta^* = \frac{1}{\sum_{l=1}^{L}|\gamma_i(l)|^2}\,\bar{\boldsymbol{s}}_i^H\mathcal{C}_i\bar{\boldsymbol{s}}_i - \beta^*.$$

If we now define

$$\epsilon_{ij}^N := \frac{1}{\sum_{l=1}^{L}|\gamma_i(l)|^2}\,\bar{\boldsymbol{s}}_i^H\mathcal{C}_{(ij)}\bar{\boldsymbol{s}}_i - \beta^*,$$

and

$$\theta_{ij}^N := \frac{1}{\sum_{l=1}^{M}|\gamma_i(l)|^2}\,\frac{T_j(\bar{\boldsymbol{s}}_i^H\mathcal{C}_{(ij)}\bar{\boldsymbol{s}}_j)^2}{1+T_j\bar{\boldsymbol{s}}_j^H\mathcal{C}_{(ij)}\bar{\boldsymbol{s}}_j}$$

then we have, from (56)

$$\epsilon_{ij}^N = \Delta_i^N + \theta_{ij}^N.$$

Hence,

$$\begin{aligned}|\mathbb{E}[(\epsilon_{ij}^N)^2] - \mathbb{E}[(\Delta_i^N)^2]| &\le 2|\mathbb{E}[\Delta_i^N \theta_{ij}^N]| + \mathbb{E}[(\theta_{ij}^N)^2] \\ &\le 2\sqrt{\mathbb{E}[(\Delta_i^N)^2]}\sqrt{\mathbb{E}[(\theta_{ij}^N)^2]} + \mathbb{E}[(\theta_{ij}^N)^2]. \qquad (57)\end{aligned}$$

Now, we can write $\bar{\boldsymbol{s}}_i = \mathcal{F}_i s_i$, where s_i is the original N-dimensional signature sequence of user i, and

$$\mathcal{F}_i = [\gamma_i(1)I, \ldots, \gamma_i(L)I]^t$$

is LN by N. And

$$\mathbb{E}[(\theta_{ij}^N)^2] \le \mathbb{E}[(s_i^H \mathcal{G} s_j)^4]$$

where

$$\mathcal{G} := \frac{T_j}{\sum_{l=1}^L |\gamma_i(l)|^2} \mathcal{F}_i^H \mathcal{C}_{(ij)} \mathcal{F}_i^H.$$

We have

$$\begin{aligned}|\lambda_{\max}(\mathcal{G})| &= \left| \frac{T_j}{\sum_{l=1}^L |\gamma_i(l)|^2} \lambda_{\max}\left(\mathcal{F}_i \mathcal{F}_i^H \mathcal{C}_{(ij)}\right)\right| \\ &\le \left| \frac{T_j}{\sum_{l=1}^L |\gamma_i(l)|^2} \lambda_{\max}(\mathcal{F}_i \mathcal{F}_i^H) \cdot \lambda_{\max}(\mathcal{C}_{(ij)})\right| \\ &\le \left| \frac{T_j}{\sum_{l=1}^L |\gamma_i(l)|^2} \lambda_{\max}(\mathcal{F}_i^H \mathcal{F}_i) \cdot \frac{1}{\sigma^2}\right| \\ &= \left| \frac{T_j}{\sigma^2 \sum_{l=1}^L |\gamma_i(l)|^2} \sum_{l=1}^L \gamma_i^*(l)\gamma_j(l)\right|\end{aligned}$$

which is uniformly bounded for all i, j, and N. Hence, we can now apply Lemma 18 and obtain

$$\mathbb{E}[(\theta_{ij}^N)^2] \le \mathbb{E}[(s_i^H \mathcal{G} s_j)^4] \le \frac{B_2}{N^2} \qquad (58)$$

for some constant B_1. Similarly, one can show that $\mathbb{E}[(\Delta_i^N)^2] \le B_2$ for another constant B_2. Substituting this and inequality (58) into (57), we get

