Diversity and Freedom:
A Fundamental Tradeoff in Multiple Antenna Channels

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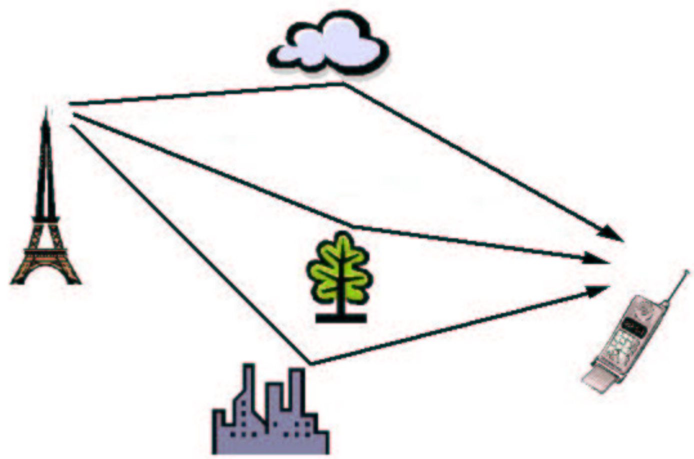
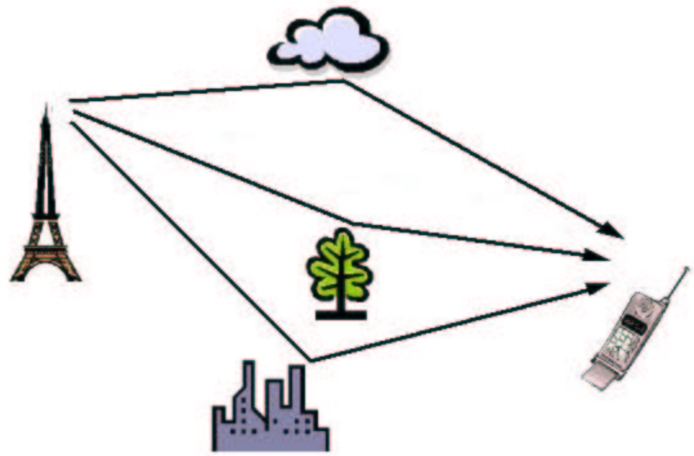
EPFL

Wireless Fading Channel



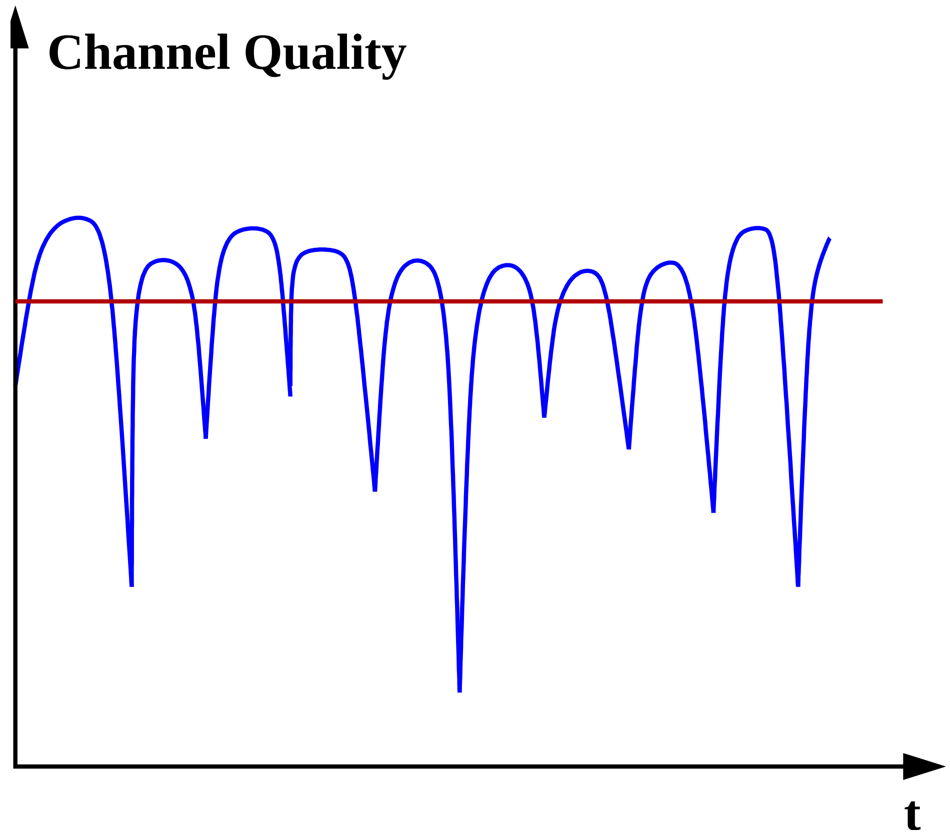
- Fundamental characteristic of wireless channels: multi-path fading.

Wireless Fading Channel



- Fundamental characteristic of wireless channels: multi-path fading.
- Two different views of fading have emerged.

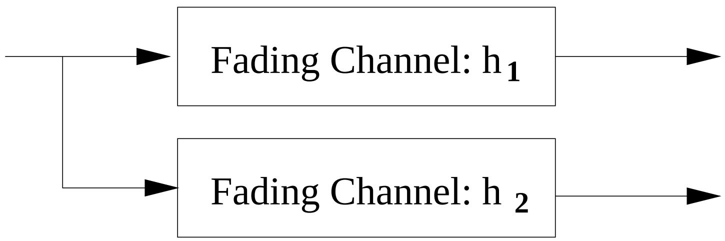
View #1: Fading is Bad



Fading Channels have poor reliability

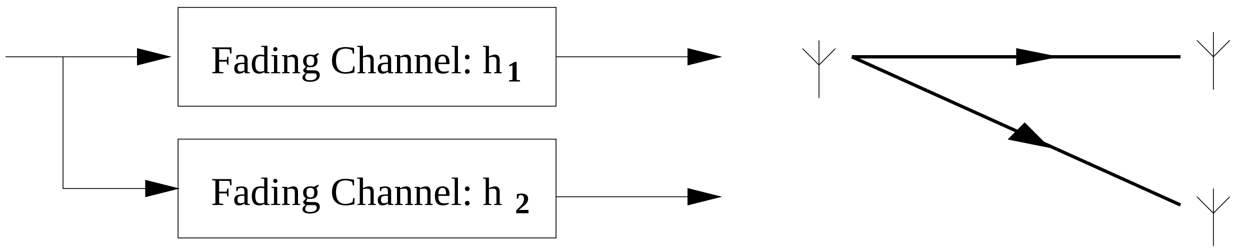
Diversity

Diversity



- Additional independent fading channels increase diversity.

Diversity



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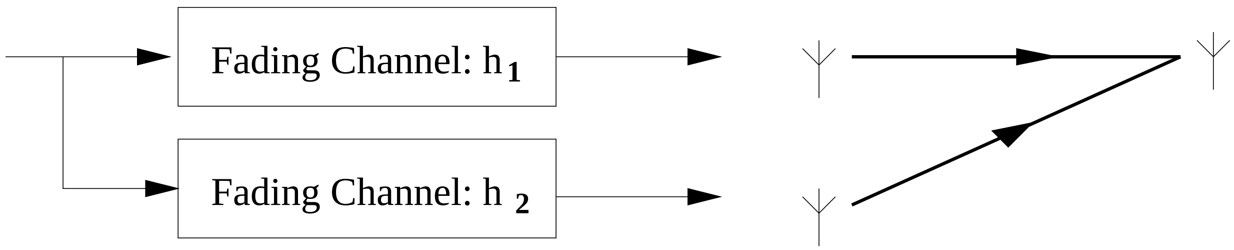
- Spatial diversity

Diversity



- Additional independent fading channels increase diversity.
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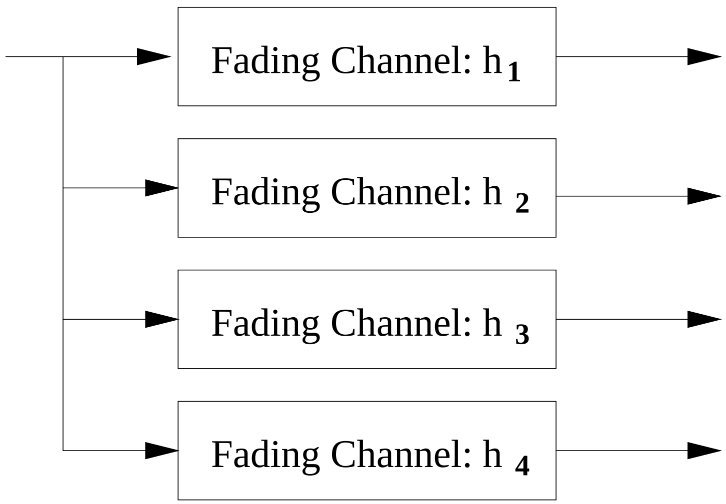
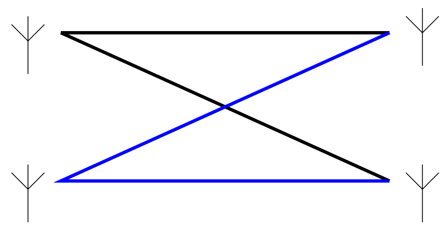
Diversity





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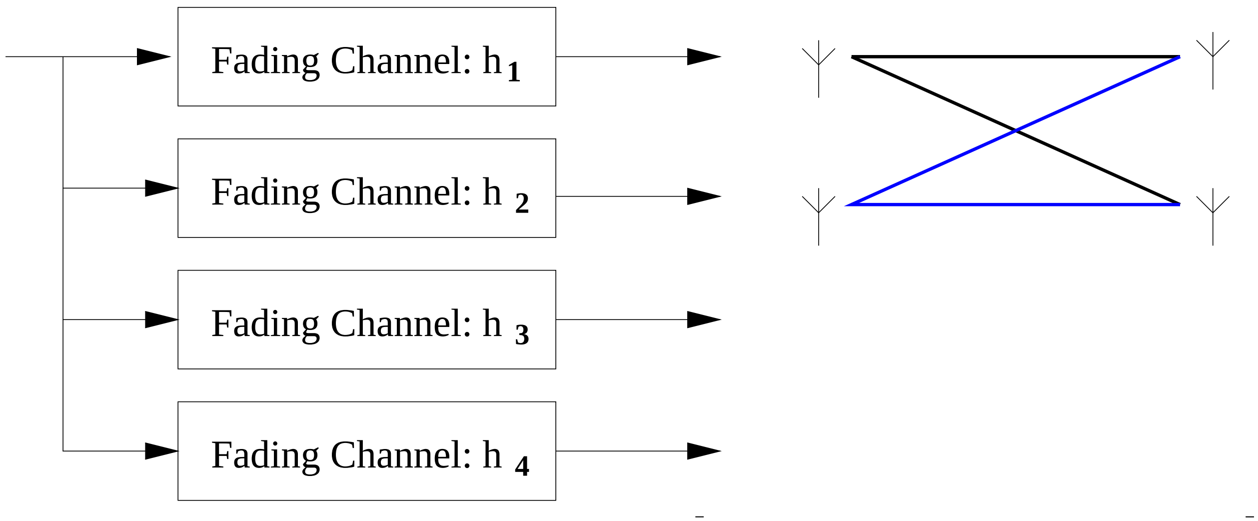
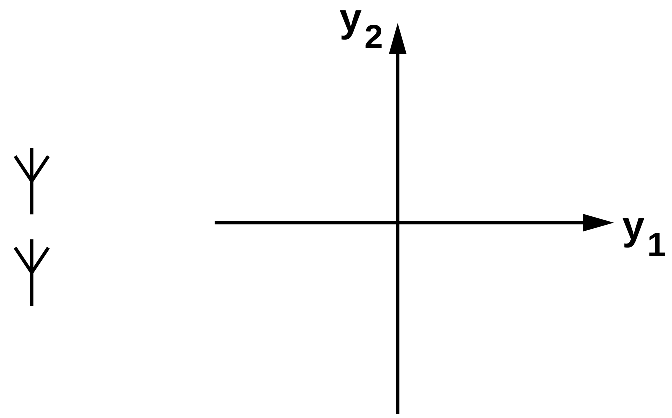
Diversity



- Additional independent fading channels increase diversity.
- Spatial diversity: receive, transmit or both.
- Repeat and Average: compensate against channel unreliability.

View #2: Fading is Good



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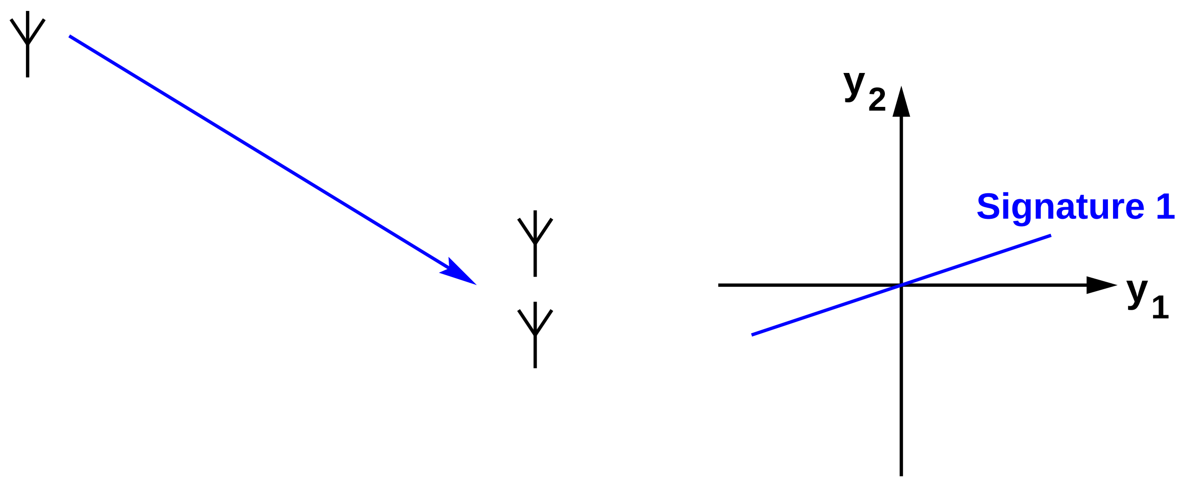

