Diversity-Multiplexing Tradeoff in MIMO Channels

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Intel Smart Antenna Workshop

Two objectives of the talk:

- Present a new performance metric for evaluating MIMO coding schemes.
- Give some examples of new coding schemes designed to optimize the metric.

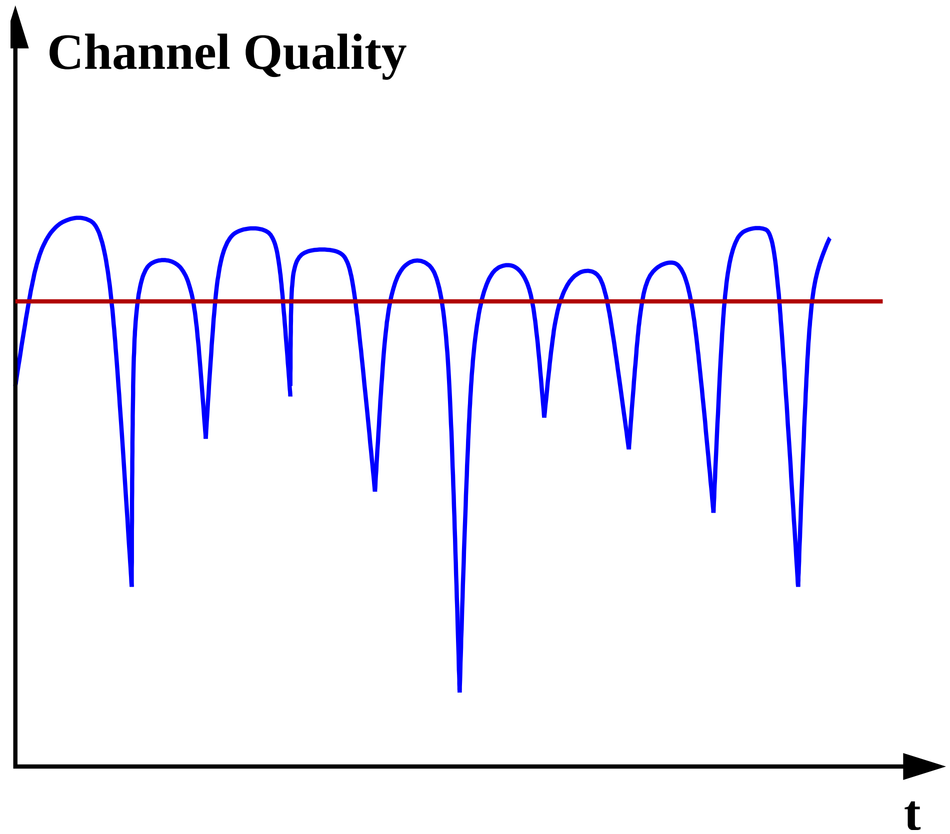
Diversity and Freedom

Two fundamental resources of a MIMO fading channel:

diversity

degrees of freedom

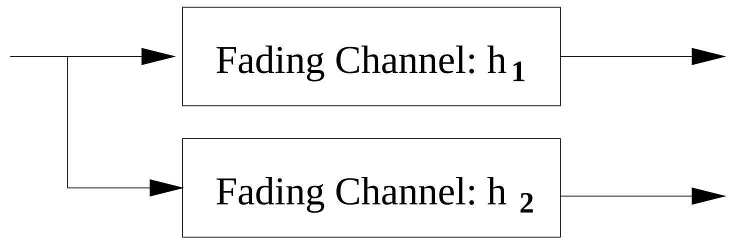
Diversity



A channel with more diversity has smaller probability in deep fades.

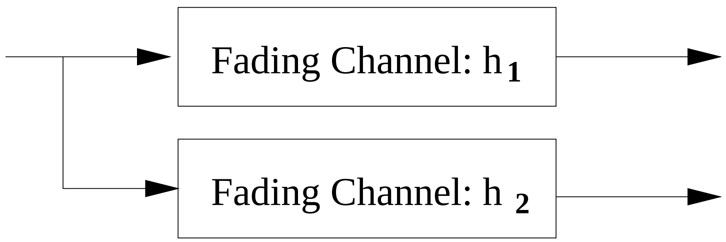
Diversity

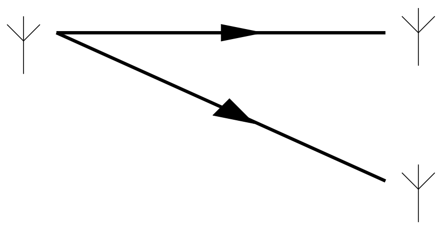
Diversity



- Additional independent fading channels increase diversity.

Diversity





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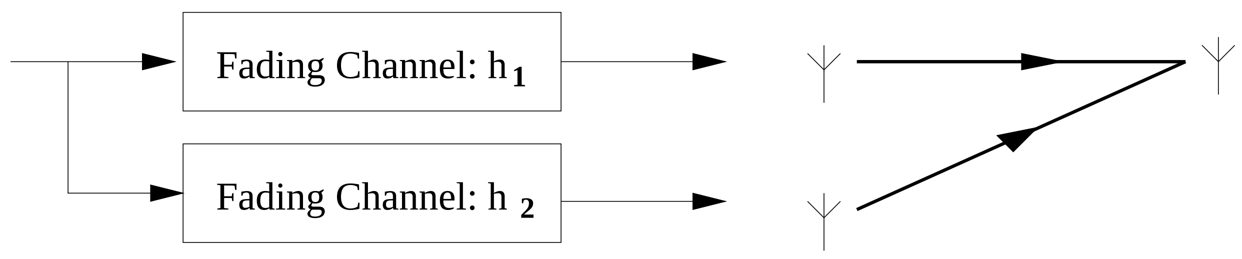
- Spatial diversity

Diversity



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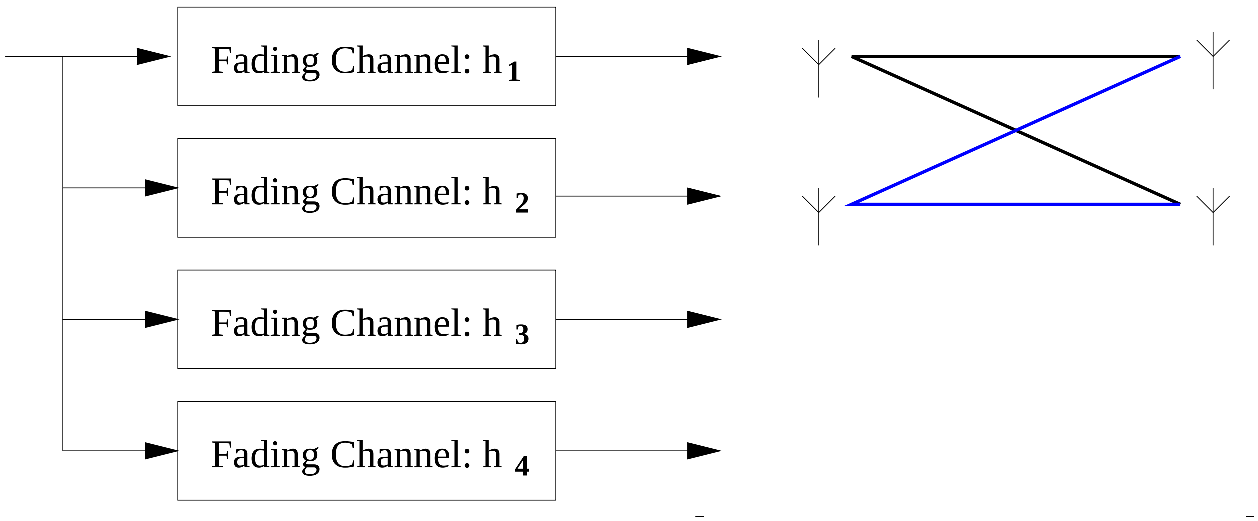
- Spatial diversity: receive, transmit

Diversity



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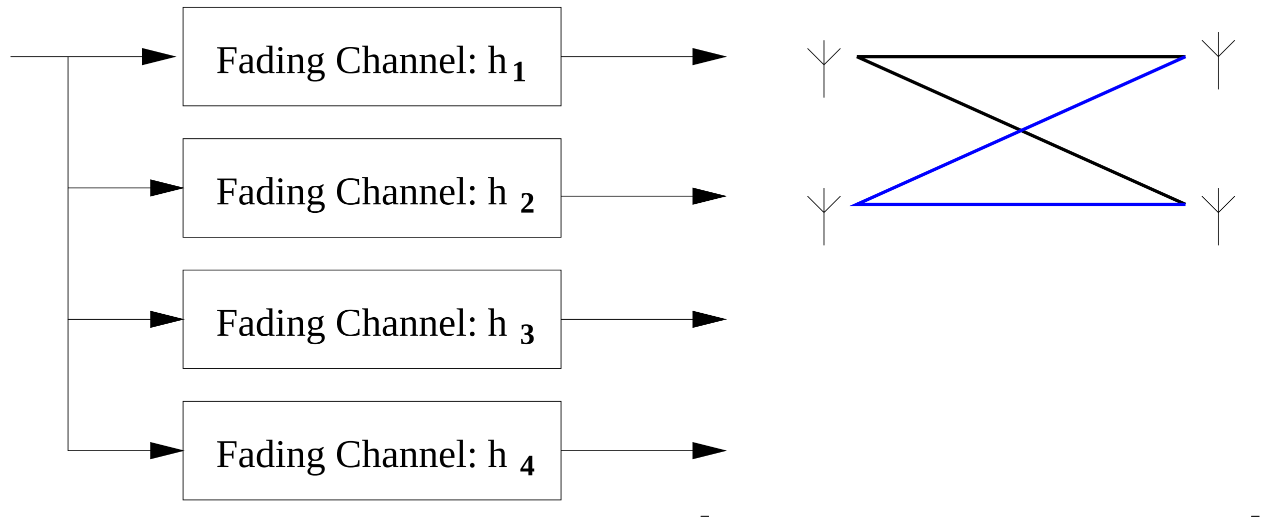
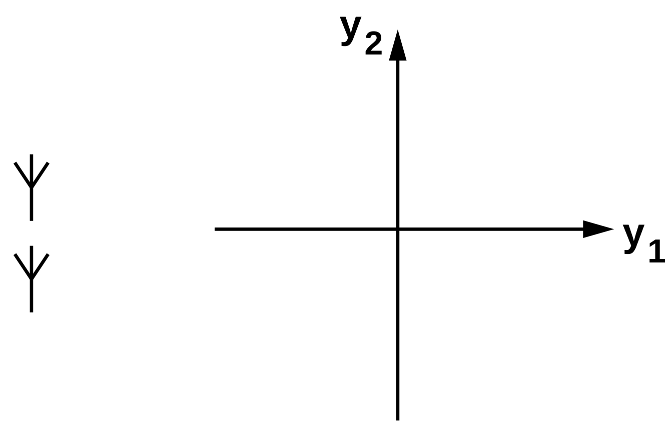


- Additional independent fading channels increase diversity.
- Spatial diversity: receive, transmit or both.
- For a m by n channel, diversity is mn.