$$|\mathbb{E}[(\epsilon_{ij}^N)^2] - \mathbb{E}[(\Delta_i^N)^2]| < \frac{B_3}{N}$$

for constant B_3 independent of i, j, N. It now follows that

$$|\mathbb{E}[(\Delta_j^N)^2] - \mathbb{E}[(\Delta_i^N)^2]| < \frac{2B_3}{N} \qquad (59)$$

for all i, j. Combining this with (55) yields

$$\lim_{N\to\infty} \mathbb{E}\left[(\Delta_1^N)^2 \frac{1}{LN} \sum_{i=1}^K \frac{P_i^2}{(1+P_i\xi_i^N)^3}\right] = 0. \qquad (60)$$

Now

$$0 \le \xi_i^N \le \max\left\{\frac{\beta_i^N}{P_i}, \beta^*\right\}$$

and

$$\frac{\beta_i^N}{P_i} \le \frac{1}{\sigma^2 \sum_{l=1}^L |\gamma_i(l)|^2} \|\bar{\boldsymbol{s}}_i\|^2 \xrightarrow{\text{a.s.}} \frac{1}{\sigma^2}$$

uniformly in i as well. So we have that

$$\frac{1}{LN} \sum_{i=1}^K \frac{P_i^2}{(1+P_i\xi_i^N)^3}$$

is greater than

$$\frac{1}{LN} \sum_{i=1}^K \frac{P_i^2}{\left(1+\max\left\{\frac{P_i}{\sum_{l=1}^L |\gamma_i(l)|^2}\|\bar{\boldsymbol{s}}_i\|^2, \beta^*\right\}\right)}$$

and we can conclude that the left-hand side is asymptotically lower-bounded by some constant $B_4 > 0$ as $N \to \infty$. This, together with (60) and an application of Dominated Convergence Theorem yields the conclusion that

$$\mathbb{E}[(\Delta_1^N)^2] = \mathbb{E}[(\beta_1^N/P_1 - \beta^*)^2] \to 0, \qquad \text{as } N \to \infty.$$

Hence β_1^N/P_1 converges to β^* in probability, completing the proof. □

C. Proof of Theorem 7

Let us begin by defining

$$E^{(K)}(l) := \mathbb{E}\left[\frac{1}{N}\sum_{k=2}^K T_k|\gamma_k(l)|^2\right], \qquad l = 1, 2, \ldots, L$$

where the expectation is over the random fadings of the interferers. Since the interferers are independent of user 1, we can also consider this expectation as taken conditional on user 1's fading levels. Then the matched filter $\boldsymbol{c}_1$ is given by

$$\boldsymbol{c}_1 = \left[\frac{\gamma_1^*(1)}{\sigma^2 + E^{(K)}(1)}\boldsymbol{s}_1, \ldots, \frac{\gamma_1^*(L)}{\sigma^2 + E^{(K)}(L)}\boldsymbol{s}_1\right]^H$$

and we can substitute this into (3) to obtain SIR_1. Let us consider the numerator and denominator in (3) separately. First

$$T_1(\boldsymbol{c}_1^H \bar{\boldsymbol{s}}_1)^2 = T_1\left(\sum_{l=1}^L \frac{|\gamma_1(l)|^2}{\sigma^2 + E^{(K)}(l)}\right)^2 \qquad (61)$$

and the two terms in the denominator are given by

$$\sigma^2 \boldsymbol{c}_1^H \boldsymbol{c}_1 = \sigma^2 \sum_{l=1}^L \frac{|\gamma_1(l)|^2}{(\sigma^2 + E^{(K)}(l))^2} \qquad (62)$$

and

$$\sum_{k=2}^K T_k |\boldsymbol{c}_1^H \bar{\boldsymbol{s}}_k|^2 = \frac{\alpha}{K}\sum_{k=2}^K T_k \left|\sum_{l=1}^L \frac{\gamma_1^*(l)\gamma_k(l)}{(\sigma^2 + E^{(K)}(l))} a_{1,k}\right|^2 \qquad (63)$$

where $a_{1,k} := \sqrt{N}\boldsymbol{s}_1^H \boldsymbol{s}_k$. Note that $a_{1,k}$ has mean zero, and variance $\mathbb{E}[|a_{1,k}|^2] = 1$ when we average over the random sequences of both user 1 and user k. We consider the limiting asymptotics of the three separate terms (61)–(63), as we take N, K to infinity, keeping $K = \alpha N$.