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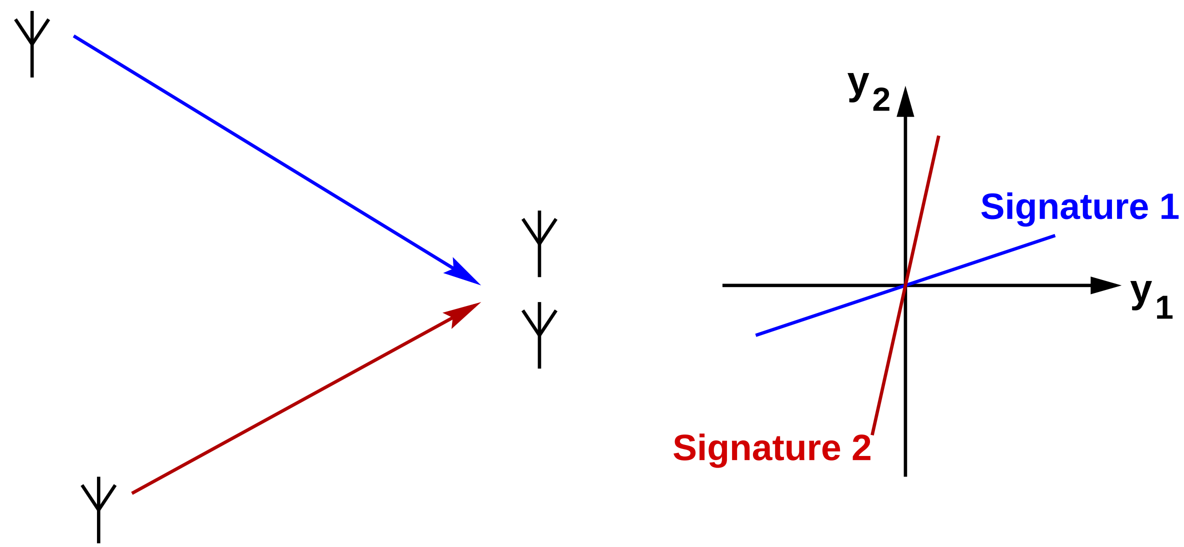

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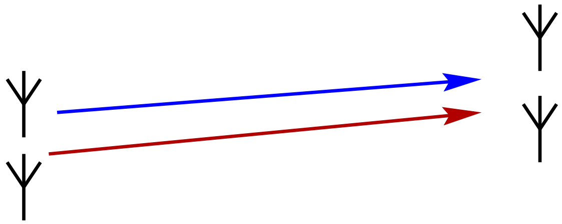

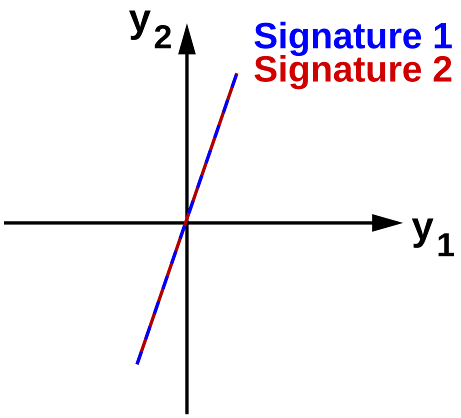

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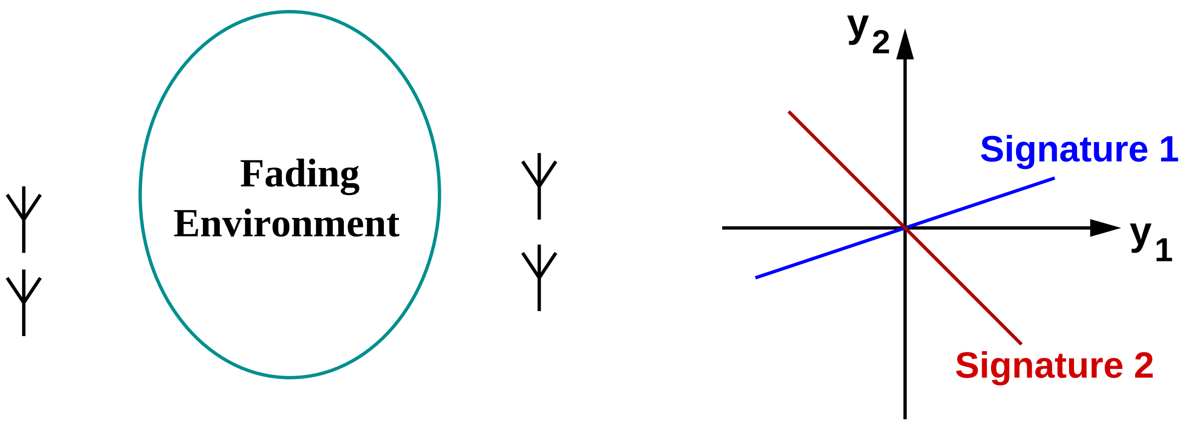
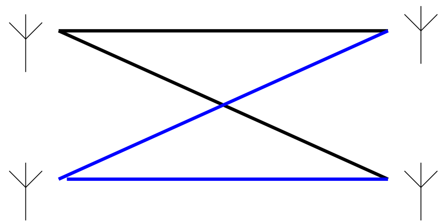


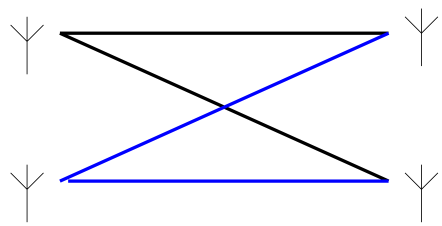
Multiple spatial channels created even when antennas are close together.

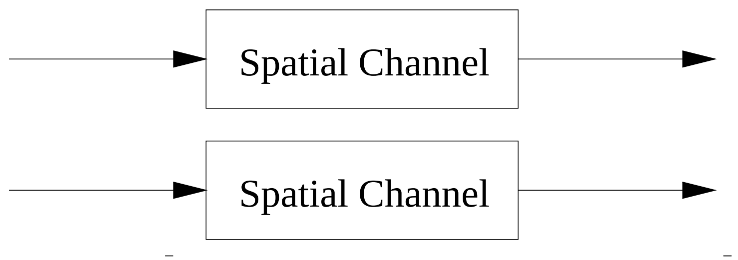
Freedom



Another way to view a 2×2 system:

Freedom

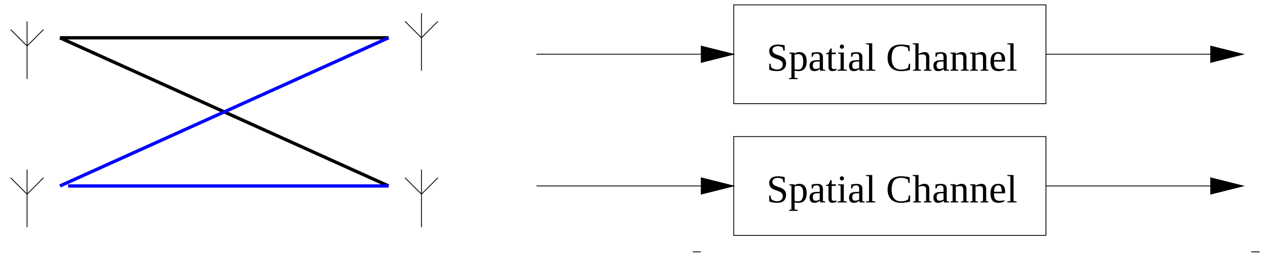




Another way to view a 2×2 system:

- Increases the degrees of freedom in the system.

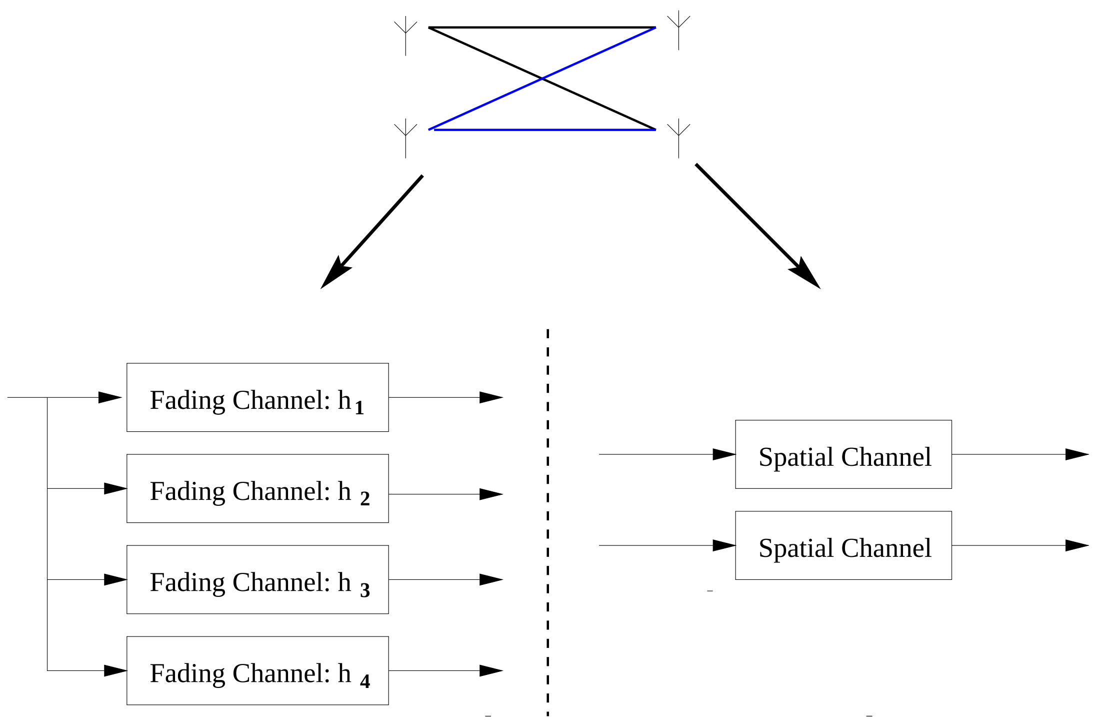
Freedom



Another way to view a 2×2 system:

- Increases the degrees of freedom in the system.
- Multiple antennas provide parallel spatial channels: spatial multiplexing

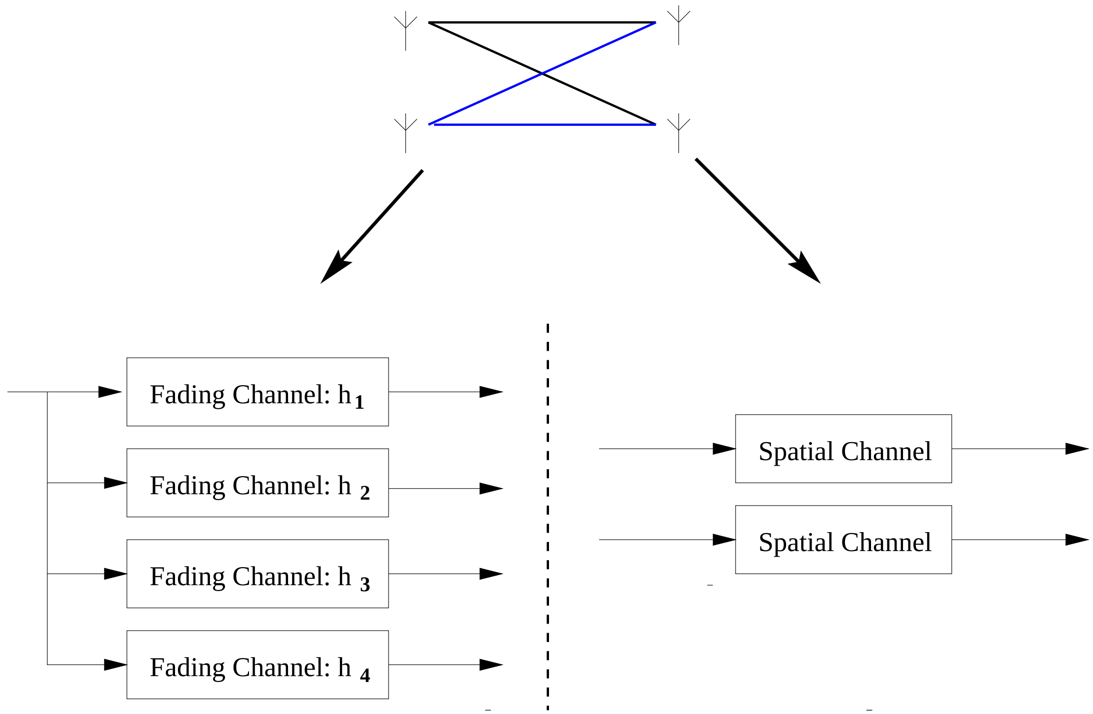
Diversity vs. Multiplexing



Multiple antenna channel provides two types of gains:

Diversity Gain vs. Spatial Multiplexing Gain

Diversity vs. Multiplexing



Multiple antenna channel provides two types of gains:

Diversity Gain vs. Spatial Multiplexing Gain

The research community has a split personality: existing schemes focus on one type of gain.

A Different Point of View

Both types of gains can be achieved simultaneously in a given multiple antenna channel

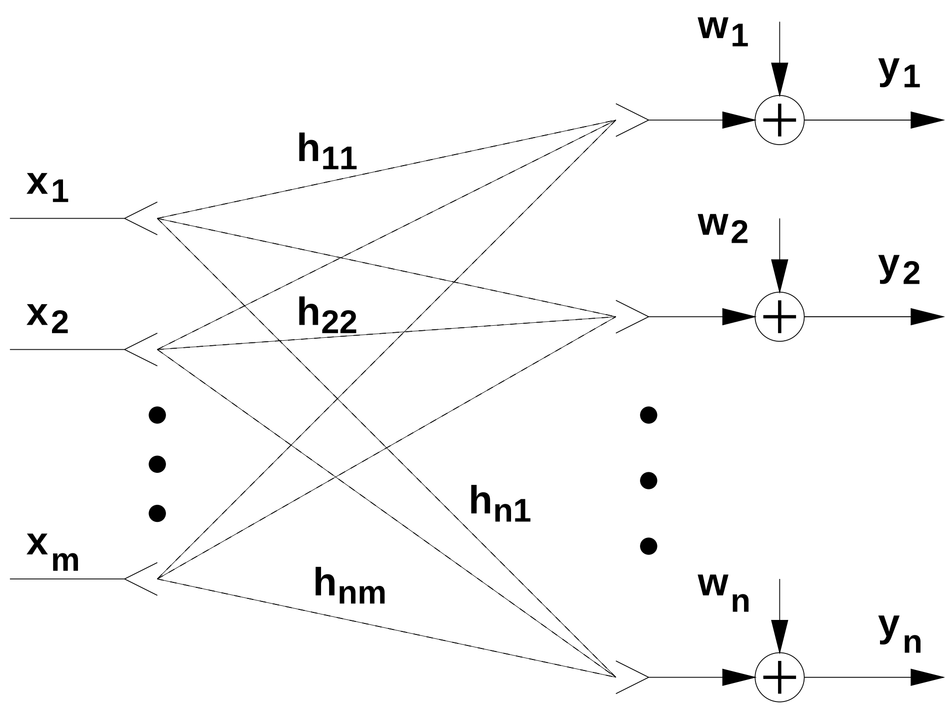
A Different Point of View

Both types of gains can be achieved simultaneously in a given multiple antenna channel, but there is a fundamental tradeoff.