Degrees of Freedom



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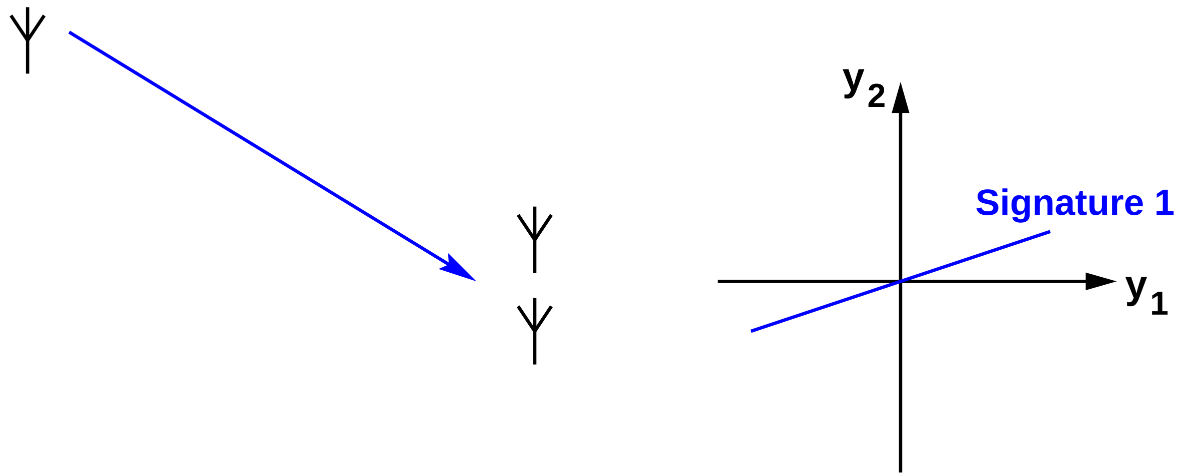

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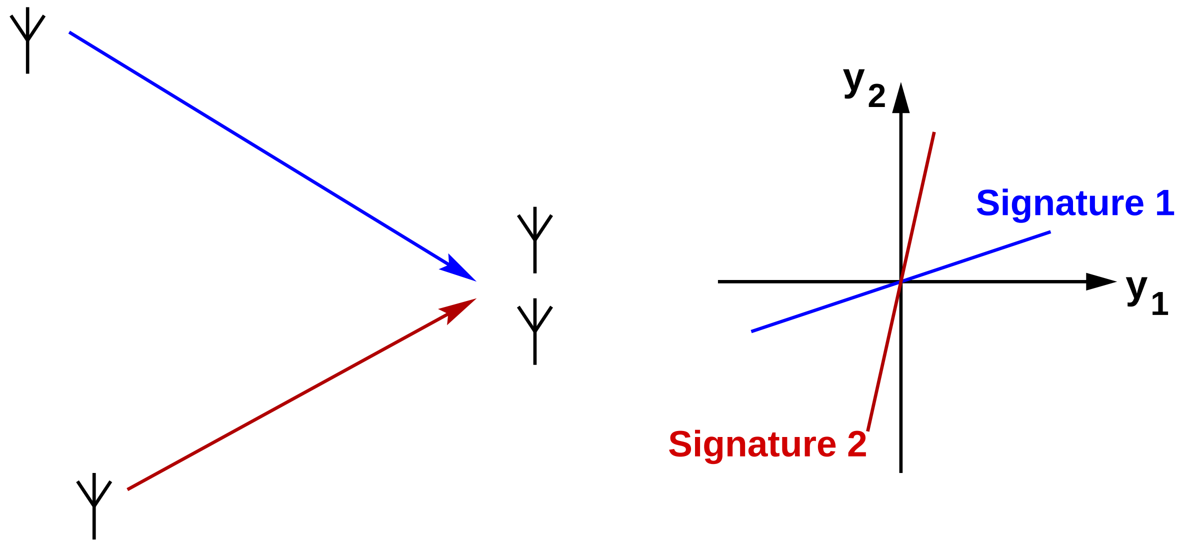
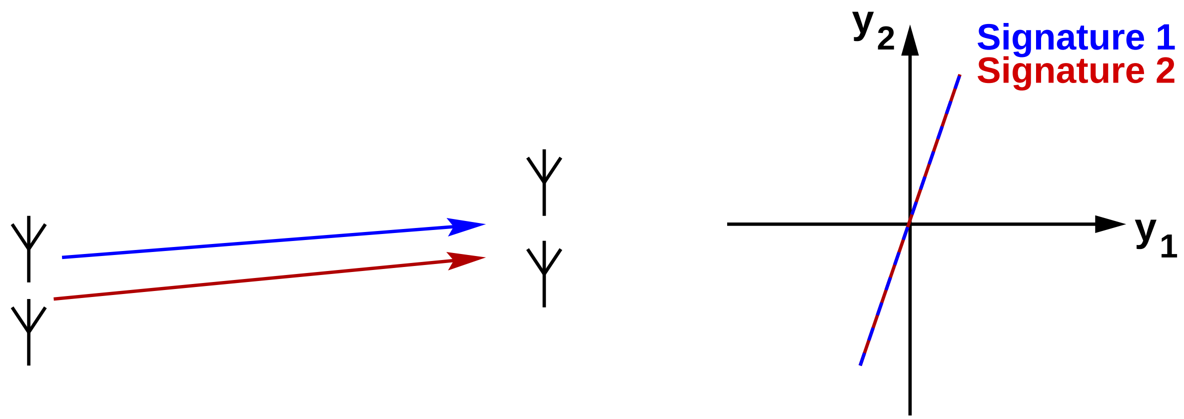


Signals arrive in multiple directions provide multiple degrees of freedom for communication.

Degrees of Freedom



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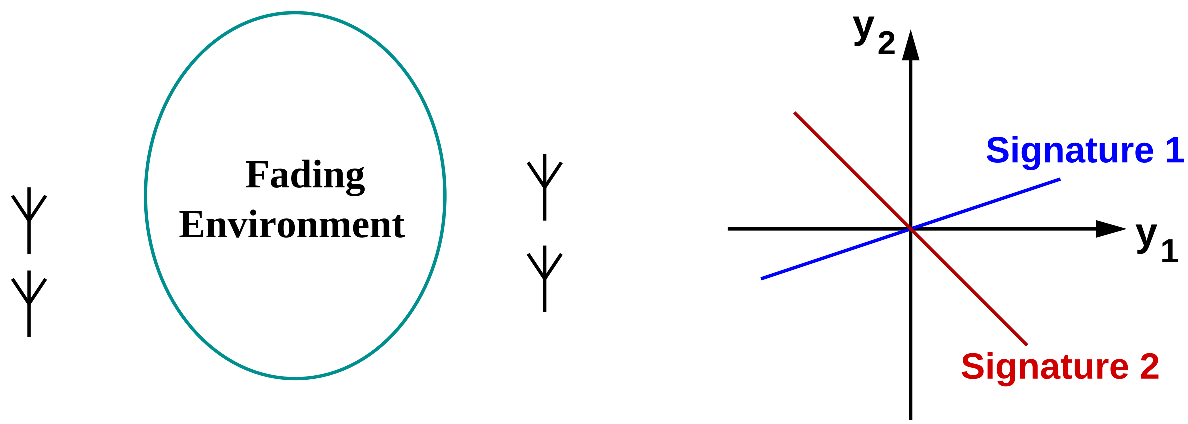

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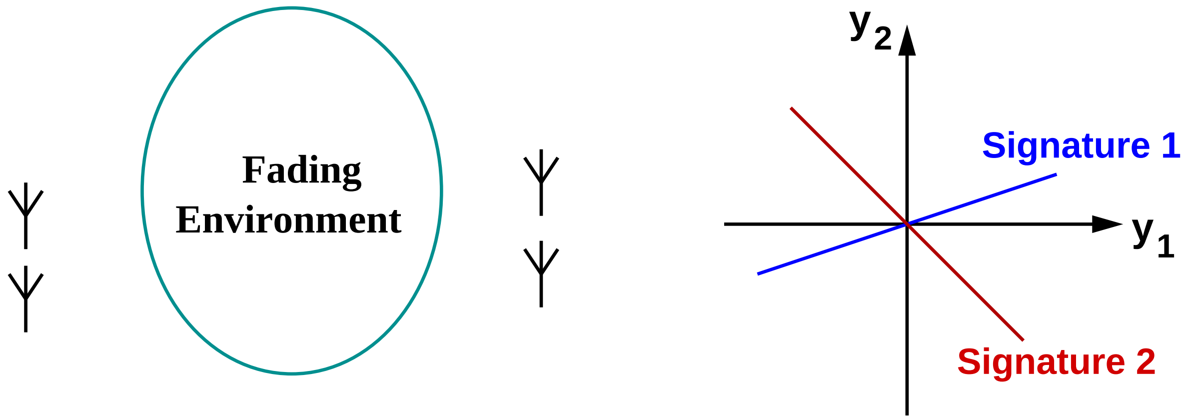
Same effect can be obtained via scattering even when antennas are close together.

Degrees of Freedom



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Same effect can be obtained via scattering even when antennas are close together.

In a m by n channel with rich scattering, there are $\min\{m, n\}$ degrees of freedom.

Diversity and Freedom

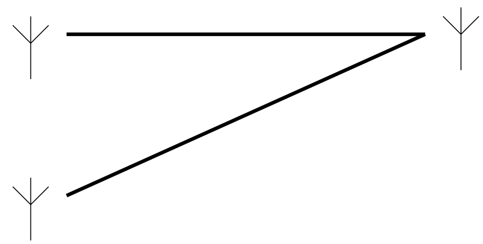
In a MIMO channel with rich scattering:

maximum diversity $= mn$

degrees of freedom $= \min\{m, n\}$

The name of the game in space-time coding is to design schemes which exploit as much of both these resources as possible.

Space-Time Code Examples: 2×1 Channel



Repetition Scheme:

time

$$X = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}$$

space

diversity: 2

data rate: 1/2 sym/s/Hz

Alamouti Scheme:

time

$$X = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$$

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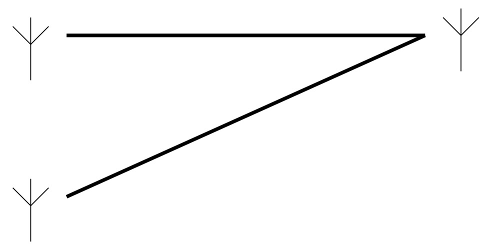
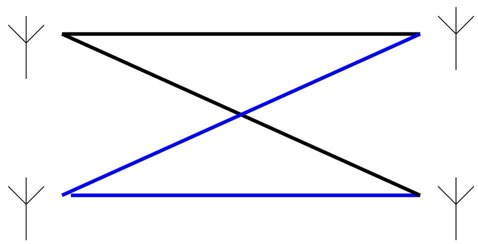
Performance Summary: 2×1 Channel



	Diversity gain	Degrees of freedom utilized /s/Hz
Repetition	2	$1/2$
Alamouti	2	1
channel itself	2	1

Space-Time Code Examples: 2×2 Channel



Repetition Scheme:

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diversity gain : 4

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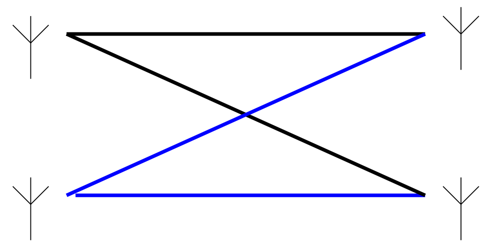

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space

diversity: 4

data rate: 1 sym/s/Hz

But the 2×2 channel has 2 degrees of freedom!