By the weak convergence of the empirical distribution of the received powers, assumed in the statement of the theorem, we have that

$$\lim_{K\to\infty} \frac{1}{K-1}\sum_{k=2}^K T_k|\gamma_k(l)|^2$$

is constant, almost surely, taking the value $\mathbb{E}[P(l)]$. This is still true if we condition on the fading levels of user 1. We have also assumed that the received powers are uniformly bounded over all k and l, so the dominated convergence theorem applies, and we have that

$$\lim_{K\to\infty} E^{(K)}(l) = \alpha \mathbb{E}[P(l)].$$

We conclude that the asymptotics for (61) and (62) are

$$T_1(\boldsymbol{c}_1^H \overline{\boldsymbol{s}}_1)^2 \to T_1 \left(\sum_{l=1}^{L} \frac{|\gamma_1(l)|^2}{\sigma^2 + \alpha\mathbb{E}[P(l)]} \right)^2 \quad \mathbb{P} \text{ a.s.} \quad (64)$$

$$\sigma^2(\boldsymbol{c}_1^H \boldsymbol{c}_1) \to \sigma^2 \sum_{l=1}^{L} \frac{|\gamma_1(l)|^2}{(\sigma^2 + \alpha\mathbb{E}[P(l)])^2} \quad \mathbb{P} \text{ a.s.} \quad (65)$$

To analyze (63), we split it into two terms

$$\frac{\alpha}{K} \sum_{k=2}^{K} T_k \sum_{l=1}^{L} \frac{|\gamma_1(l)|^2 |\gamma_k(l)|^2}{(\sigma^2 + E^{(K)}(l))^2} |a_{1,k}|^2 \qquad (66)$$

and

$$2\frac{\alpha}{K} \sum_{k=2}^{K} T_k \sum_{l_1=1}^{L} \sum_{l_2>l_1} \frac{\gamma_1^*(l_1)\gamma_k(l_1)\gamma_1(l_2)\gamma_k^*(l_2)}{(\sigma^2+E^{(K)}(l_1))(\sigma^2+E^{(K)}(l_2))} |a_{1,k}|^2. \qquad (67)$$

It is clear from the weak convergence of the empirical distributions of received powers, and the convergence of the $E^{(K)}(l)$'s, that the term (66) converges almost surely to

$$\sum_{l=1}^{L} \frac{|\gamma_1(l)|^2}{(\sigma^2 + \alpha\mathbb{E}[P(l)])^2} \alpha\mathbb{E}[P(l)].$$

Let us, therefore, focus on the term (67). It follows from the circular symmetry of the $\gamma_k(l)$'s, that if we average over the random fading levels of the interferers, and their signature sequences, the expectation of (67) is zero. The variance of (67) is, therefore, the second moment, and this can be reduced via circular symmetry to

$$4\left(\frac{\alpha}{K}\right)^2 \sum_{k=2}^{K} \sum_{l_1=1}^{L} \sum_{l_2>l_1} \cdot \frac{\mathbb{E}[|\gamma_1(l_1)|^2 |\gamma_k(l_1)|^2 |\gamma_1(l_2)|^2 |\gamma_k(l_2)|^2 |a_{1,k}|^4]}{(\sigma^2 + E^{(K)}(l_1))^2 (\sigma^2 + E^{(K)}(l_2))^2}$$

which is $O(1/N)$ as $N \to \infty$. We conclude that the expression in (67) converges to zero almost surely, for any realization of signature sequences and fading levels of the interferers.