Outline

- Precise problem formulation.
- A simple characterization of the optimal tradeoff.
- Sketch of proof.
- Tradeoff as a unified framework to compare different schemes.

Channel Model



$$\mathbf{y}_t = \mathbf{H}_t \mathbf{x}_t + \mathbf{w}_t, \qquad \mathbf{w}_t \sim \mathcal{CN}(0, 1)$$

- Rayleigh flat fading i.i.d. across antenna pairs ($h_{ij} \sim \mathcal{CN}(0, 1)$).
- SNR is the average signal-to-noise ratio at each receive antenna.

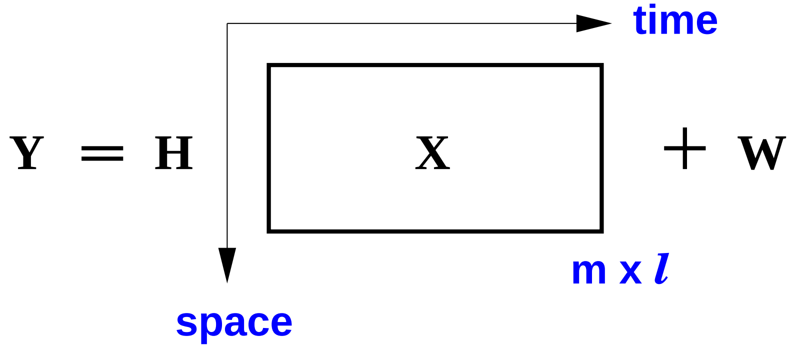
Coherent Block Fading Model

- Focus on codes over l symbols, where $\mathbf{H}$ remains constant.
- $\mathbf{H}$ is known to the receiver but not the transmitter.
- Assumption valid as long as

$$l \ll \text{coherence time} \times \text{coherence bandwidth.}$$

Space-Time Block Code

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$$



Focus on coding over a single block of length l.

How to Define Diversity Gain

Motivation: Binary Detection

$$\mathbf{y} = \mathbf{h}\mathbf{x} + \mathbf{w} \qquad P_e \approx P(\|\mathbf{h}\| \text{ is small }) \propto \mathsf{SNR}^{-1}$$

$$\left.\begin{aligned} \mathbf{y}_1 &= \mathbf{h}_1\mathbf{x} + \mathbf{w}_1 \\ \mathbf{y}_2 &= \mathbf{h}_2\mathbf{x} + \mathbf{w}_2 \end{aligned}\right\} \qquad \begin{aligned} P_e &\approx P(\|\mathbf{h}_1\|, \|\mathbf{h}_2\| \text{ are both small}) \\ &\propto \mathsf{SNR}^{-2} \end{aligned}$$

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General Definition

A space-time coding scheme achieves diversity gain d, if

$$P_e(\text{SNR}) \sim \text{SNR}^{-d}$$

How to Define Spatial Multiplexing Gain

Motivation: Channel capacity (Telatar '95, Foschini'96)

$$C(\mathsf{SNR}) \approx \min\{m,n\} \log \mathsf{SNR} (bps/Hz)$$

$\min\{m,n\}$ degrees of freedom to communicate.

How to Define Spatial Multiplexing Gain

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$$C(\mathsf{SNR}) \approx \min\{m, n\} \log \mathsf{SNR} (bps/Hz)$$

$\min\{m, n\}$ degrees of freedom to communicate.

Definition A space-time coding scheme achieves spatial multiplexing gain r, if

$$R(\mathsf{SNR}) = r \log \mathsf{SNR} (bps/Hz)$$

Fundamental Tradeoff

A space-time coding scheme achieves

Spatial Multiplexing Gain r : $R = r \log \mathsf{SNR} \quad (bps/Hz)$

and

Diversity Gain d : $P_e \approx \mathsf{SNR}^{-d}$

Fundamental Tradeoff

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Fundamental tradeoff: for any r, the maximum diversity gain achievable: $d^*(r)$.

$$r \longrightarrow d^*(r)$$

Fundamental Tradeoff

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$$r \longrightarrow d^*(r)$$

A tradeoff between data rate and error probability.

Outline

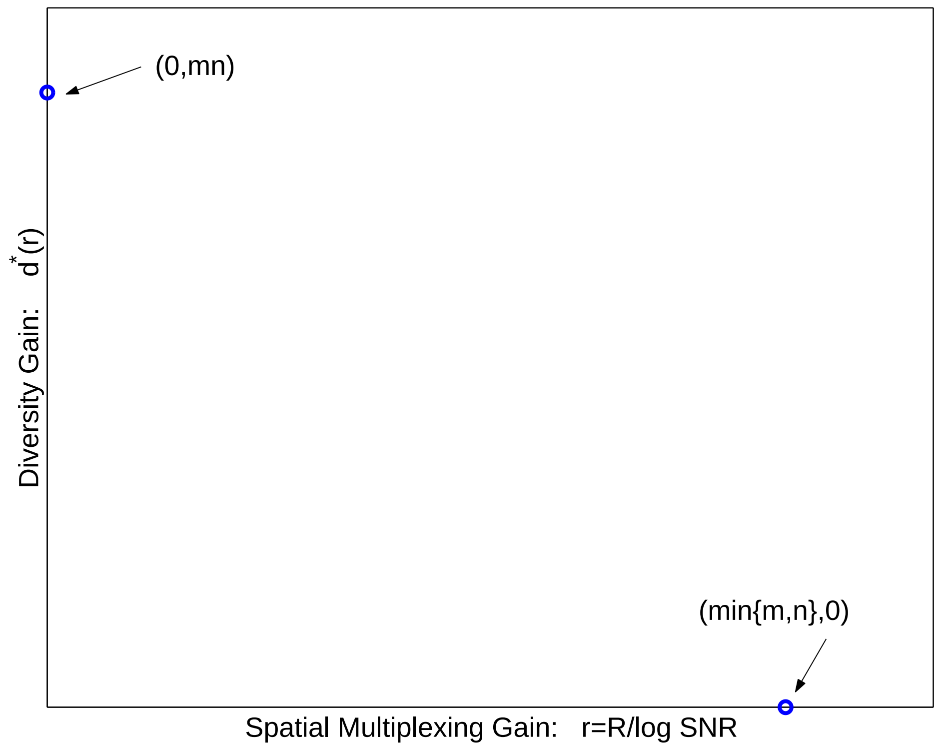
- Precise problem formulation.
- A simple characterization of the optimal tradeoff.
- Sketch of proof.
- Tradeoff as a unified framework to compare different schemes.

Main Result: Optimal Tradeoff

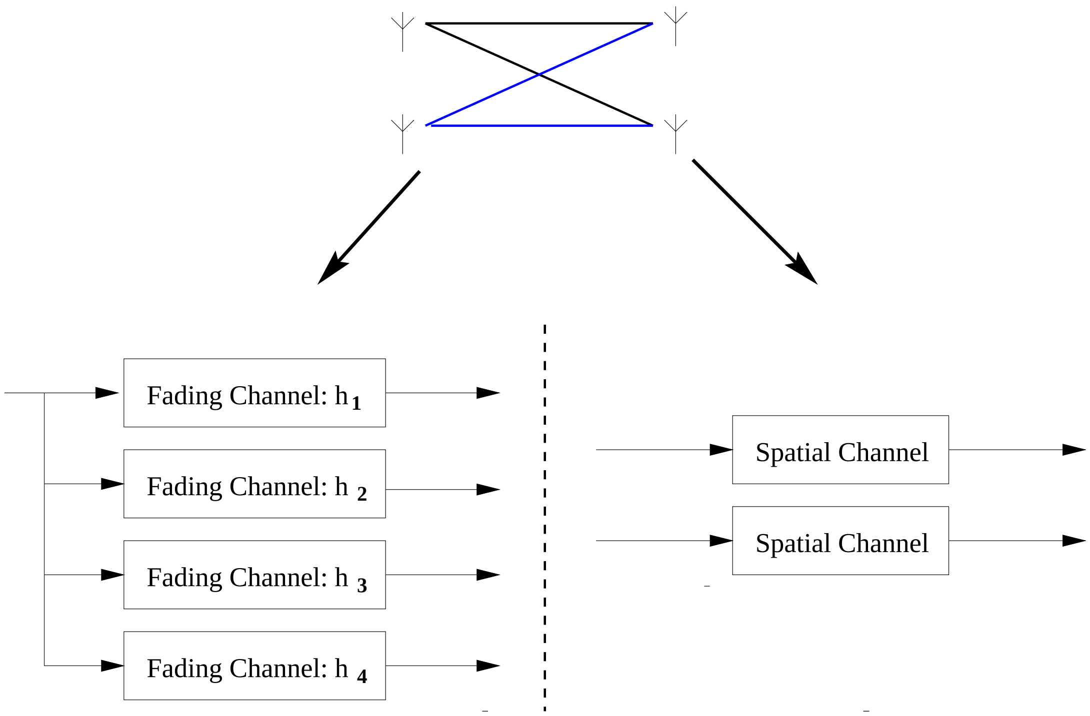
m: # of Tx. Ant.
n: # of Rx. Ant.
l: block length
$l \geq m + n - 1$

d: diversity gain
$P_e \approx \text{SNR}^{-d}$

r: multiplexing gain
$R = r \log \text{SNR}$



Recall: 2×2 system



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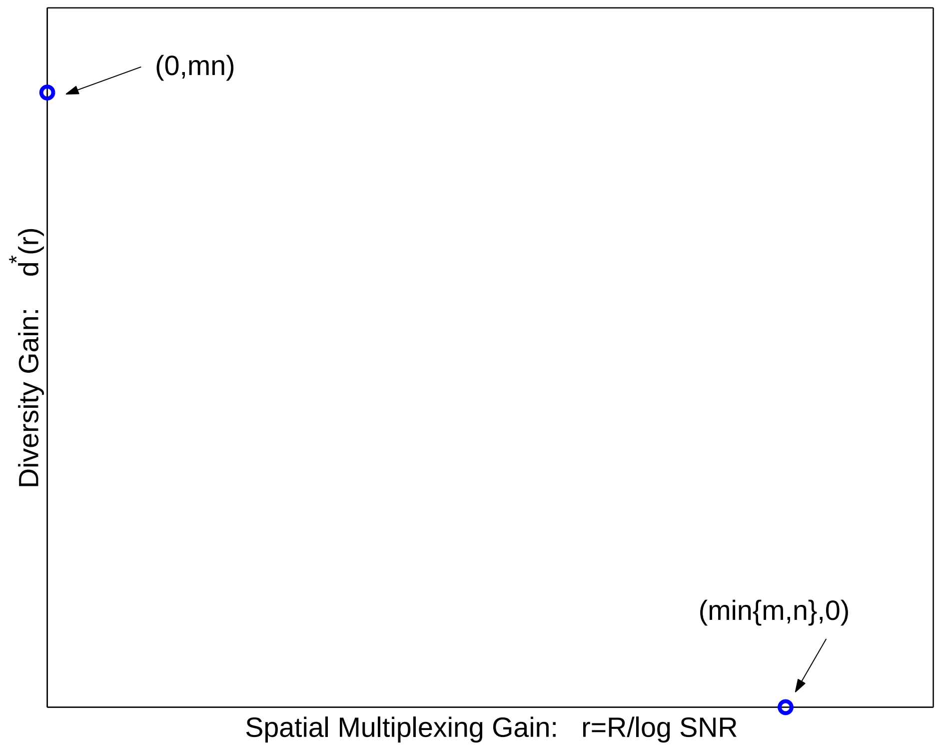

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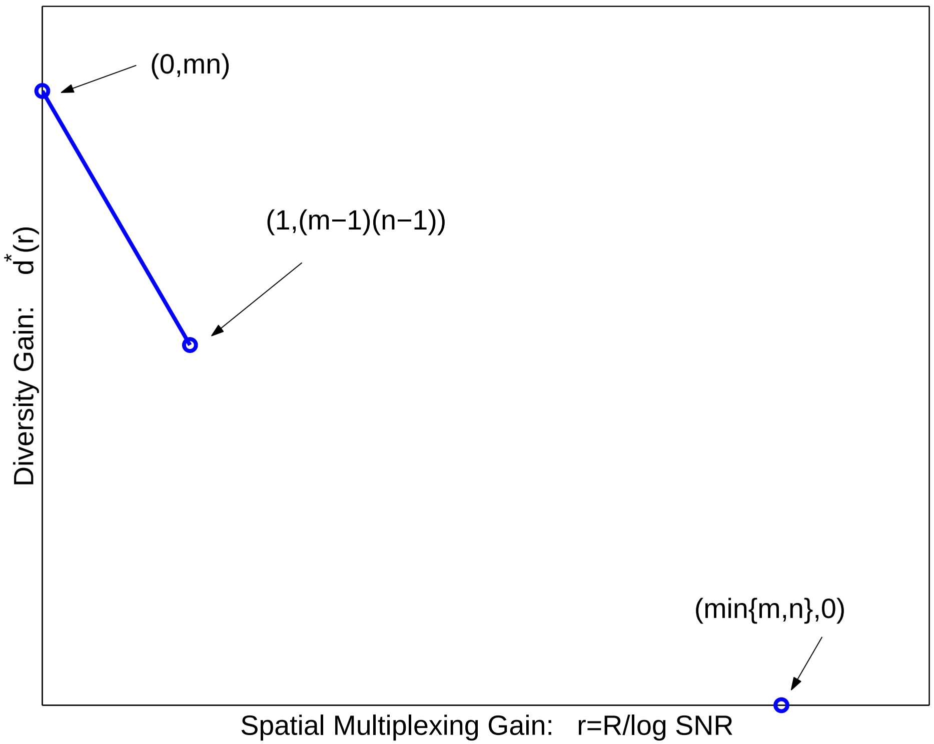