V-BLAST with Nulling

Send two independent uncoded streams over the two transmit antennas.

Demodulate each stream by nulling out the other stream.

Data rate: 2 sym/s/Hz

Diversity: 1

Winters, Salz and Gitlins 93:

Nulling out k interferers using n receive antennas yields a diversity gain of $n - k$.

Performance Summary: 2×2 Channel

	Diversity gain	d.o.f. utilized /s/Hz
Repetition	4	$1/2$
Alamouti	4	1
V-Blast with nulling	1	2
channel itself	4	2

Questions:

- Alaomuti is clearly better than repetition, but how can it be compared to V-Blast?

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Questions:

- Alaomuti is clearly better than repetition, but how can it be compared to V-Blast?
- How does one quantify the "optimal" performance achievable by any scheme?
- We need to make the notions of "fiversity gain" and "d.o.f. utilized" precise and enrich them.

Classical Diversity Gain

Motivation: PAM

$$\mathbf{y} = \mathbf{h}\mathbf{x} + \mathbf{w} \qquad P_e \approx P(\|\mathbf{h}\| \text{ is small }) \propto \mathsf{SNR}^{-1}$$

$$\left.\begin{array}{l} \mathbf{y}_1 = \mathbf{h}_1\mathbf{x} + \mathbf{w}_1 \\ \mathbf{y}_2 = \mathbf{h}_2\mathbf{x} + \mathbf{w}_2 \end{array}\right\} \qquad \begin{array}{rl} P_e & \approx P(\|\mathbf{h}_1\|, \|\mathbf{h}_2\| \text{ are both small}) \\ & \propto \mathsf{SNR}^{-2} \end{array}$$

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General Definition

A space-time coding scheme achieves (classical) diversity gain $d_{\max}$, if

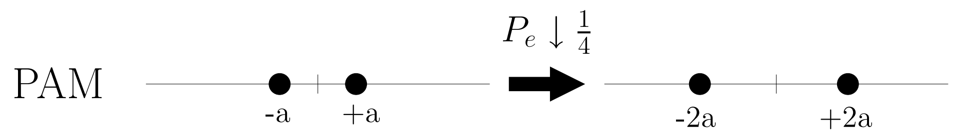
$$P_e(\mathsf{SNR}) \sim \mathsf{SNR}^{-d_{\max}}$$

for a fixed data rate.

i.e. error probability deceases by $2^{-d_{\max}}$ for every 3 dB increase in SNR, by $4^{-d_{\max}}$ for every 6dB increase, etc.

Example: PAM vs QAM in 1 by 1 Channel

Every 6 dB increase in SNR doubles the distance between constellation points for a given rate.



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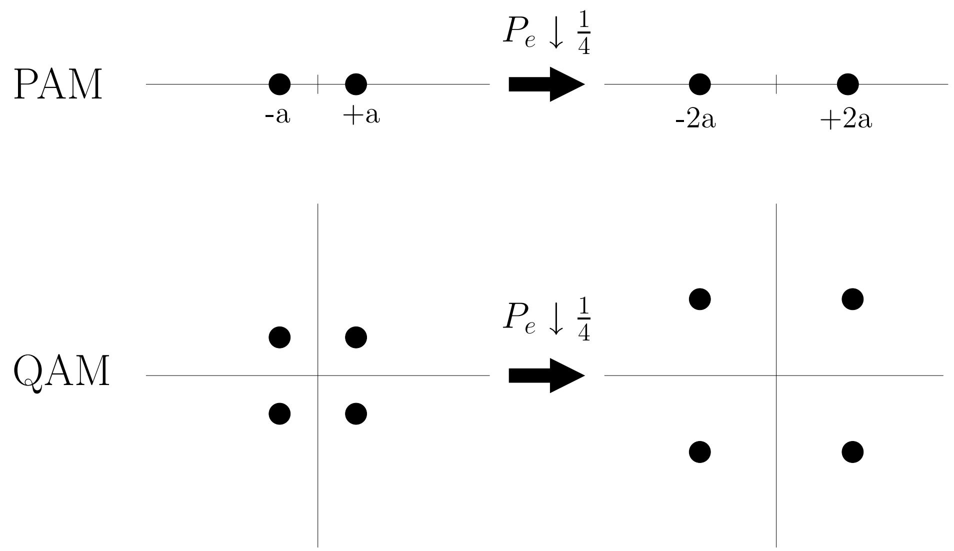

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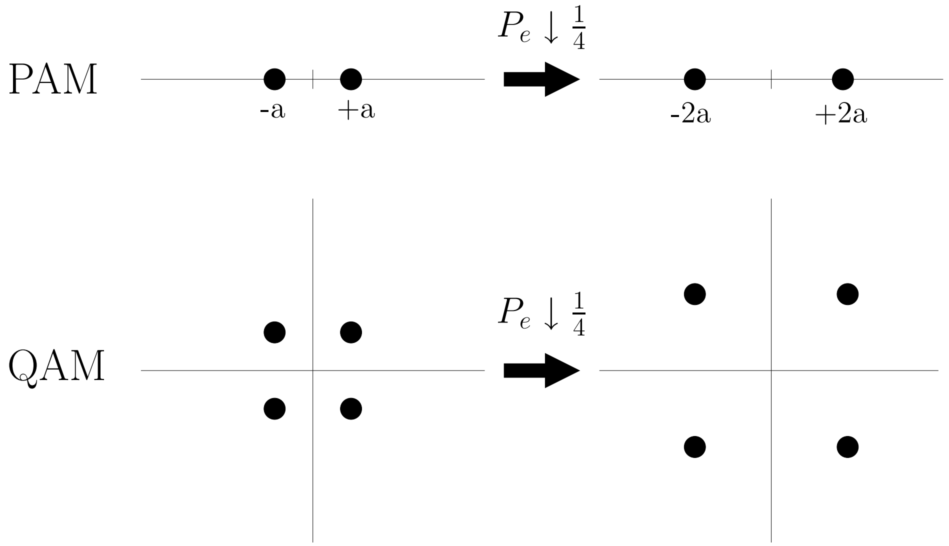
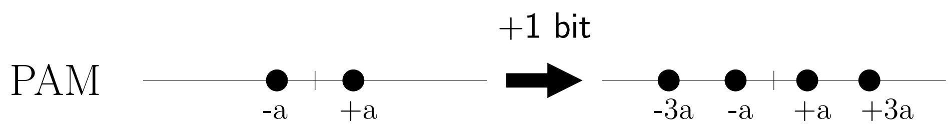


Both PAM and QAM have the same (classical) diversity gain of 1.

(classical) diversity gain does not say anything about the d.o.f. utilized by the scheme.

Ask a Dual Question

Every 6 dB doubles the constellation size for a given reliability, for PAM.



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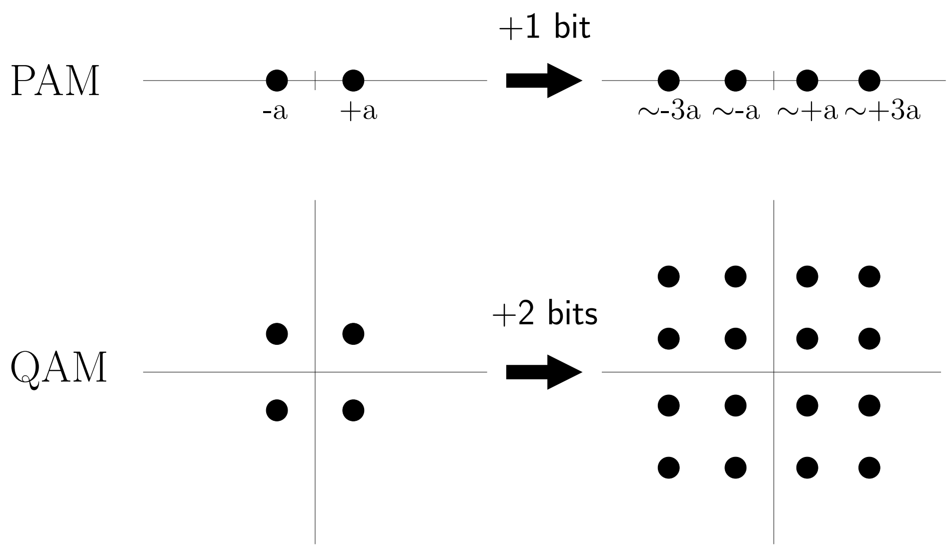

But for QAM, every 6 dB quadruples the constellation size.

Degrees of Freedom Utilized

Definition:

A space-time coding scheme utilizes $r_{\max}$ degrees of freedom/s/Hz if the data rate scales like

$$R(\mathsf{SNR}) \sim r_{\max} \log_2 \mathsf{SNR} \quad \text{bits/s/Hz}$$

for a fixed error probability (reliability)

In a 1×1 channel, $r_{\max} = 1/2$ for PAM, $r_{\max} = 1$ for QAM.

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Note: A space-time coding scheme is a family of codes within a certain structure, with varying symbol alphabet as a function of SNR.

Diversity-Multiplexing Tradeoff

Every 3 dB increase in SNR yields

either

a $2^{-d_{\max}}$ decrease in error probability for a fixed rate;

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More generally, one can increase reliability and the data rate at the same time.

Diversity-Multiplexing Tradeoff of A Scheme

(Zheng and Tse 03)

Definition

A space-time coding scheme achieves a diversity-multiplexing tradeoff curve $d(r)$ if for each multiplexing gain r, simultaneously

$$R(\mathsf{SNR}) \sim r \log_2 \mathsf{SNR} \quad \text{bits/s/Hz}$$

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The largest multiplexing gain is $r_{\max}$, the d.o.f. utilized by the scheme.

The largest diversity gain is $d_{\max} = d(0)$, the classical diversity gain.

Diversity-Multiplexing Tradeoff of the Channel

Definition

The diversity-multiplexing tradeoff $d^*(r)$ of a MIMO channel is the best possible diversity-multiplexing tradeoff achievable by any scheme.

$r^*_{\max}$ is the largest multiplexing gain achievable in the channel.

$d^*_{\max} = d^*(0)$ is the largest diversity gain achievable.