Putting the pieces of the denominator of (3) together, we see that it is asymptotically given by

$$\sum_{l=1}^{L} \frac{|\gamma_1(l)|^2}{(\sigma^2 + \alpha\mathbb{E}[P(l)])^2} (\sigma^2 + \alpha\mathbb{E}[P(l)]) = \left(\sum_{l=1}^{L} \frac{|\gamma_1(l)|^2}{\sigma^2 + \alpha\mathbb{E}[P(l)]} \right).$$

Applying this with (64) we obtain the convergence result stated in the theorem.

D. Proof of Theorem 10

The issue of power control for the MMSE receiver, and (32), have already been considered in [27], where it is shown that if a solution for transmit powers exists, then it is unique. We, therefore, focus in our proof on existence.

Let $\boldsymbol{T}$ be an arbitrary vector of positive powers,

$$\mathcal{D} = \operatorname{diag}(T_1, T_2, \ldots, T_K)$$

and $\Sigma = \mathcal{S}\mathcal{D}\mathcal{S}^H$, recalling that $\mathcal{S}$ is the $NL \times K$ matrix of "super" signature sequences, each column consisting of a stack of repeated signature sequences, one for each antenna, and each multiplied by the appropriate complex channel gain to that antenna. Let us now consider the sufficiency condition in the theorem. We assume that (31) holds, and establish the existence of a solution $\boldsymbol{T}^*$ to the K nonlinear equations (32) by direct construction. In particular, we generate a component-wise increasing sequence of transmit power vectors $\boldsymbol{T}^{(n)}$, such that $\boldsymbol{T}^{(n)} \to \boldsymbol{T}^*$.

The definition of the sequence $(\boldsymbol{T}^{(n)})_{n=1}^{\infty}$ is as follows. Start with $\boldsymbol{T}^{(0)} = 0$. Clearly, for $\boldsymbol{T}^{(0)} = 0$ we have $\beta^{(0)} = 0$, where $\beta^{(n)}$ is the vector of *achieved* SIR values when $\boldsymbol{T} = \boldsymbol{T}^{(n)}$. To get $\boldsymbol{T}^{(1)}$, let $T_k^{(1)}$ be the minimum power required by user k to obtain β_k (desired), under the assumption that $T_j = T_j^{(0)}$ for all other $j \neq k$. Since $\beta_k^{(0)} < \beta_k$, it follows that $T_k^{(1)} > T_k^{(0)}$, and this is true for all $k = 1, 2, \ldots, K$. Moreover, since $T_j^{(1)} > T_j^{(0)}$ $\forall j \neq k$, it follows that $\beta_k^{(1)} < \beta_k$ for all k. We repeat this power adaptation procedure for $n = 2, 3, \ldots$. By induction on the above argument, we obtain that $\boldsymbol{T}^{(n)}$ is component-wise increasing, and

$$\beta^{(n)} < \beta, \qquad \text{for all } n = 1, 2, \ldots. \qquad (68)$$

To obtain our solution $\boldsymbol{T}^*$ as the limit of the $\boldsymbol{T}^{(n)}$, we must first show that none of these powers tend to infinity. Let $\mathcal{D}^{(n)}$ be the $K \times K$ diagonal matrix with diagonal entries given in $\boldsymbol{T}^{(n)}$. Define $\Sigma^{(n)} := \mathcal{S}\mathcal{D}^{(n)}\mathcal{S}^H$ and let $\Lambda^{(n)} := (\lambda_1^{(n)}, \ldots, \lambda_{LN}^{(n)})$ denote some ordering of its eigenvalues. We will show that it is impossible for the eigenvalues to tend to infinity, and then that this implies that the transmit powers cannot tend to infinity.