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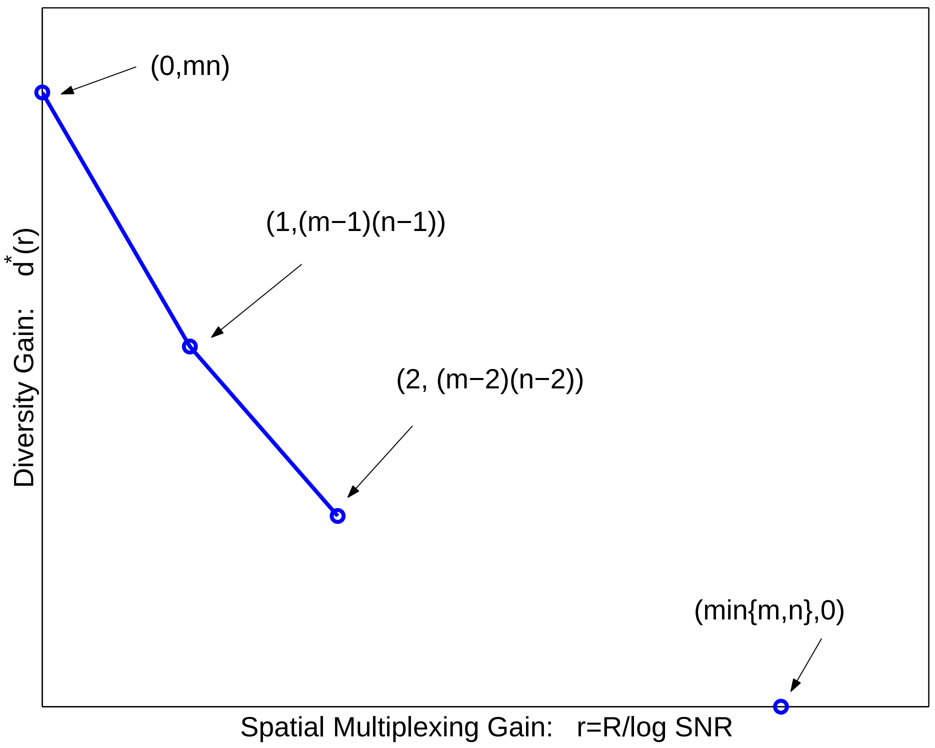

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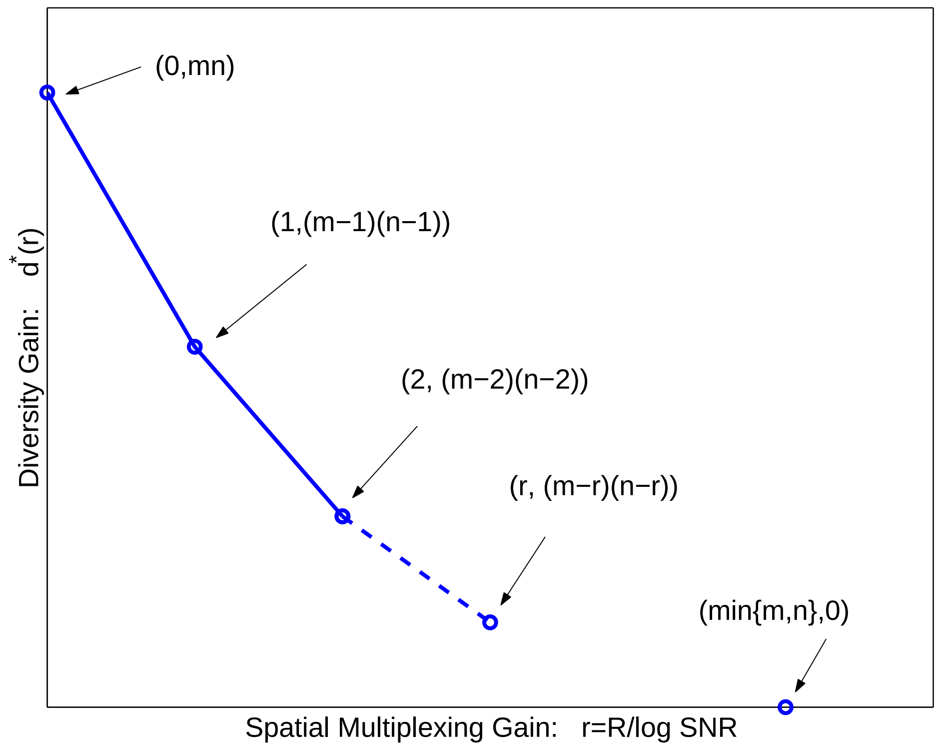

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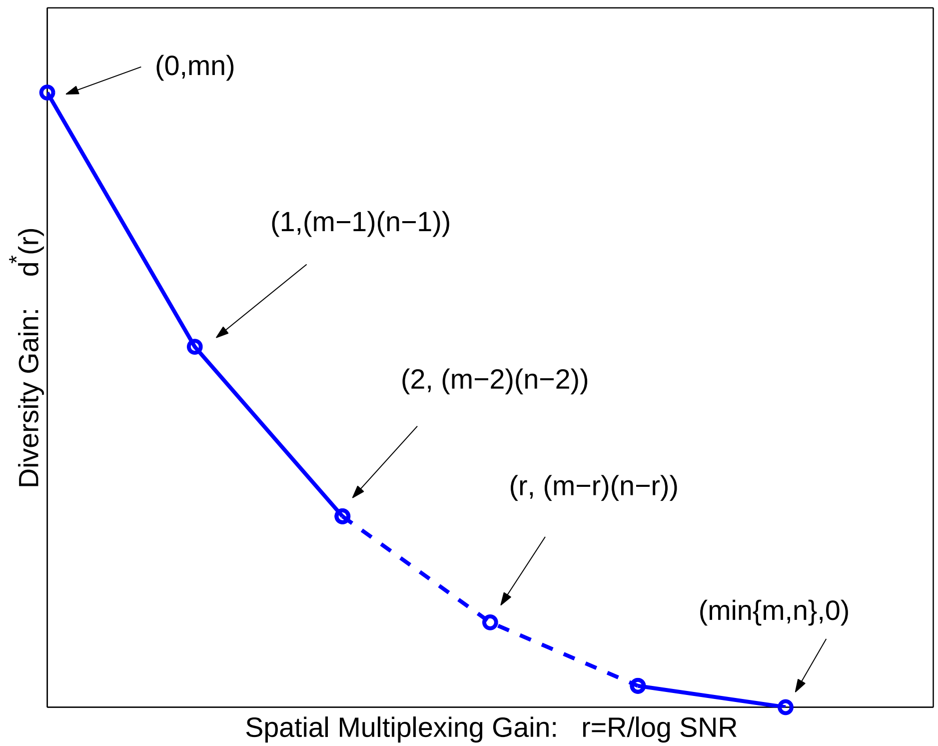

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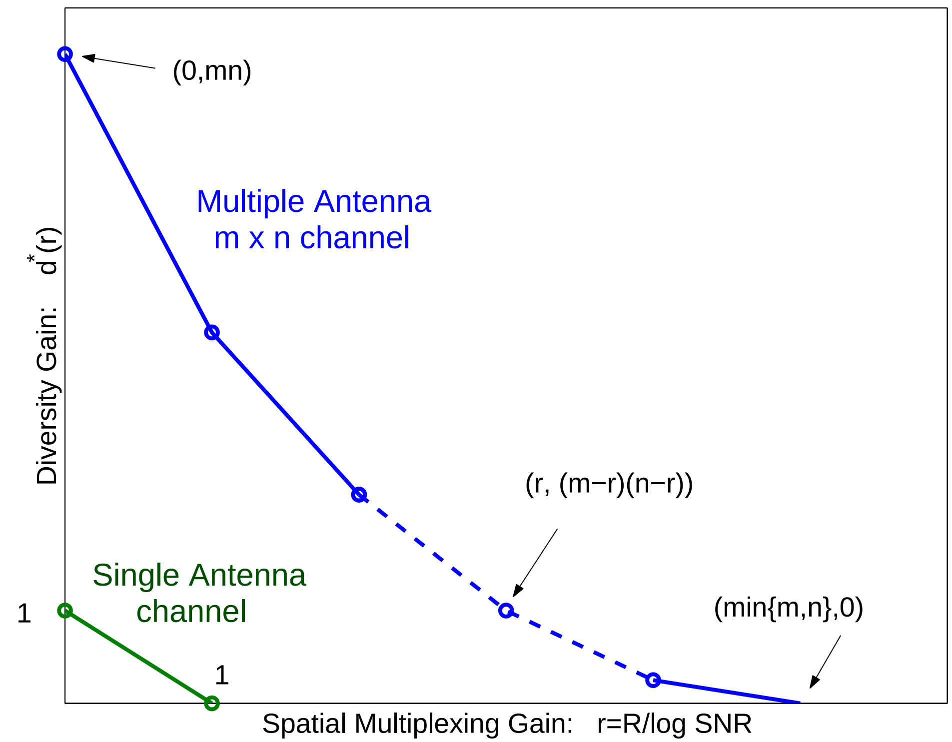

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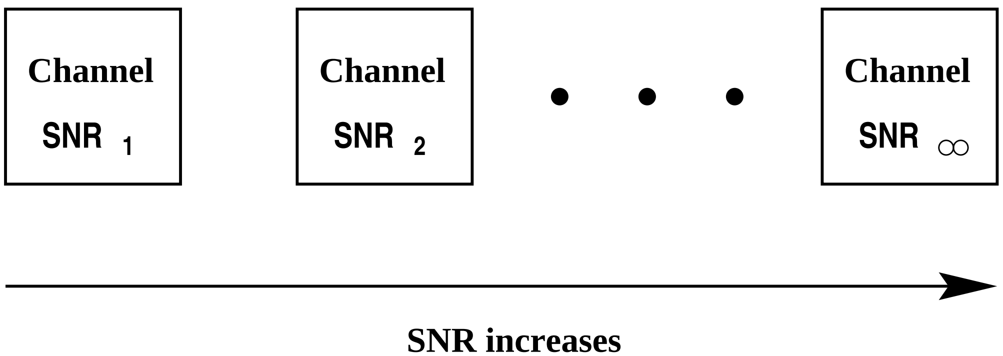
Main Result: Optimal Tradeoff

Theorem: In a channel with m transmit and n receive antennas, for any space-time coding scheme with block length $l \geq m+n-1$ that achieves spatial multiplexing gain of r, the error probability is lower bounded by:

$$\lim_{\mathsf{SNR}\to\infty} \frac{\log P_e(\mathsf{SNR})}{\log \mathsf{SNR}} \geq -d^*(r)$$

Furthermore, there exists a coding scheme that achieves this lower bound with equality.

A Conceptual Picture



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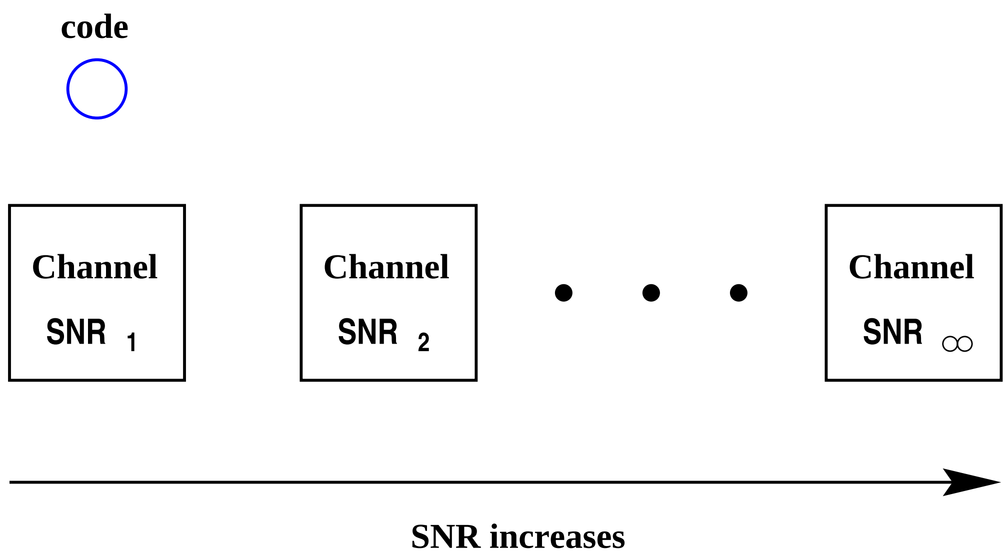

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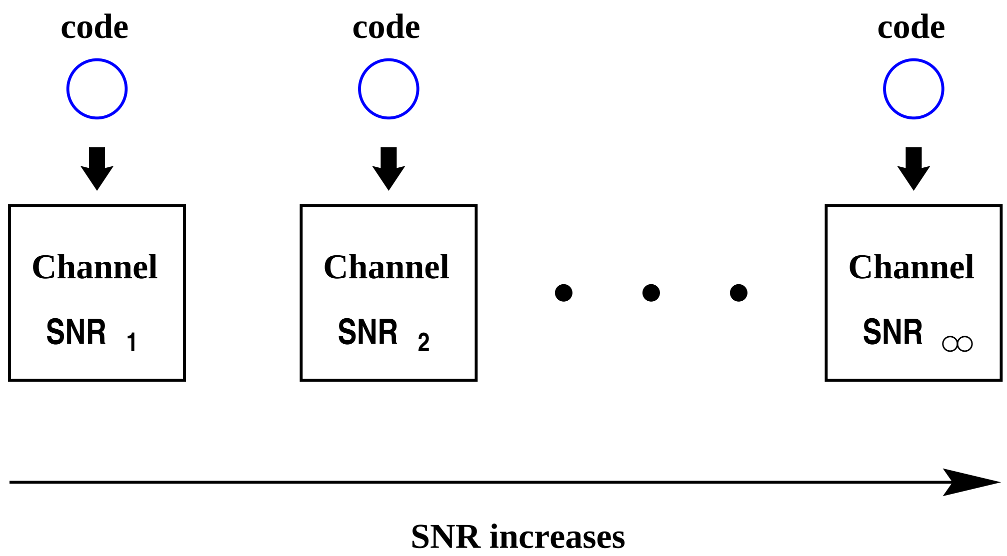