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$d^*_{\max} = d^*(0)$ is the largest diversity gain achievable.

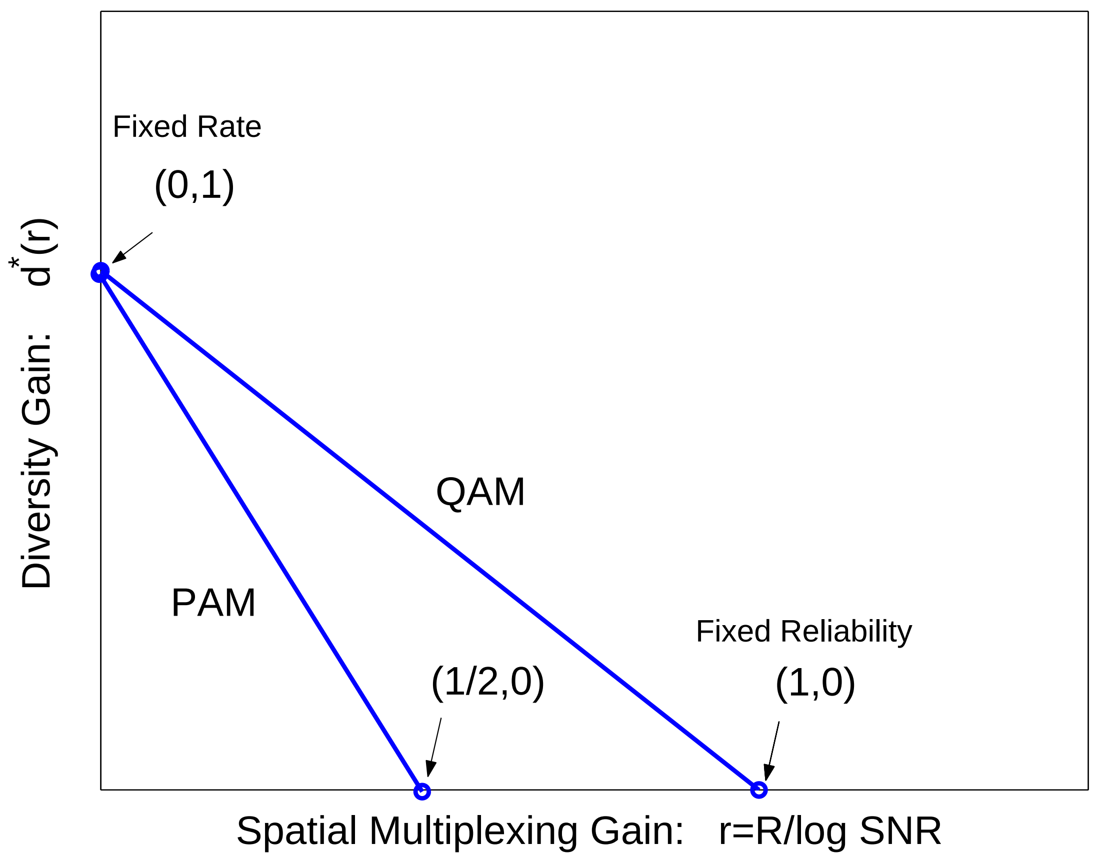
For a $m \times n$ MIMO channel, it is not difficult to show:

$$r^*_{\max} = \min\{m, n\}$$

$$d^*_{\max} = mn$$

What is more interesting is how the entire curve looks like.

Example: 1×1 Channel



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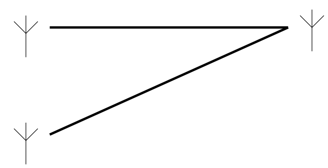

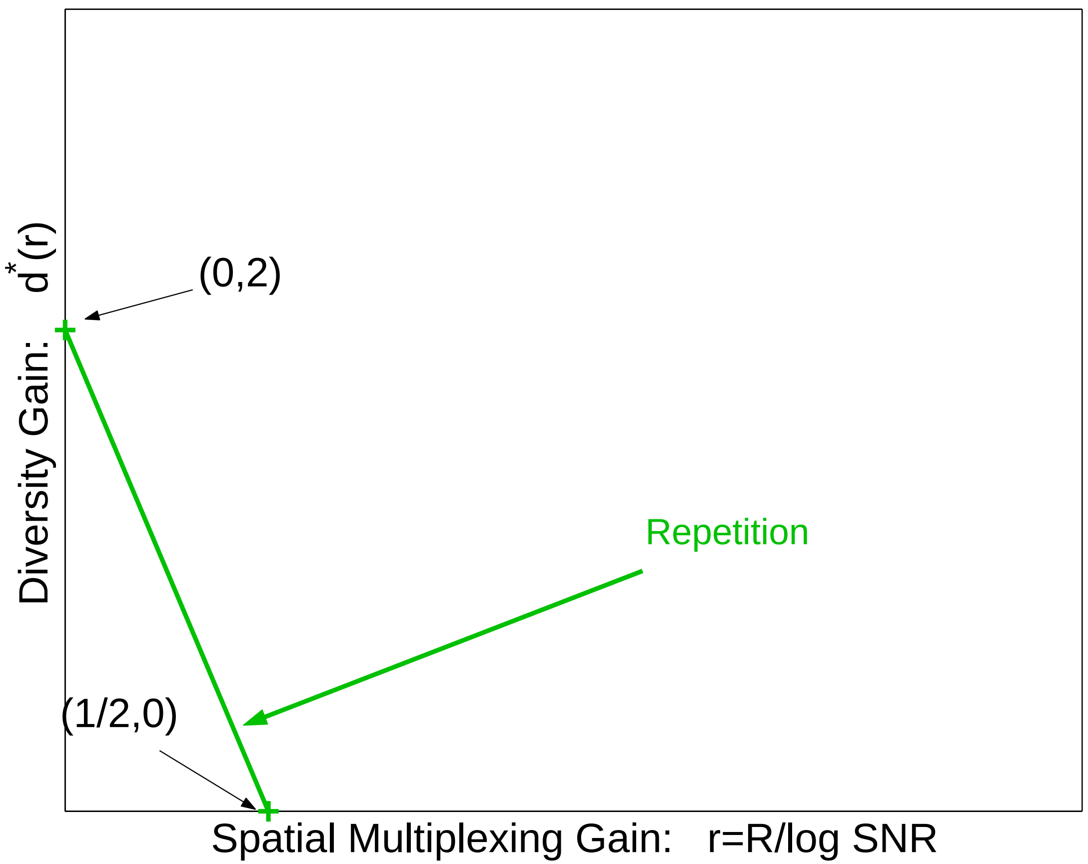

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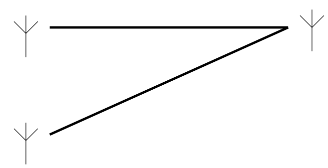

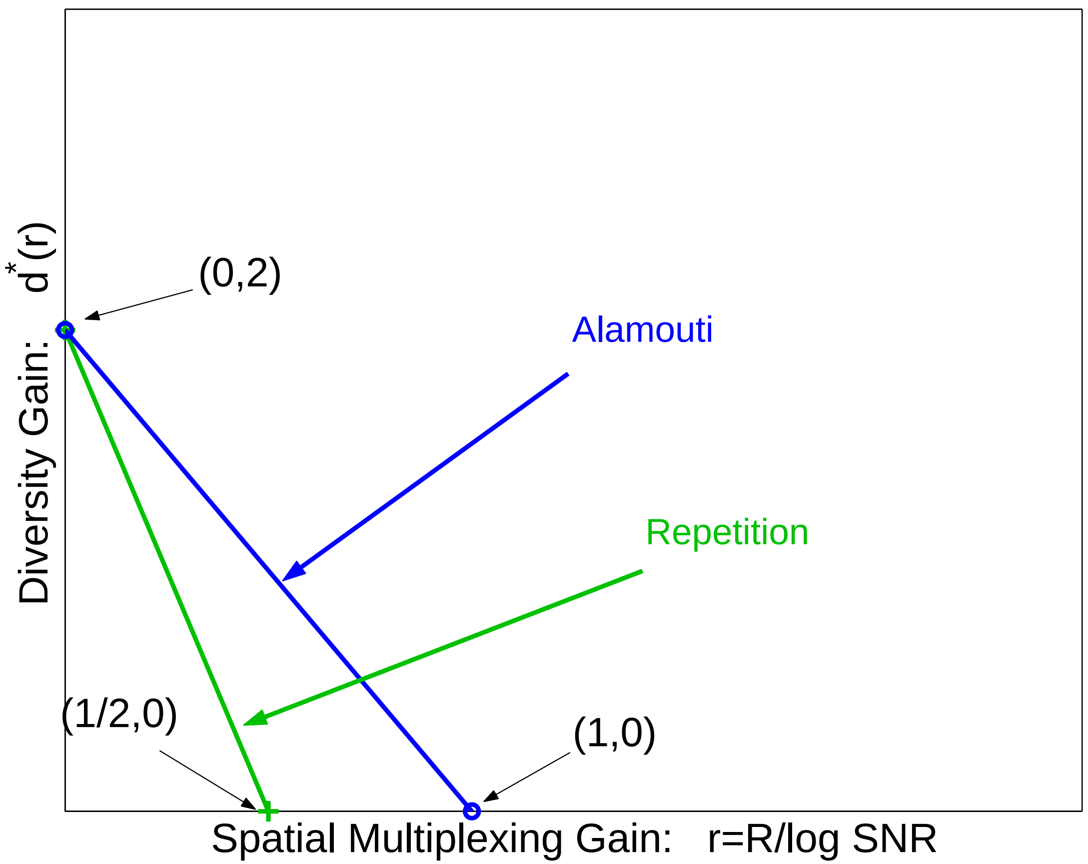

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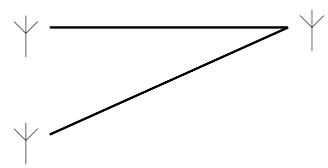