Our approach will be to normalize the powers in such a way that they remain bounded, and then take convergent subsequences. To this end, let U be the unique subset of $\{1, 2, \ldots, K\}$ such that

$$\limsup_{n\to\infty} \max_{u_1, u_2 \in U} \frac{T_{u_1}^{(n)}}{T_{u_2}^{(n)}} < \infty \qquad (69)$$

and

$$\limsup_{n\to\infty} \max_{u_1 \in U^c, u_2 \in U} \frac{T_{u_1}^{(n)}}{T_{u_2}^{(n)}} = 0.$$

Thus, the ratios of transmit powers in U remain bounded, and the other users' powers become relatively negligible in the limit. It shall be useful to refer to the precise value of the limsup in (69), a number that is at least unity; let us denote this value by δ.

To obtain our normalization constant, define

$$c(n) := \max_{u\in U} T_u^{(n)}.$$

Note that the condition that at least one user's transmit power becomes unbounded in the limit is then equivalent to $c(n) \to \infty$. We focus on users in U, and define the normalized $|U| \times |U|$ diagonal matrix of transmit powers

$$\bar{\mathcal{D}}^{(n)} = \mathrm{diag}\left(T_i^{(n)} c(n)^{-1},\, i \in U\right).$$

Let $\mathcal{S}(U)$ be the $LN \times |U|$ matrix consisting of the super-signature sequences in U, and define $\bar{\Sigma}^{(n)} = \mathcal{S}(U)\bar{\mathcal{D}}^{(n)}\mathcal{S}(U)^H$. From the definition of U, the entries in $\bar{\mathcal{D}}^{(n)}$ are bounded above and below by 1 and δ^{-1}, respectively, so there exists a $|U| \times |U|$ diagonal matrix $\bar{\mathcal{D}}$ with strictly positive diagonal entries, and a subsequence $(n_k)_{k=1}^{\infty}$, along which $\bar{\mathcal{D}}^{(n_k)} \to \bar{\mathcal{D}}$. It follows that

$$\bar{\Sigma}^{(n_k)} \to \bar{\Sigma} := \mathcal{S}(U)\bar{\mathcal{D}}\mathcal{S}(U)^H$$

which is a matrix of the same rank as $(\mathcal{S}(U))$. The limiting eigenvalues of $\bar{\Sigma}^{(n_k)}$ are therefore given by the eigenvalues of $\bar{\Sigma}$

$$\bar{\Lambda}^{(n_k)} \to \bar{\Lambda}. \tag{70}$$

There are precisely $\mathrm{rank}\,(\mathcal{S}(U))$ elements of $\bar{\Lambda}$ that are strictly positive, and the rest are zero.

Clearly, there is a relationship between the eigenvalues of $\Sigma^{(n)}$ and $\bar{\Sigma}^{(n)}$. The intuition is that $\bar{\Sigma}^{(n)}$ provides all the information about users in U in the limit as $n \to \infty$, since the other users are asymptotically negligible. To make this precise, consider the fictitious channel in which users in U^c are not present, and users in U are allocated powers as prescribed by the diagonal matrix $c(n)\bar{\mathcal{D}}^{(n)}$. The corresponding interference covariance matrix is then $c(n)\bar{\Sigma}^{(n)}$, with eigenvalues $c(n)\bar{\Lambda}^{(n)}$. For all n, precisely $\mathrm{rank}\,(\mathcal{S}(U))$ of these eigenvalues are positive, and the rest are zero. The above convergent subsequence result therefore tells us about the limiting ratios of the eigenvalues of $c(n)\bar{\Sigma}^{(n)}$.

We now apply the conservation law (30) to this fictitious channel. First, let $\bar{\beta}^{(n)}$ be the vector of SIRs achieved at iteration n of the power adaptation. Then

$$\sum_{i\in U} \frac{\bar{\beta}_i^{(n)}}{1+\bar{\beta}_i^{(n)}} = LN - \sum_{j=1}^{LN} \frac{\sigma^2}{c(n)\bar{\lambda}_j^{(n)} + \sigma^2}.$$