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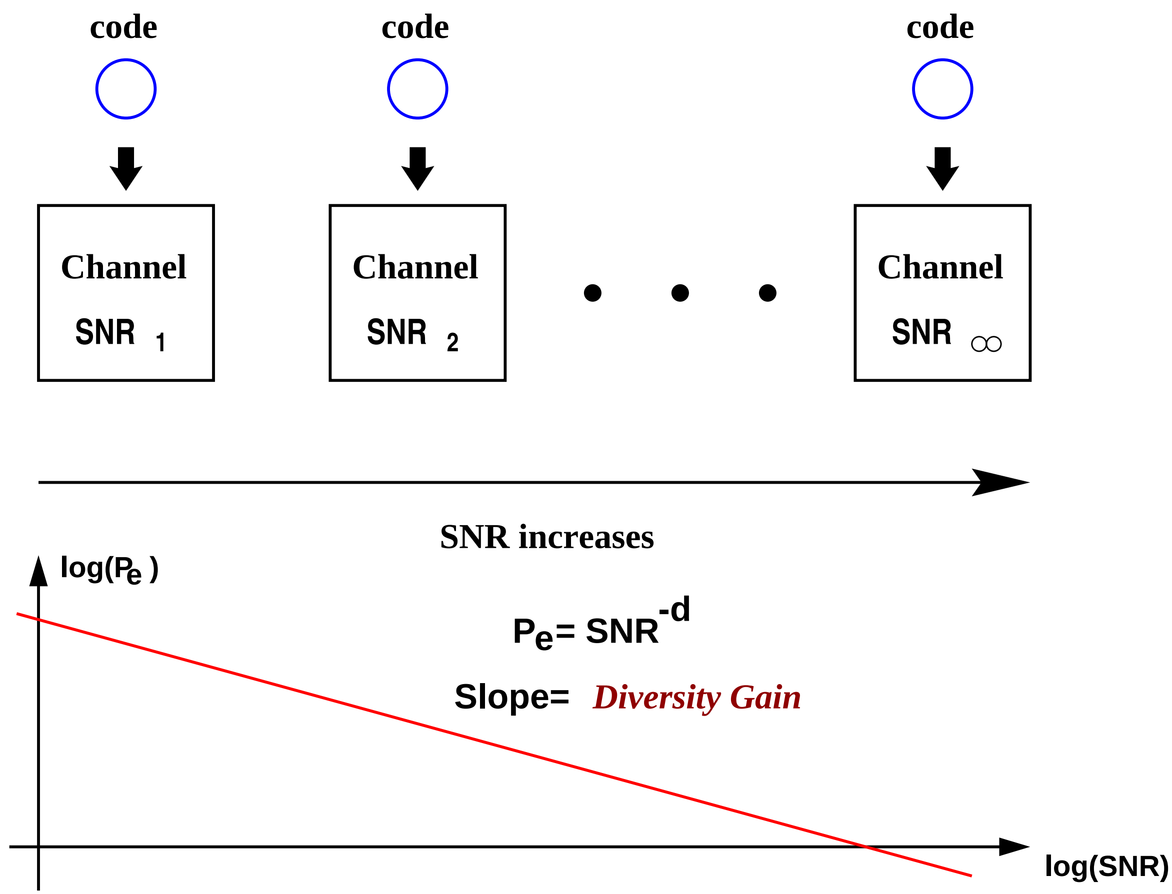

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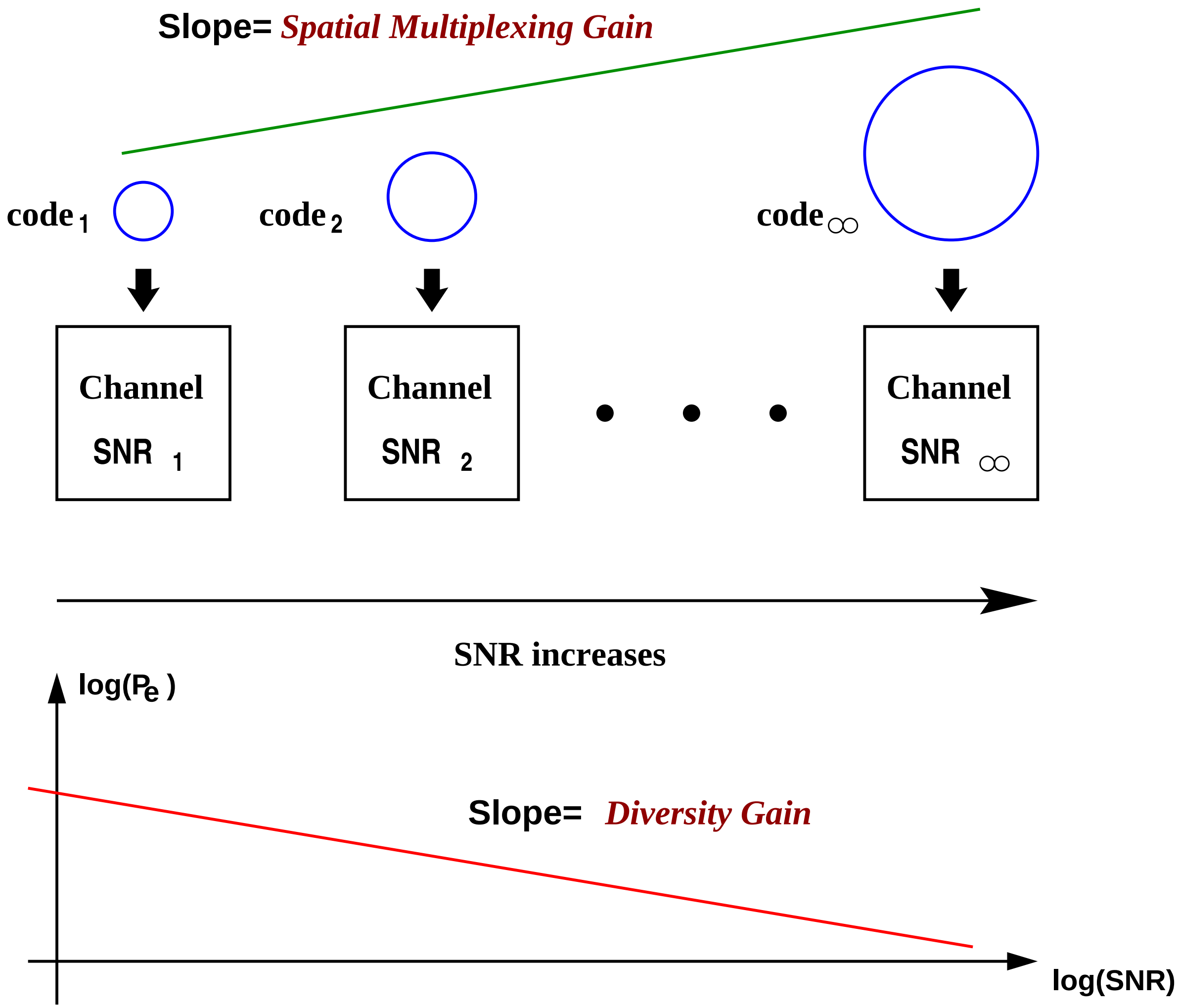

What do I get by adding one more antenna?

Adding More Antennas

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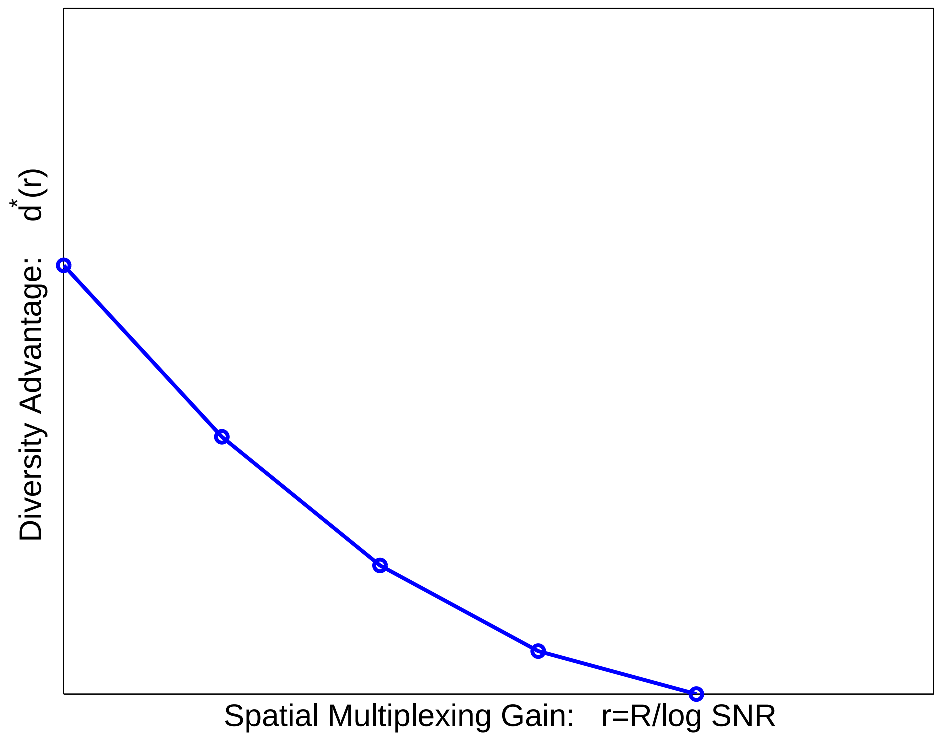
r: multiplexing gain



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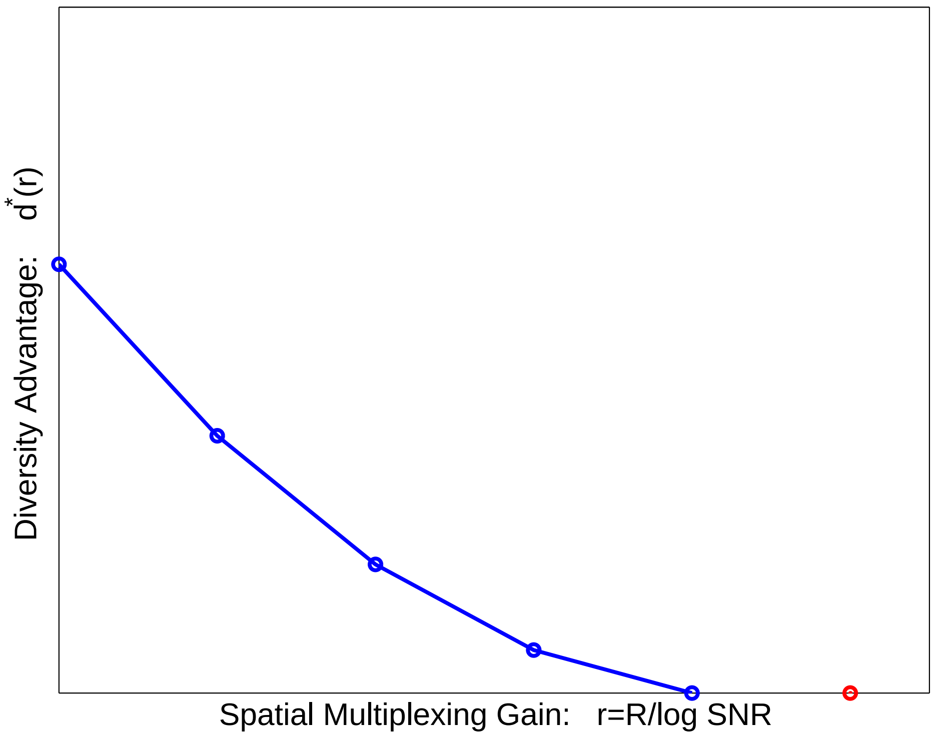

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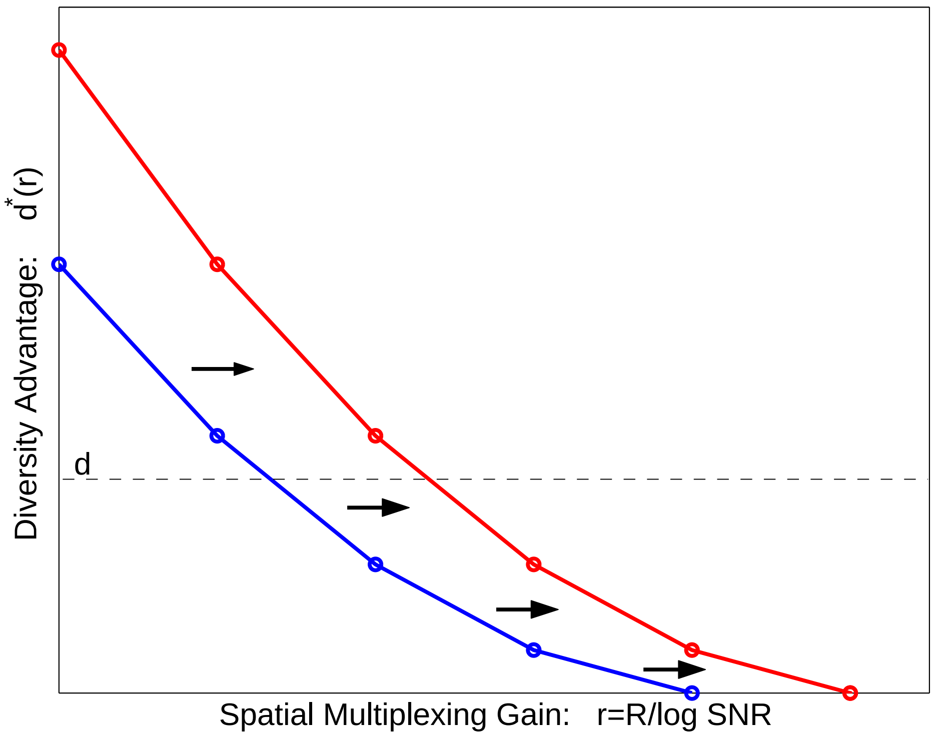

- Capacity result: increasing $\min\{m, n\}$ by 1 adds 1 more degree of freedom.
- Tradeoff curve: increasing both m and n by 1 yields multiplexing gain $+1$ for any diversity requirement d.

To Fade or Not to Fade?