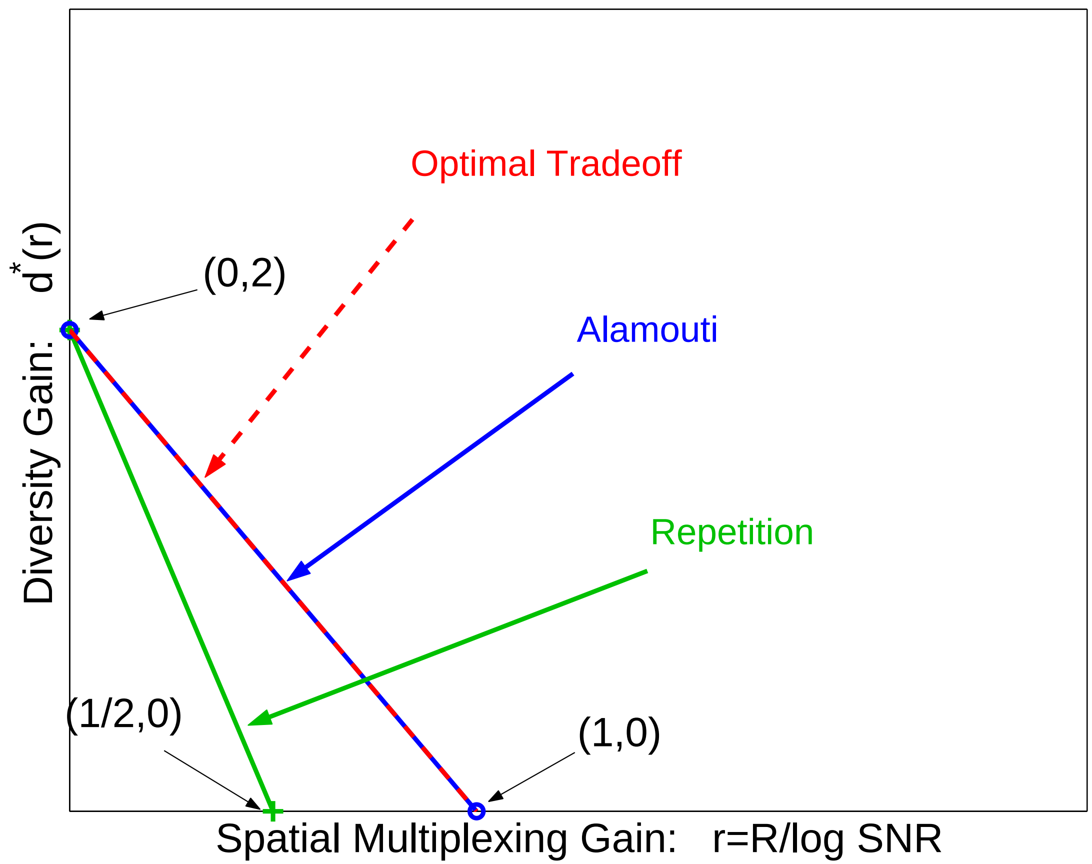

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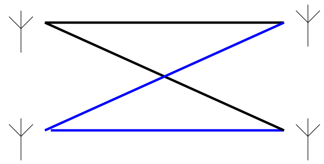

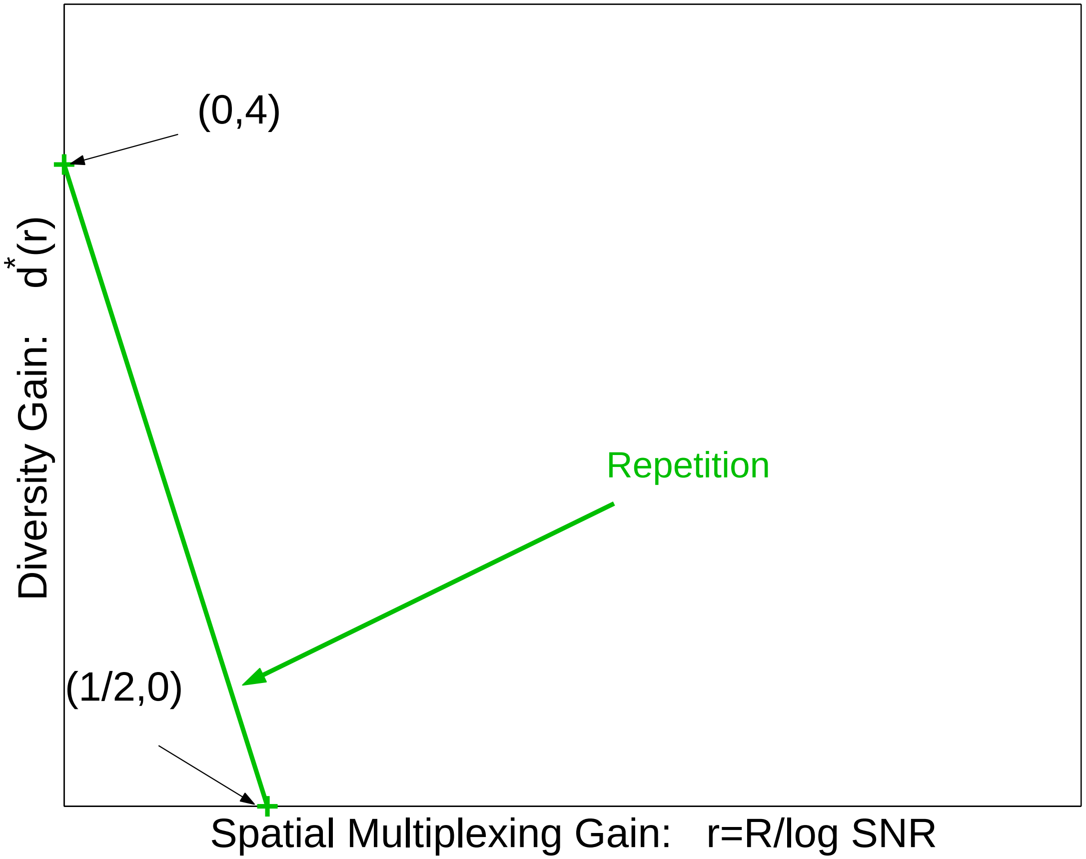

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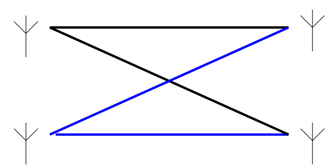

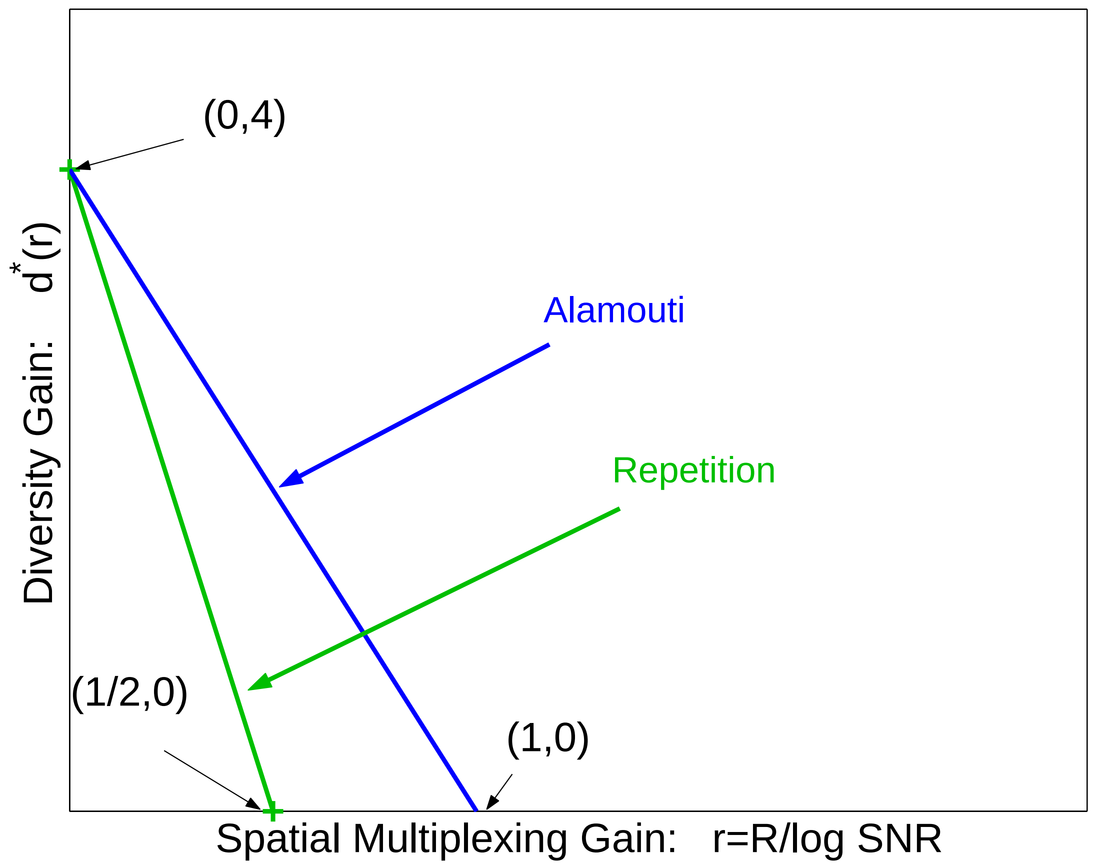

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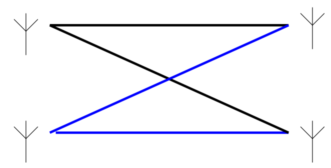

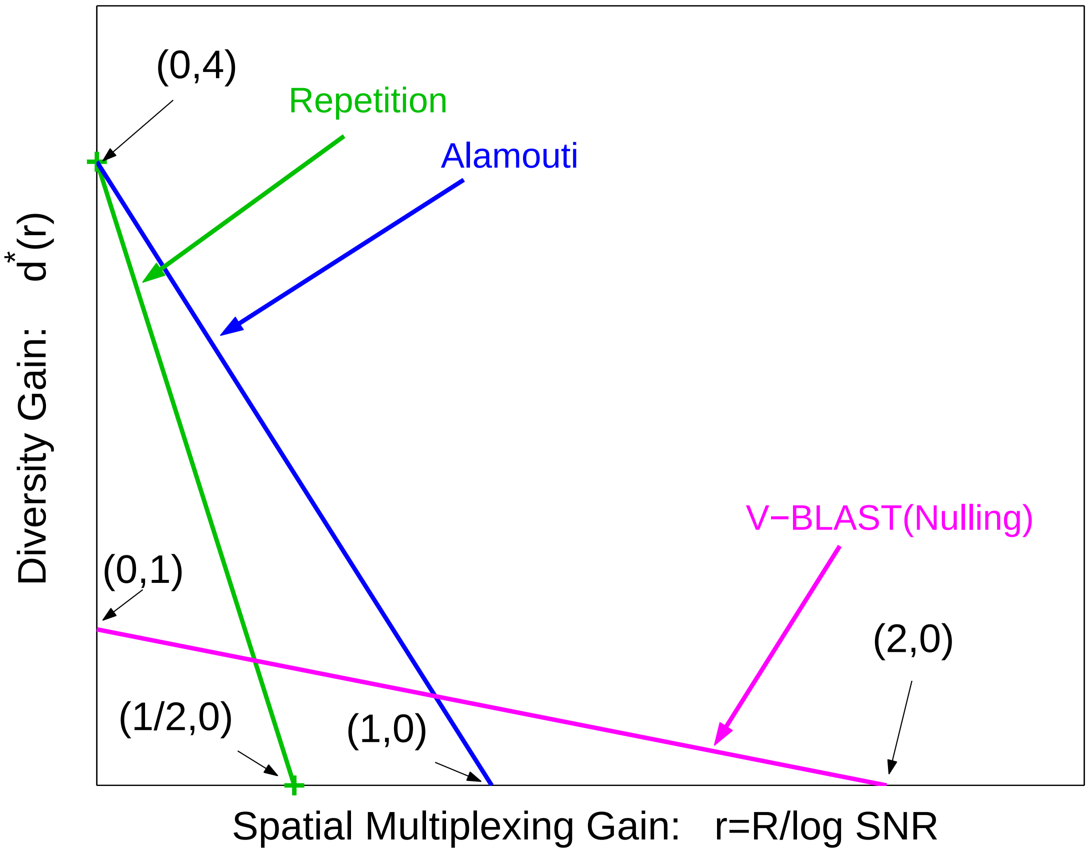

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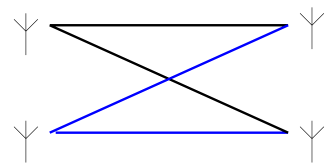