Since exactly $LN - K$ of the $\bar{\lambda}_j^{(n)}$'s are zero, we can instead write

$$\sum_{i\in U} \frac{\bar{\beta}_i^{(n)}}{1+\bar{\beta}_i^{(n)}} = \mathrm{rank}\,(\mathcal{S}(U)) - \sum_{j=1}^{LN} \frac{\sigma^2}{c(n)\bar{\lambda}_j^{(n)} + \sigma^2}\, I\left[\bar{\lambda}_j^{(n)} > 0\right]. \tag{71}$$

Now, in the original channel, users in U^c have asymptotically no effect on the users in U in the limit $n \to \infty$. It follows that

$$\limsup_{n\to\infty} \sum_{i\in U} \frac{\bar{\beta}_i^{(n)}}{1+\bar{\beta}_i^{(n)}} = \limsup_{n\to\infty} \sum_{i\in U} \frac{\beta_i^{(n)}}{1+\beta_i^{(n)}}.$$

But by (31) and (68), it then follows that

$$\limsup_{n\to\infty} \sum_{i\in U} \frac{\bar{\beta}_i^{(n)}}{1+\bar{\beta}_i^{(n)}} < \mathrm{rank}\,(\mathcal{S}(U)),$$

and hence, by (71), that

$$\liminf_{n\to\infty} \sum_{j=1}^{LN} \frac{\sigma^2}{c(n)\bar{\lambda}_j^{(n)} + \sigma^2}\, I\left[\bar{\lambda}_j^{(n)} > 0\right] > 0.$$

In particular, it is not possible for all the positive $c(n)\bar{\lambda}_j^{(n)}$ to be unbounded along any subsequence $(n_k)_{k=1}^{\infty}$. However, we observed in (70) that there exists a particular subsequence $(n_k)_{k=1}^{\infty}$ along which all the positive $\bar{\lambda}_j^{(n)}$'s converge, so along this subsequence, $c(n_k)$ must remain bounded. However, the powers $\boldsymbol{T}^{(n)}$ are an increasing sequence, so they must be bounded along the whole sequence $n = 1, 2, \ldots$. We conclude that there exists a positive vector $\boldsymbol{T}^*$ of transmit powers for which $\boldsymbol{T}^{(n)} \to \boldsymbol{T}^*$. Returning to (69), we conclude that when all the constraints in (31) hold, U is, in fact, the set of all users.

The monotonicity of our sequence $\boldsymbol{T}^{(n)}$ is, in fact, a fundamental feature of the whole problem (see [25]). For example, if we have a feasible channel of users labeled $1, 2, \ldots, K$, then a smaller channel with users from a subset $U \subseteq \{1, 2, \ldots, K\}$ is also feasible, and the transmit powers that achieve feasibility in the larger channel, $\boldsymbol{T}^*$, dominate the transmit powers that achieve feasibility in the smaller channel $\bar{\boldsymbol{T}}$. That is,

$$\forall\, u \in U, \qquad \bar{T}_u \le T_u^*.$$

To see why this is the case, start with $\bar{T}_u^{(0)} := T_u^* \ \forall\, u \in U$ and generate the decreasing sequence $\bar{\boldsymbol{T}}^{(n)} \downarrow \bar{\boldsymbol{T}}$.

Necessity follows almost immediately from monotonicity and the conservation law. Monotonicity implies that if our system is feasible, then so is the "fictitious" system in which users in U are the only ones present. The conservation law for the fictitious channel is

$$\sum_{i\in U} \frac{\beta_i}{1+\beta_i} = \mathrm{rank}\,(\mathcal{S}(U)) - \sum_{j=1}^{LN} \frac{\sigma^2}{\lambda_j + \sigma^2}$$

where we note that the λ_j's are the eigenvalues of the covariance matrix Σ of the fictitious channel.

The conservation law must hold when all the β_is are achieved. It is, therefore, necessary that

$$\sum_{i\in U} \frac{\beta_i}{1+\beta_i} < \mathrm{rank}(\mathcal{S}(U))$$

for all $U \subseteq \{1, 2, \ldots, K\}$.

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