Fading

LOS Non-Fading AWGN Channel

$$\mathbf{y} = \mathbf{x} + \mathbf{w}$$

Fading Channel

$$\mathbf{y} = \mathbf{h}\mathbf{x} + \mathbf{w}$$

Fading

LOS Non-Fading AWGN Channel

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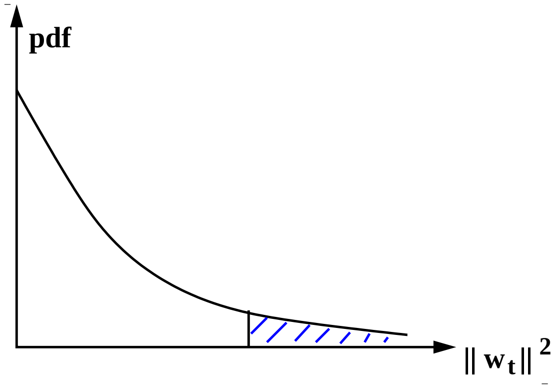
Error occurs when noise is large, with probability

$$P_e \approx \exp(-\mathsf{SNR})$$



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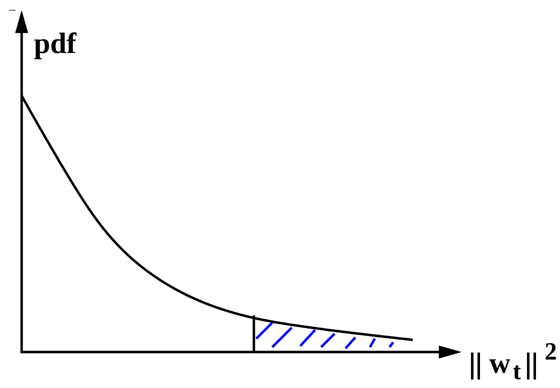
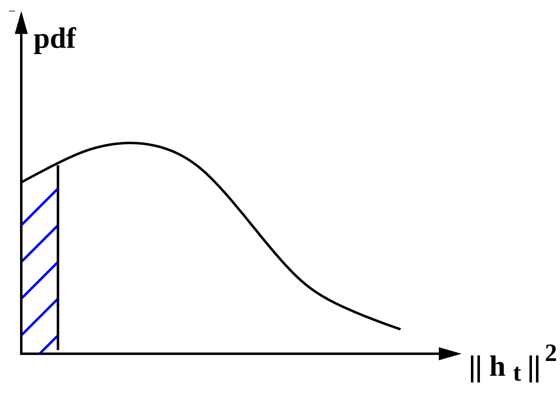


Fading Channel

$$\mathbf{y} = \mathbf{h}\mathbf{x} + \mathbf{w}$$

Error occurs when the channel is in deep fade, with probability

$$P_e \approx \mathsf{SNR}^{-1}$$



Fading

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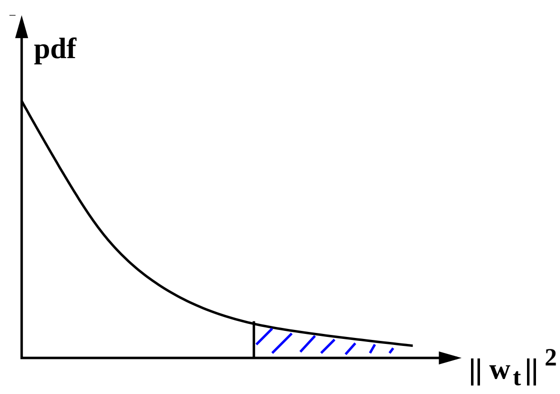

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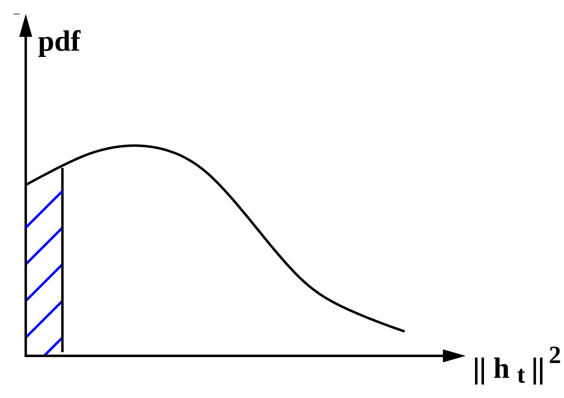

Deep fades are major cause of error

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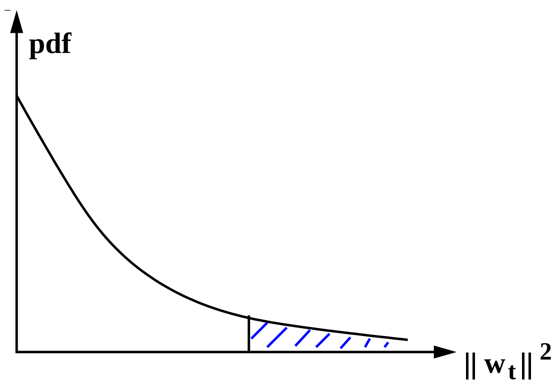

Infinite diversity.

Deep fades are major cause of error

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Error occurs when the channel is in deep fade, with probability

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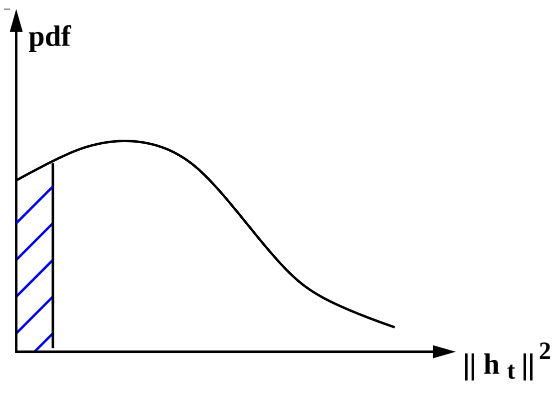
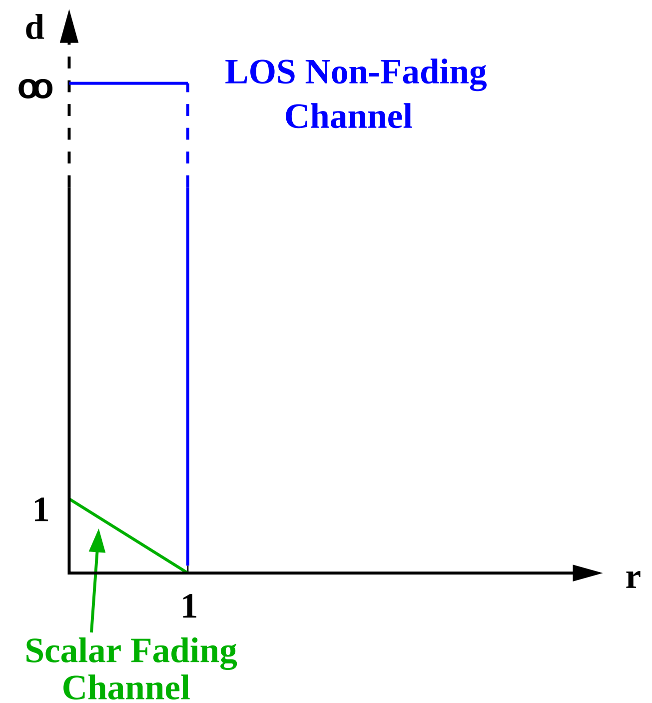


Need to increase diversity

Line-of-Sight vs Fading Channel

m: # of Tx. Ant.
n: # of Rx. Ant.

d: Diversity gain
r: Multiplexing Gain



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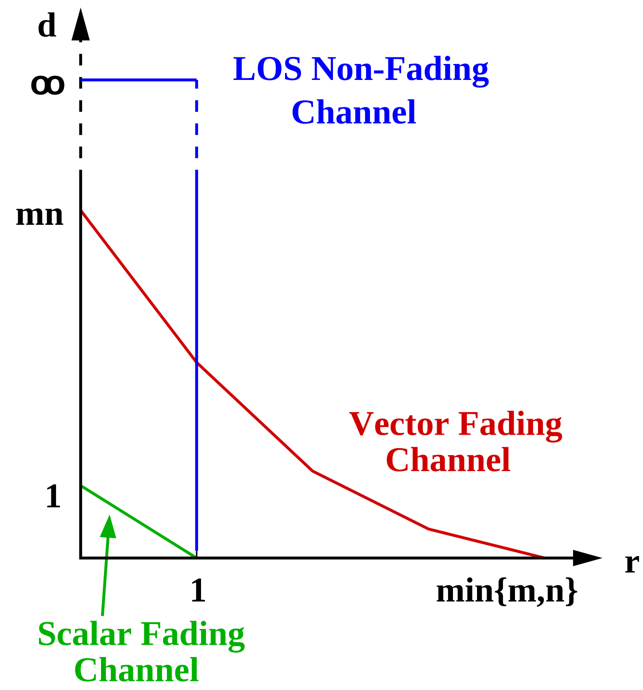

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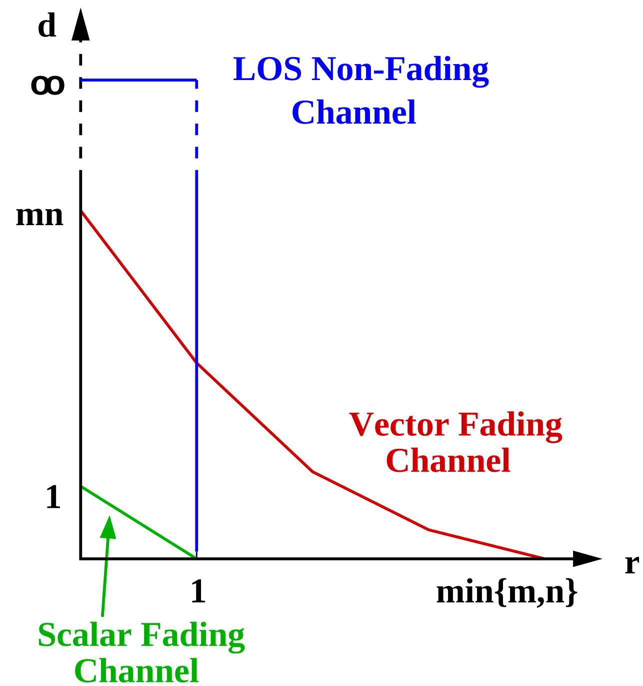

- In a scalar 1×1 system, line-of-sight AWGN is better.
- In a vector $m \times n$ system, fading is exploited as a source of randomness to provide multiple degrees of freedom.
- Whether fading is good or bad depends on where you operate.

Outline

- Precise problem formulation.
- A simple characterization of the optimal tradeoff.
- Sketch of proof.
- Tradeoff as a unified framework to compare different schemes.

Converse: Outage Bound

- Outage formulation for quasi-static scenarios. (Ozarow et al 94, Telatar 95)
- Look at the mutual information per symbol $I(Q, \mathbf{H})$ as a function of the input distribution and channel realization.

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- At high SNR, i.i.d. Gaussian input Q^* is asymptotically optimal, and

$$I(Q^*, \mathbf{H}) = \log\det\left[I + \mathsf{SNR}\mathbf{H}\mathbf{H}^* \right].$$

Outage Analysis

$$P(\text{Outage}) = P\{I(\mathbf{H}) = \log\det[I + \mathsf{SNR}\mathbf{H}\mathbf{H}^\dagger] < R\}$$

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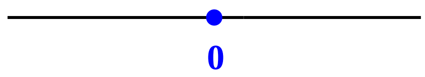
- In general $m \times n$ channel, outage occurs when some or all of the singular values of $\mathbf{H}$ are small. There are many ways for this to happen.
- Let $\mathbf{v}$ = vector of singular values of $\mathbf{H}$:
  Laplace Principle:

$$P(\text{Outage}) \approx \min_{\mathbf{v} \in \mathbf{Out}} \mathsf{SNR}^{-f(\mathbf{v})}$$

Geometric Picture (integer r)

Scalar Channel



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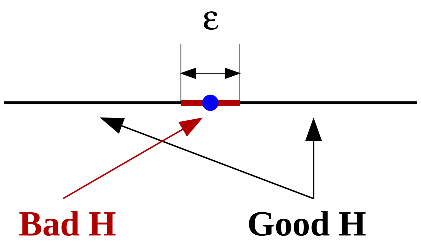

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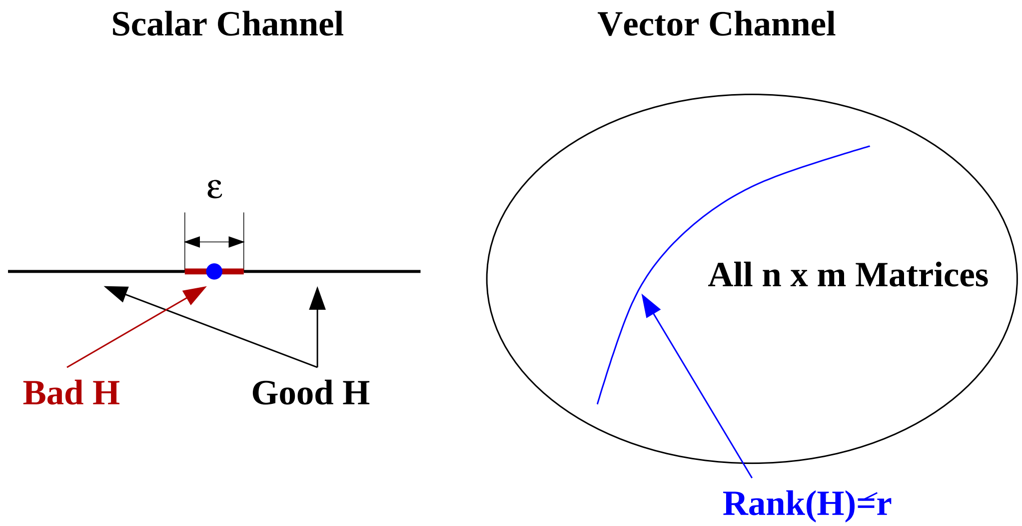

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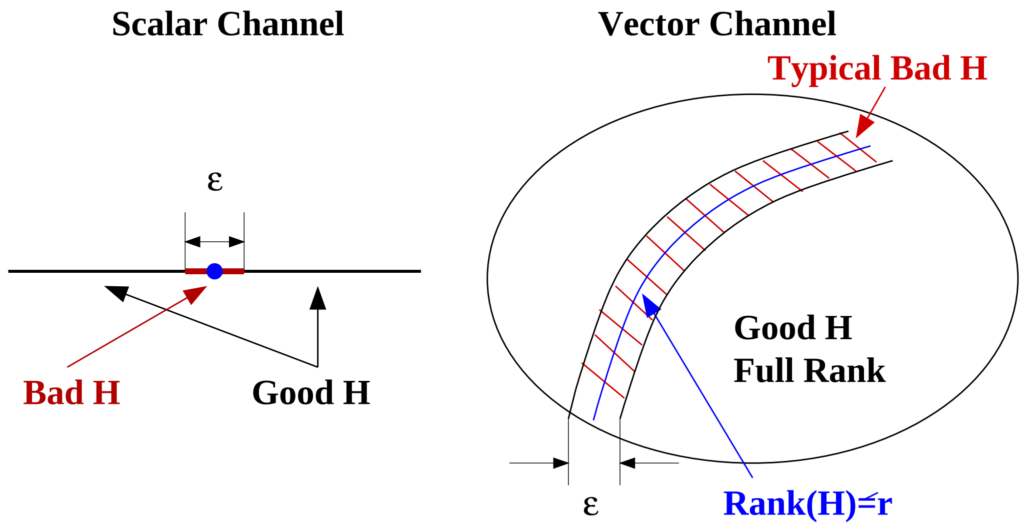