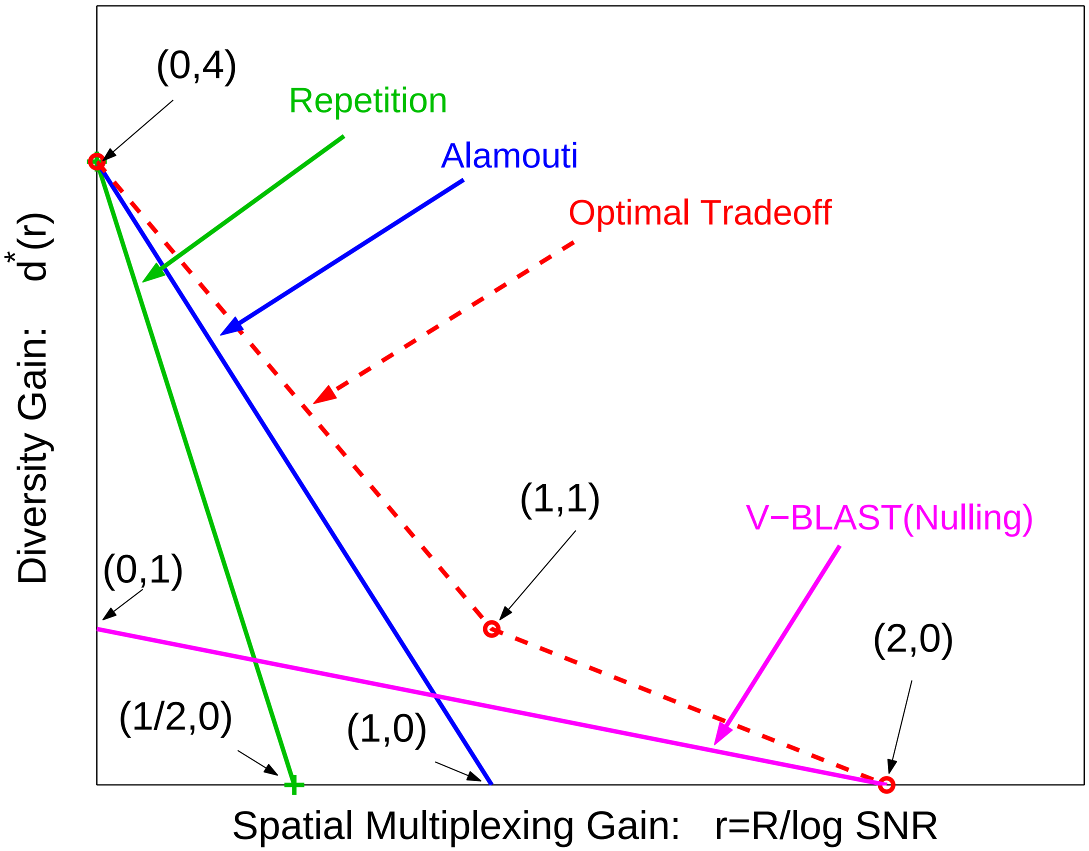

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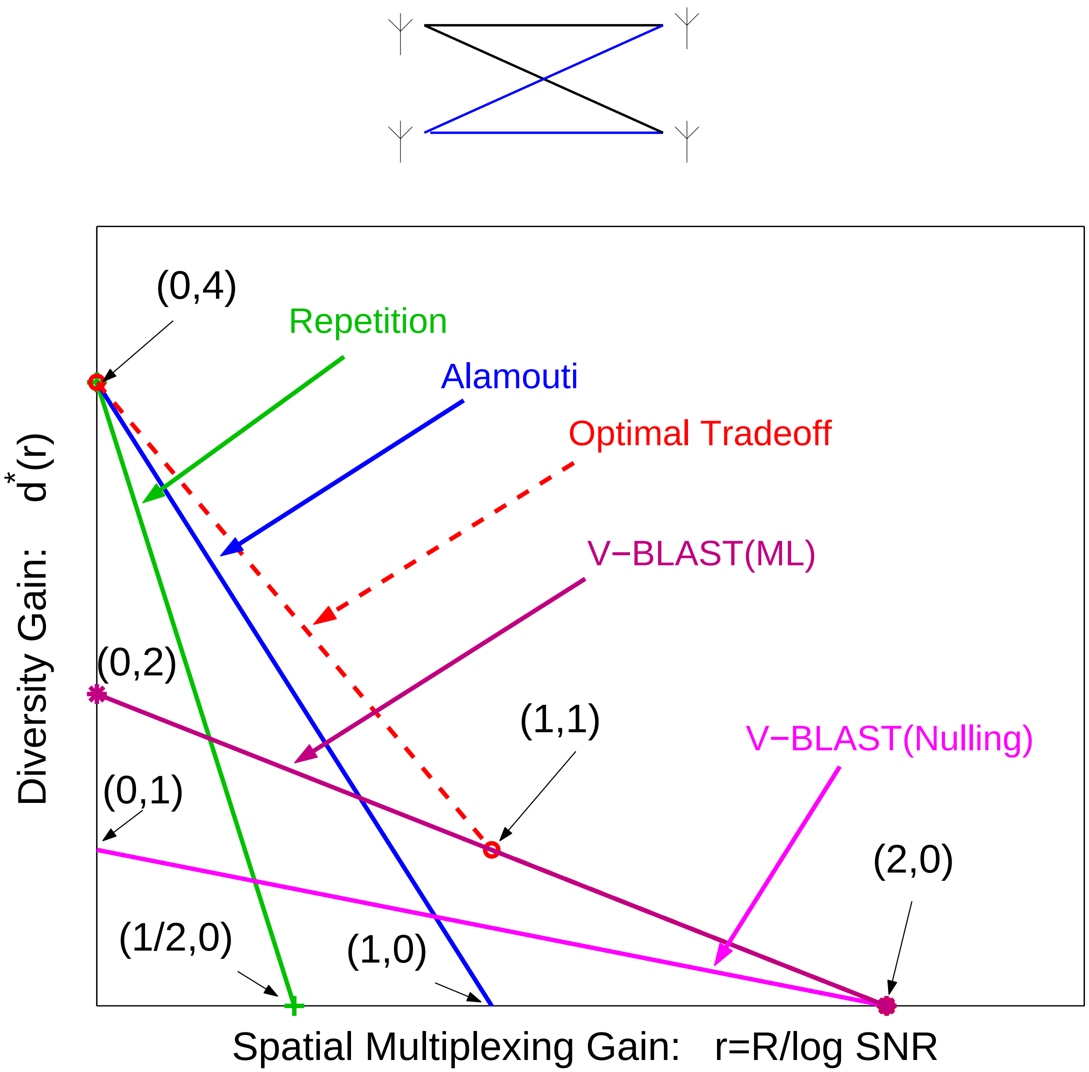
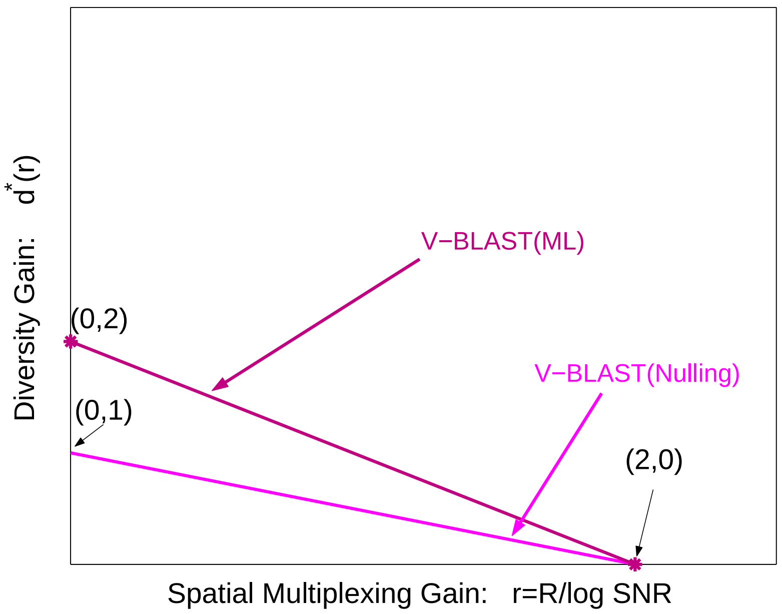


ML vs Nulling in V-Blast



Winters, Salz and Gitlins 93:

Nulling out k interferers using n receive antennas provides a diversity gain of $n - k$.

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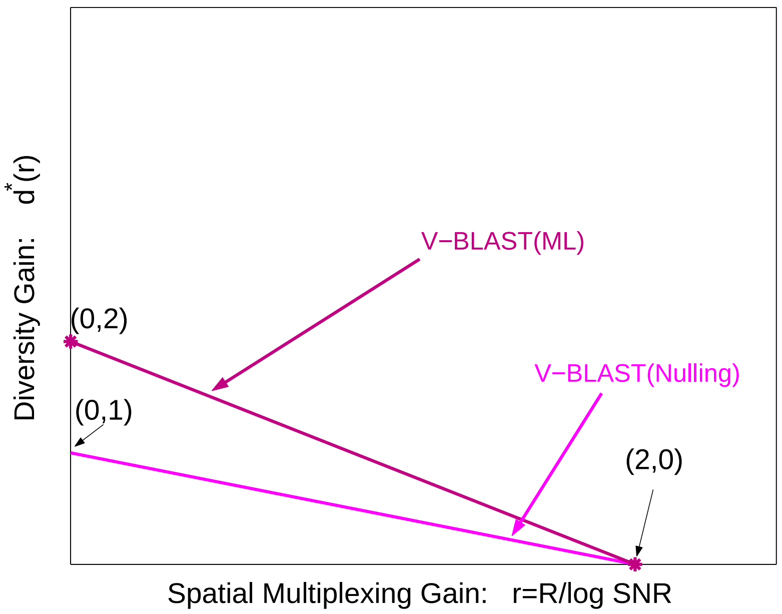