Result: At rate $R = r \log \mathsf{SNR}$, for r integer, outage occurs typically when $\mathbf{H}$ is close to the set $\{\mathbf{H} : rank(\mathbf{H}) \leq r\}$,

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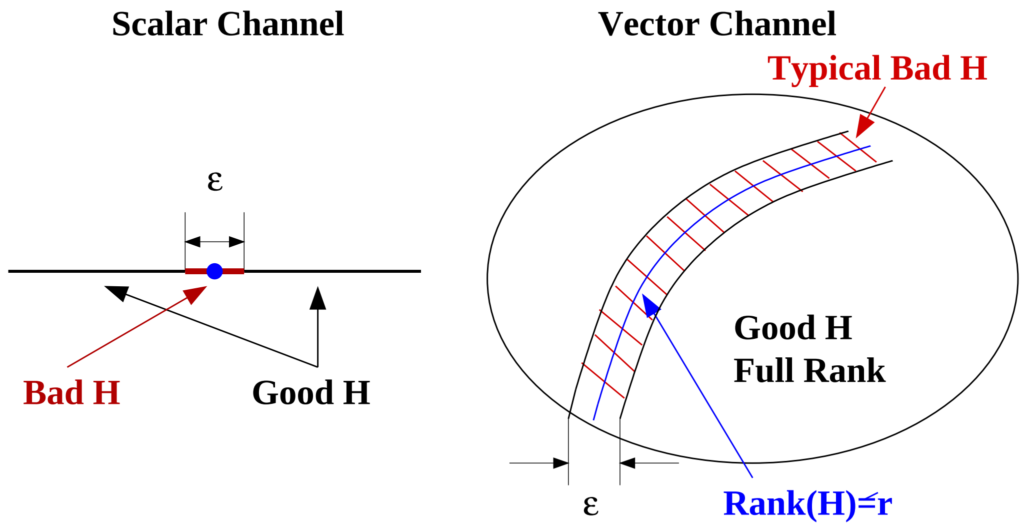

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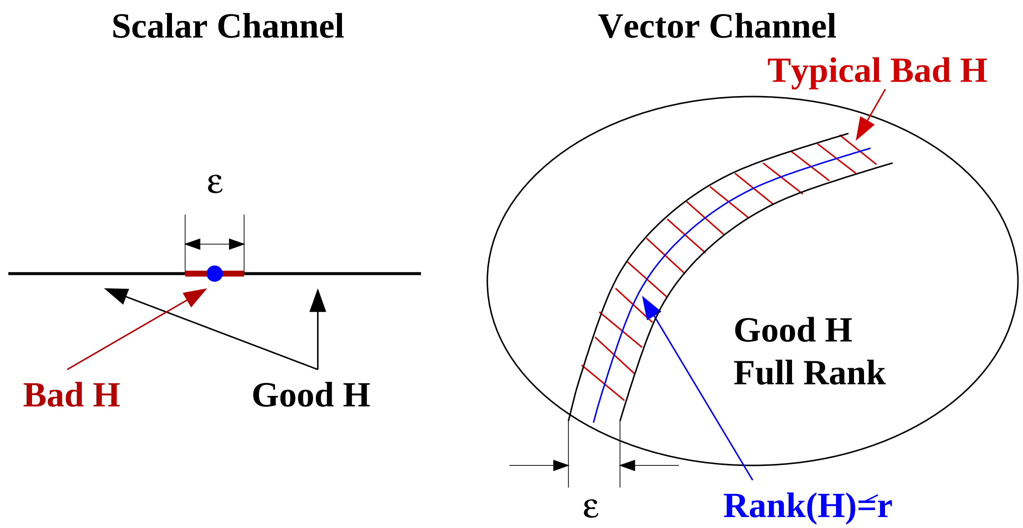

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The dimension of the normal space to the sub-manifold of rank r matrices within the set of all $m \times n$ matrices is $(m - r)(n - r)$.

$$P(\text{Outage}) \approx \mathsf{SNR}^{-(m-r)(n-r)}$$

Achievability: Random Codes

- Outage performance achievable as codeword length $l \to \infty$.
- For random codes of finite length, errors can occur in three ways:
  - Channel $\mathbf{H}$ is aytically bad (outage)
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  - Random codewords are atypically close together.
- We show that as long as $l \geq m + n - 1$, the first event is the dominating one.

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Tradeoff Performance of Specific Designs

Provide a unified framework to compare different schemes.

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Provide a unified framework to compare different schemes.

For a given scheme, compute

$$r \longrightarrow d(r)$$

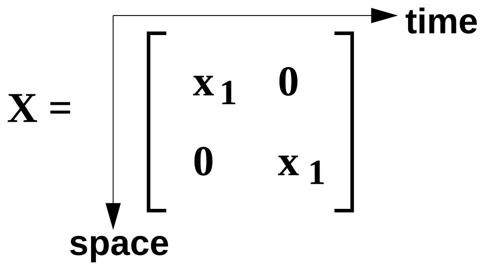
Compare with $d^*(r)$

Two Diversity-Based Schemes

Focus on two transmit antennas.

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$$

Repetition Scheme:



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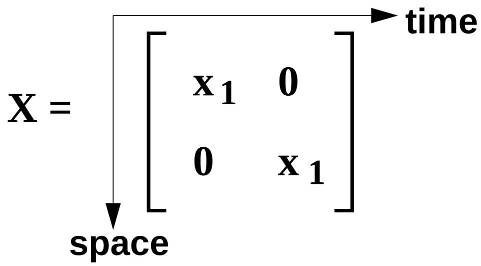

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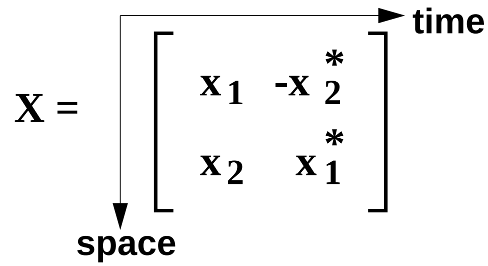
Alamouti Scheme:



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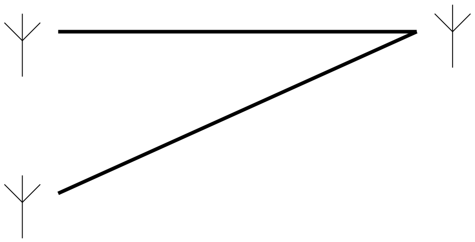
Comparison: 2×1 System



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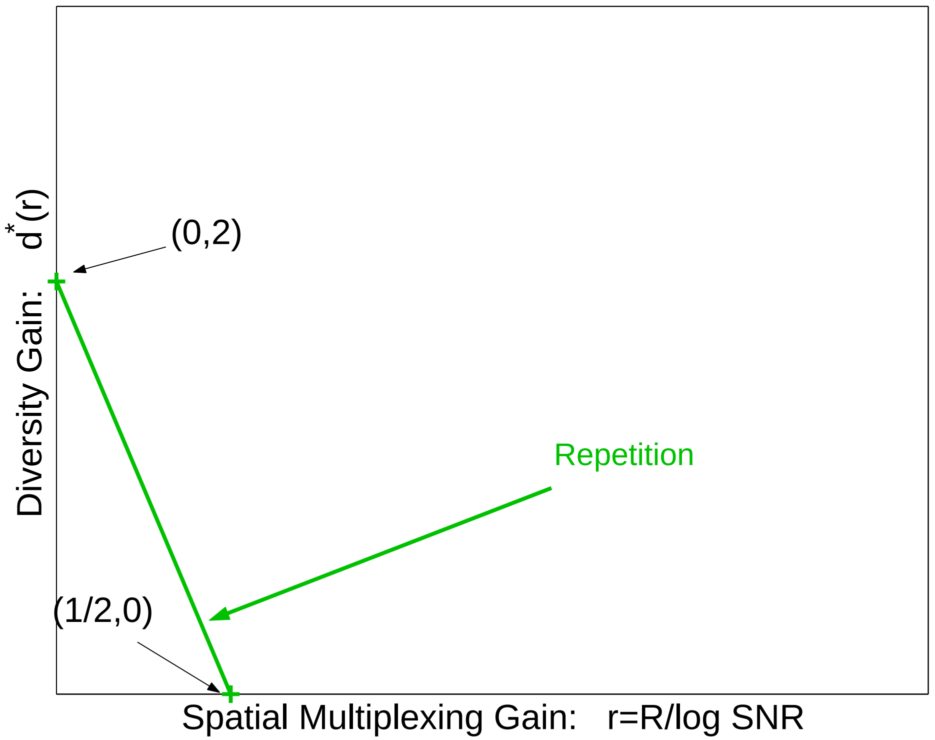

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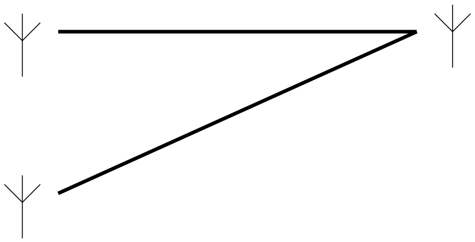

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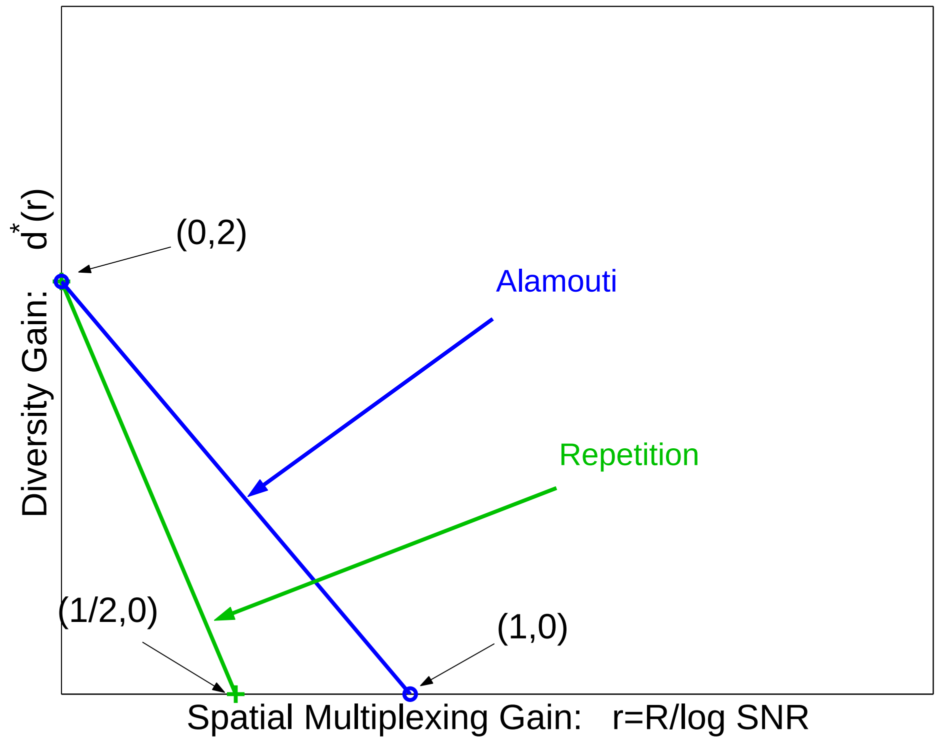

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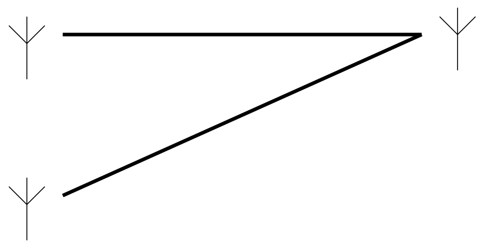

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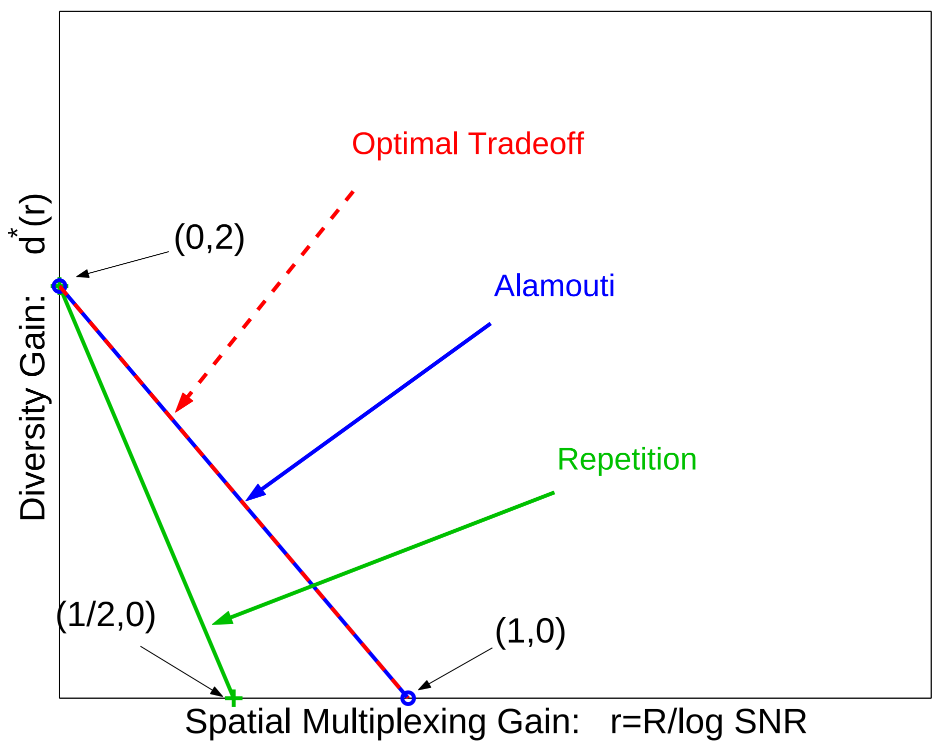
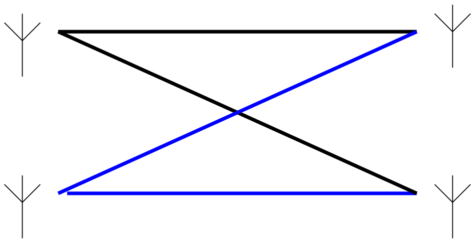


Comparison: 2×2 System



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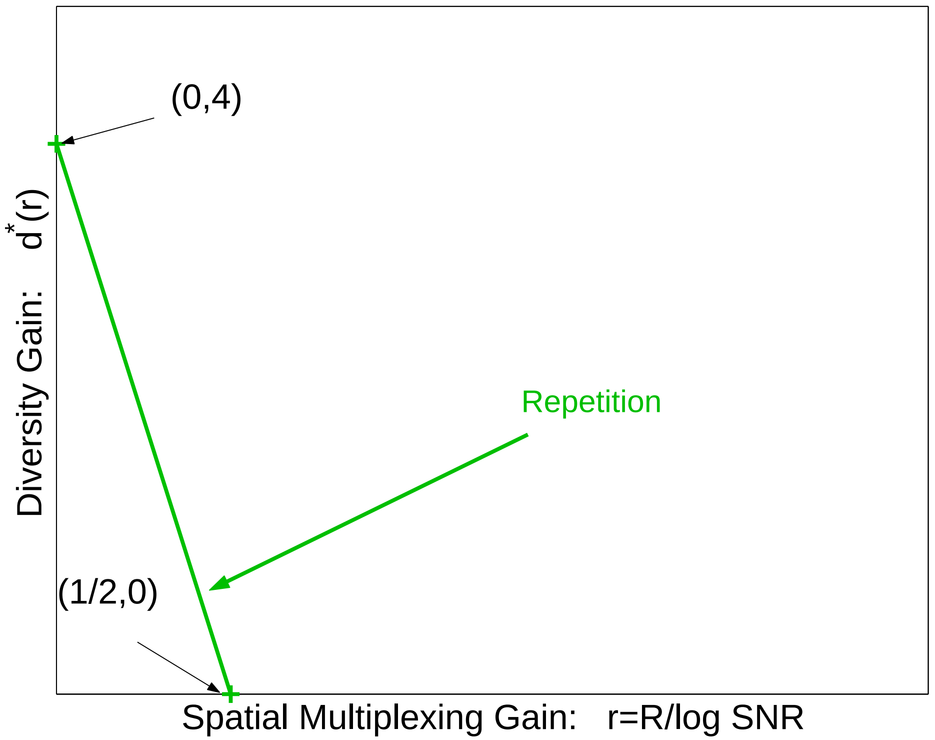

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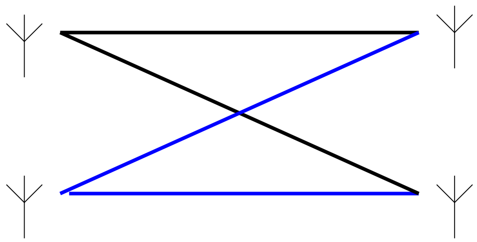

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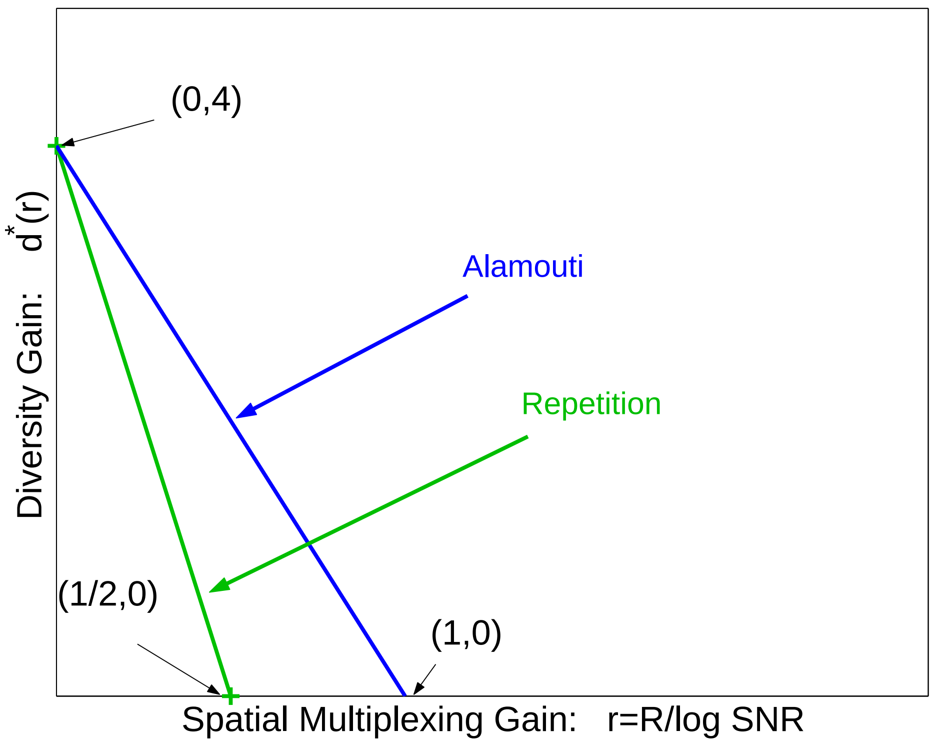

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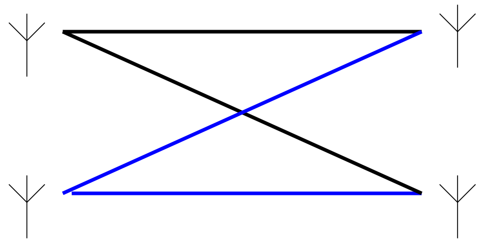

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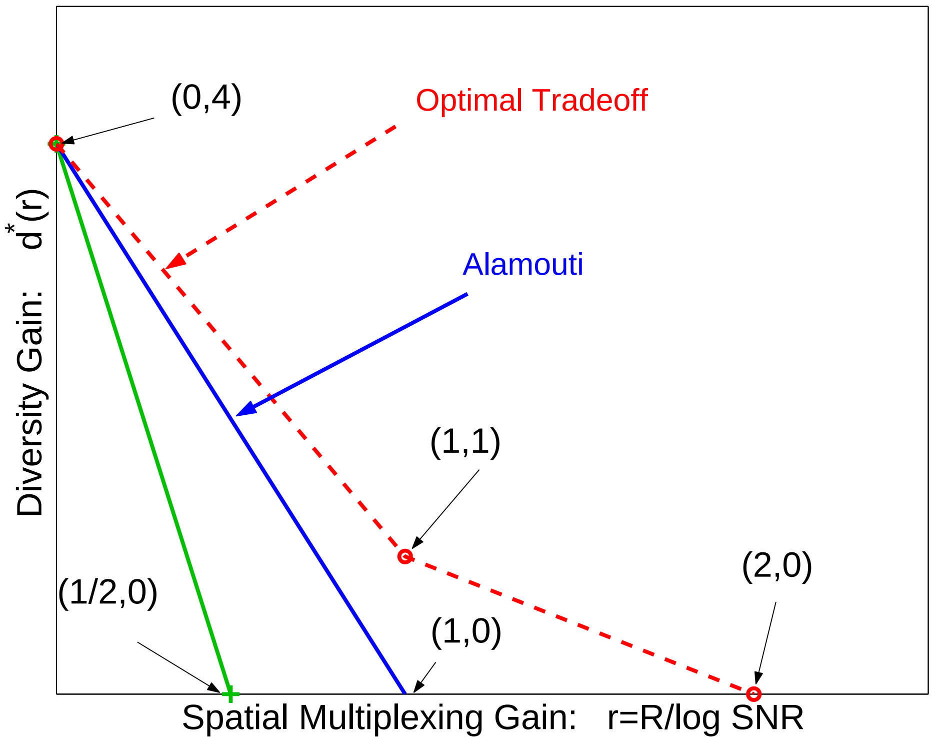
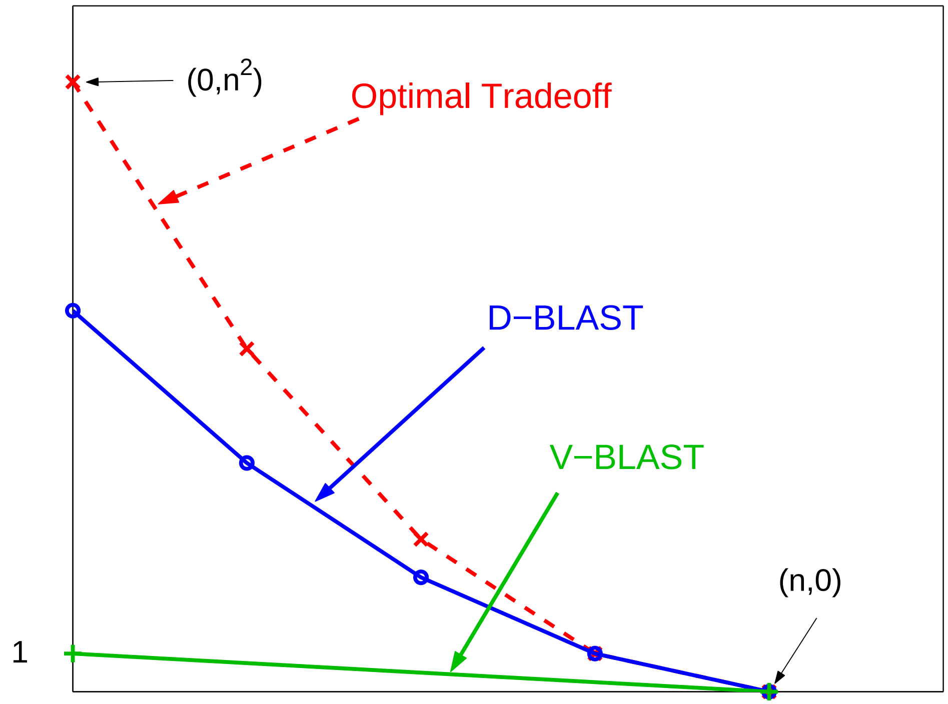


Multiplexing-Based Schemes: BLAST ($n \times n$)



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- Pinpoint the problem and improve tradeoff performance.

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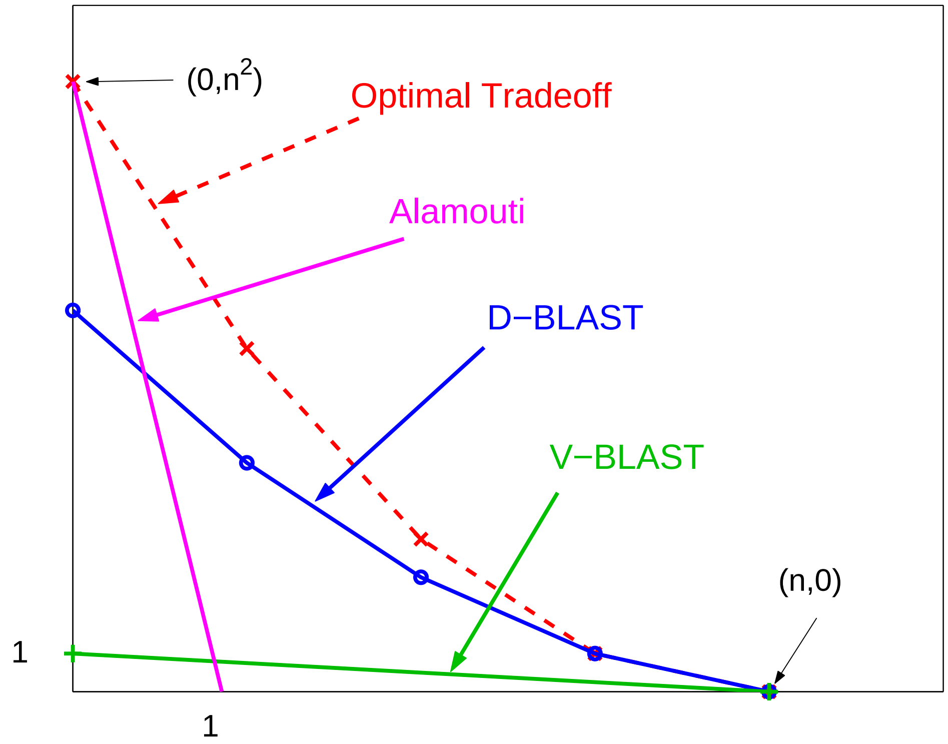

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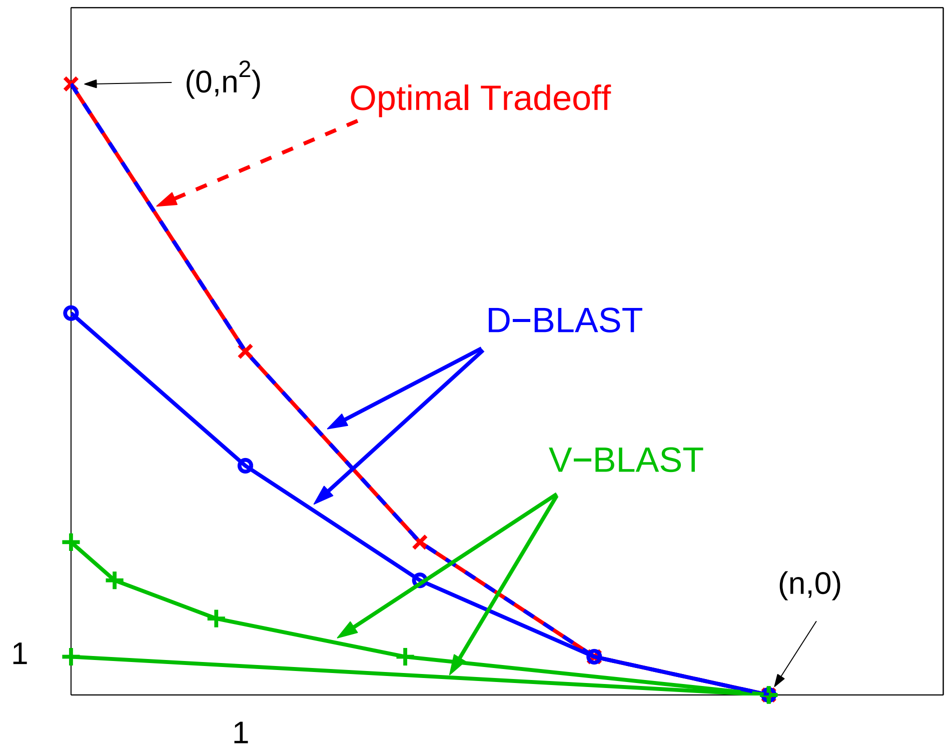

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