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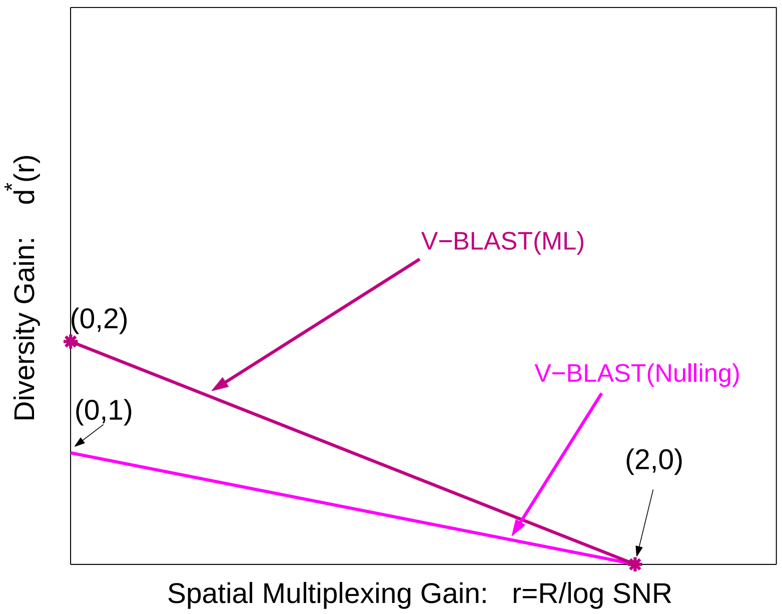

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Tse,Viswanath and Zheng 03:

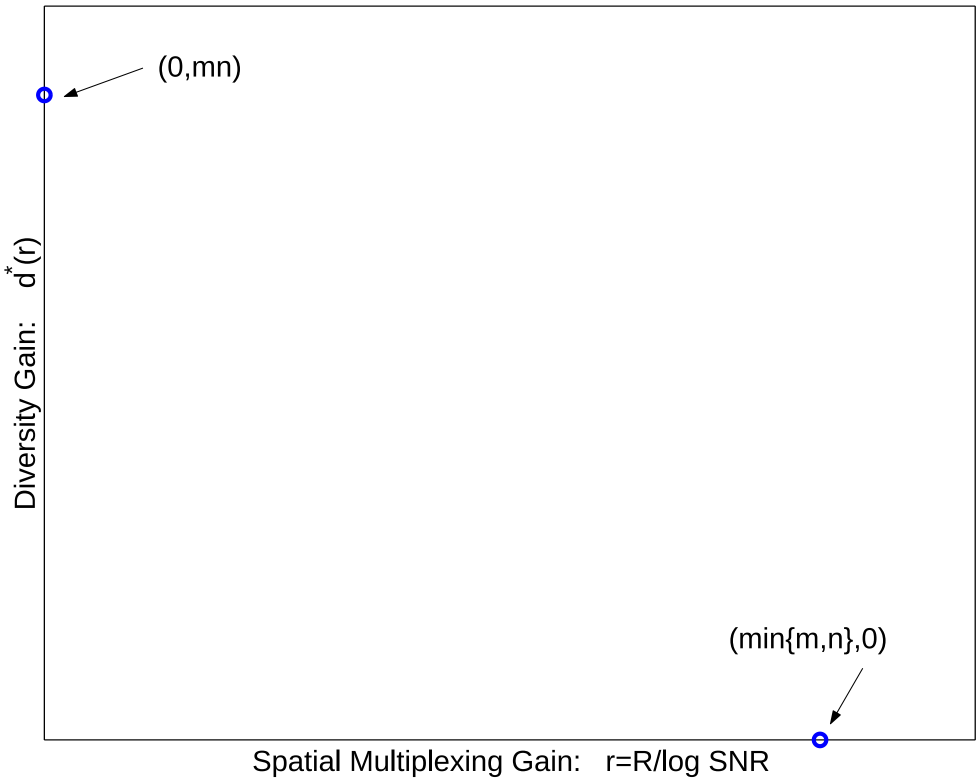
Jointly detecting all users provides a diversity gain of n to each.

There is free lunch. (?)

Optimal D-M Tradeoff for General $m \times n$ Channel

(Zheng and Tse 03)

As long as block length $l \geq m+n-1$:



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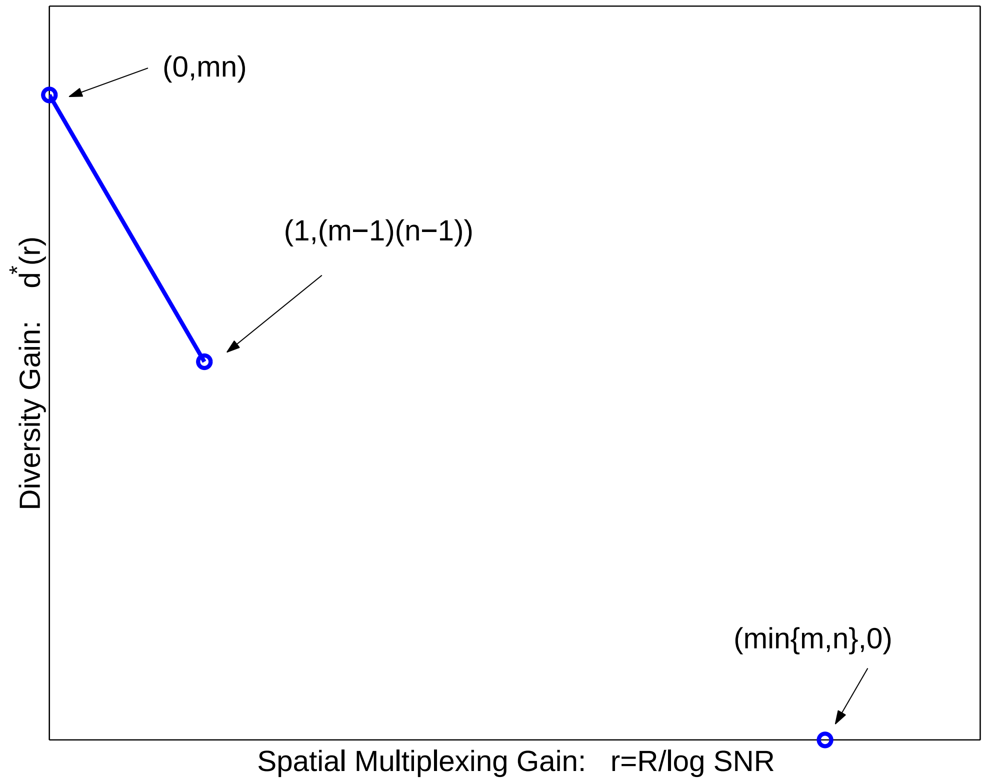

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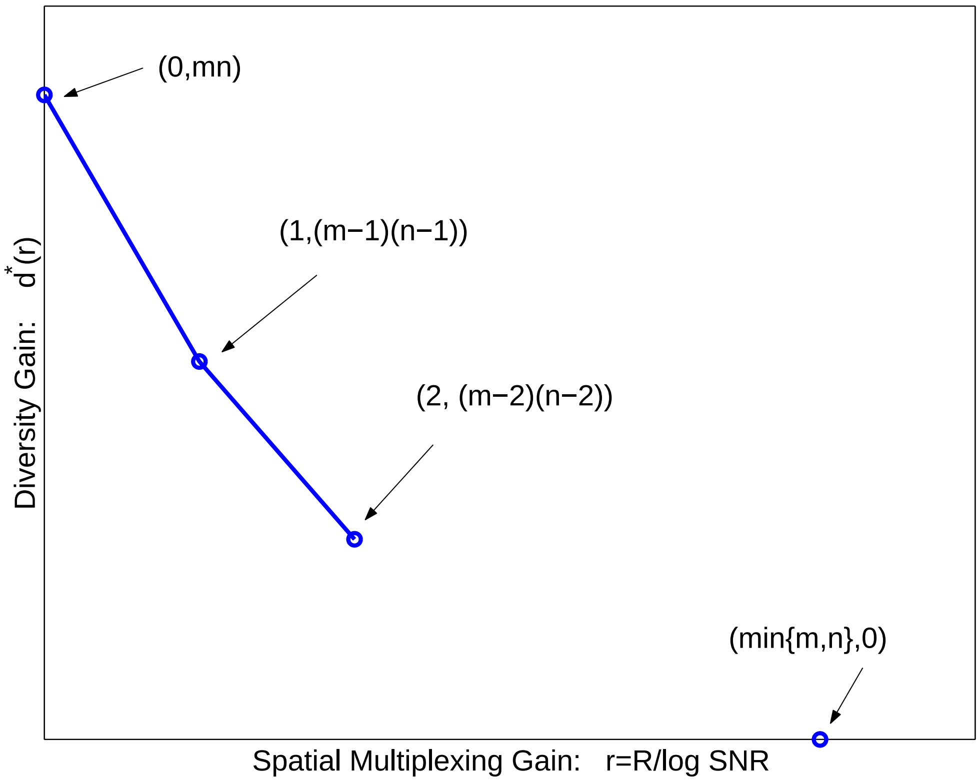
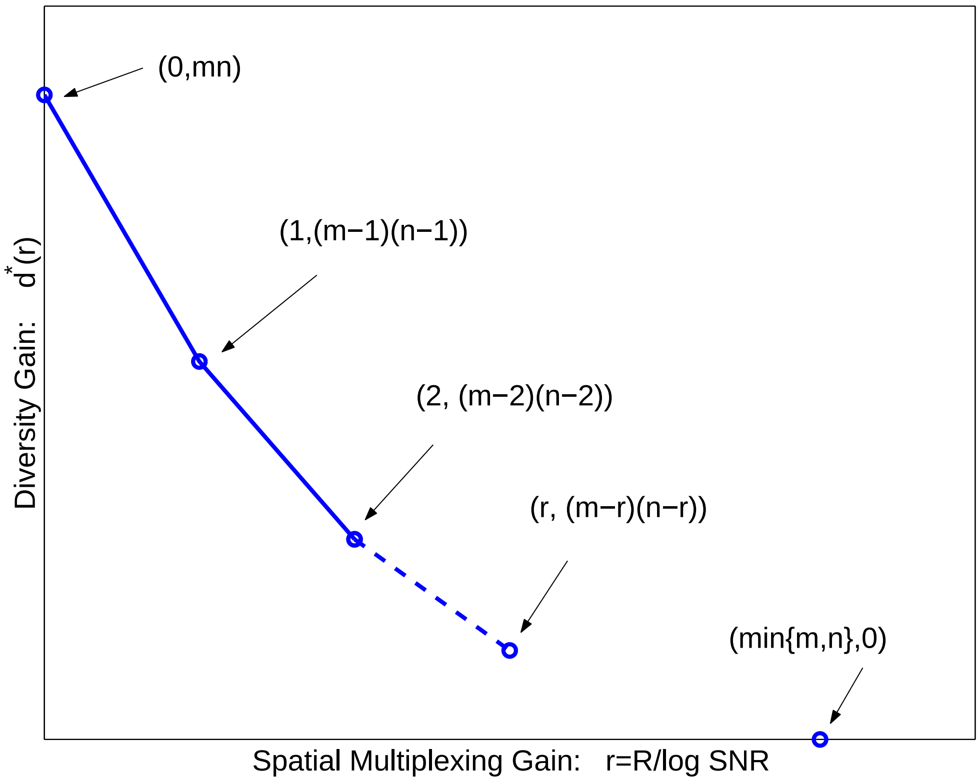


Optimal D-M Tradeoff for General $m \times n$ Channel

(Zheng and Tse 03)

As long as block length $l \geq m + n - 1$:



Optimal D-M Tradeoff for General $m \times n$ Channel

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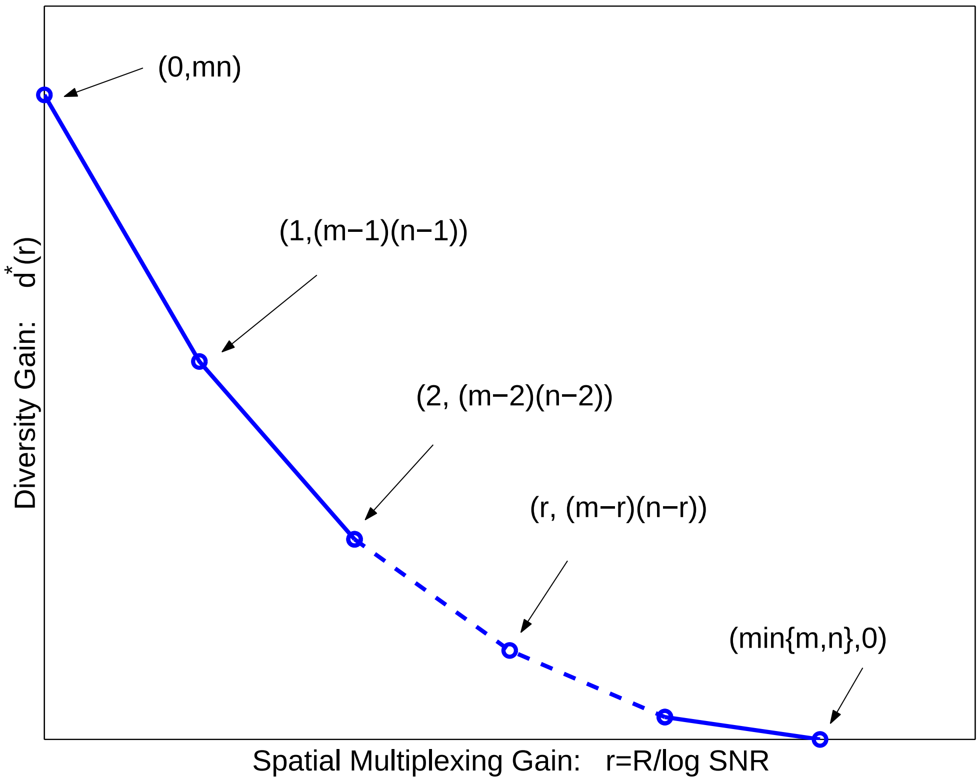
As long as block length $l \geq m + n - 1$:



For integer r, it is *as though* r transmit and r receive antennas were dedicated for multiplexing and the rest provide diversity.

Achieving Optimal Diversity-Multiplexing Tradeoff

- Hao and Wornell 03: MIMO rotation code (2×2 channel only).
- Tavildar and Viswanath 04: D-Blast plus permutation code.
- El Gamal, Caire and Damen 03: Lattice codes.

Hao and Wornell 03

Alamouti scheme:

$$\begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$$

Hao and Wornell's scheme:

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

where

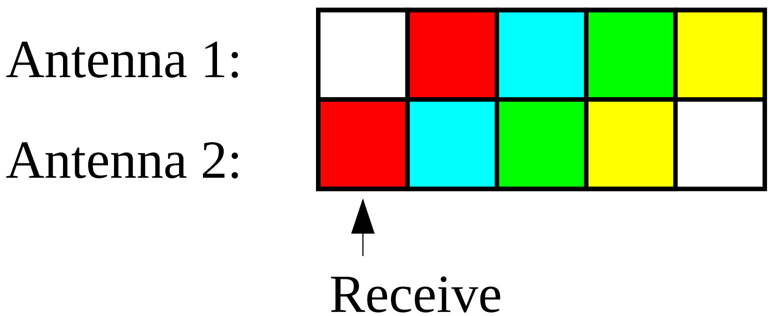
$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathrm{Rotate}(\theta_1^*) \begin{bmatrix} u_1 \\ u_4 \end{bmatrix} \qquad \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \mathrm{Rotate}(\theta_2^*) \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

and u_1, u_2, u_3, u_4 are independent QAM symbols.

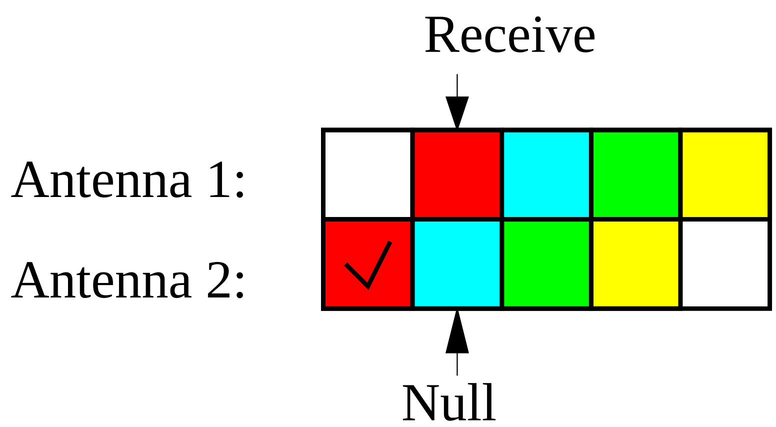
Tavildar and Viswanth 04

- First use D-Blast to convert the MIMO channel into a parallel channel.
- Then design permutation codes to achieve the optimal diversity-multiplexing tradeoff on the parallel channel.

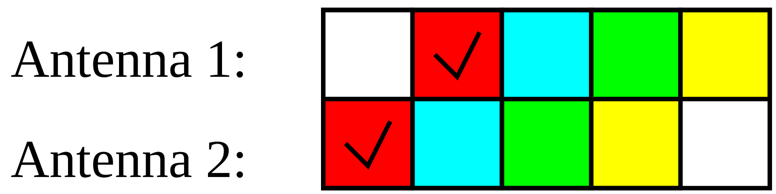
D-BLAST



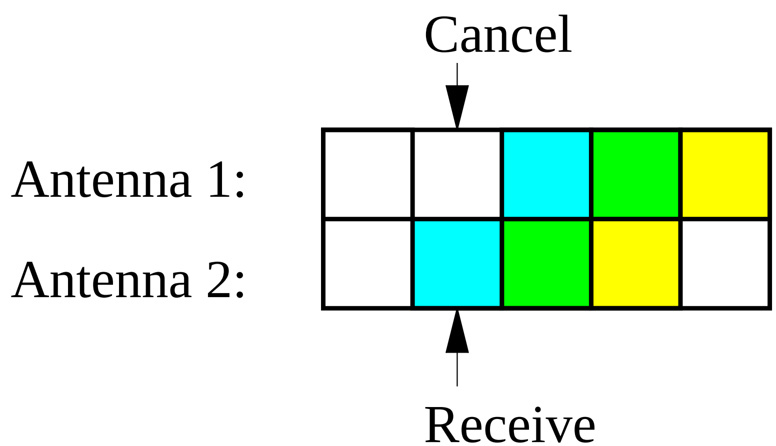
D-BLAST



D-BLAST

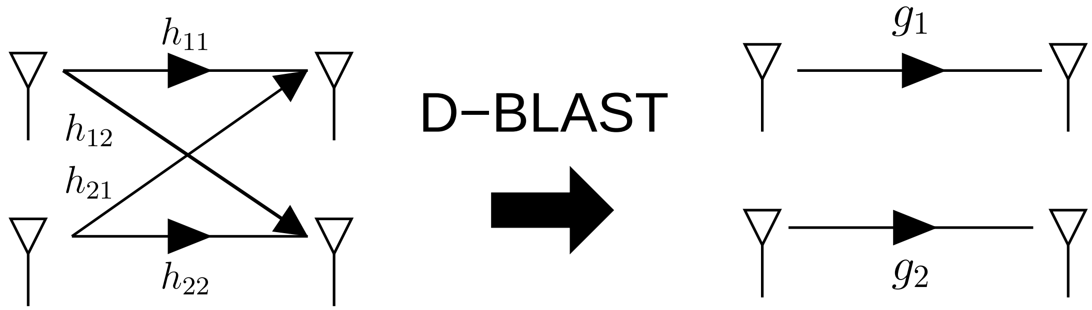


D-BLAST



Original D-Blast is sub-optimal.

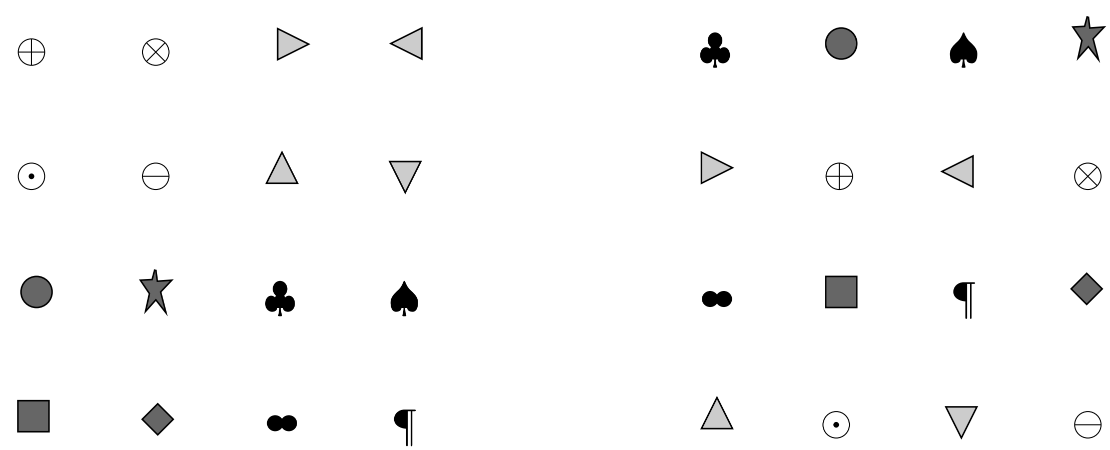
D-Blast with MMSE suppression is information lossless



Permutation Coding for Parallel Channel

The channel is parallel but the fading at the different sub-channels are correlated.

Nevertheless it is shown that the permutation codes can achieve the optimal diversity-multiplexing tradeoff of the parallel channel.



Conclusion

Diversity-multiplexing tradeoff is a unified way to look at space-time code design for MIMO channels.

It puts diversity and multiplexing on an equal footing.

It provides a framework to compare existing schemes as well as stimulates the design of new schemes.