The Approximate Capacity of the Many-to-One and One-to-Many Gaussian Interference Channels

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***Abstract*—Recently, Etkin, Tse, and Wang found the capacity region of the two-user Gaussian interference channel to within 1 bit/s/Hz. A natural goal is to apply this approach to the Gaussian interference channel with an arbitrary number of users. We make progress towards this goal by finding the capacity region of the many-to-one and one-to-many Gaussian interference channels to within a constant number of bits. The result makes use of a deterministic model to provide insight into the Gaussian channel. The deterministic model makes explicit the dimension of signal level. A central theme emerges: the use of lattice codes for alignment of interfering signals on the signal level.**

***Index Terms*—Capacity, interference alignment, interference channel, lattice codes, multiuser channels.**

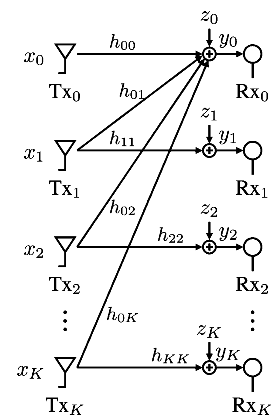

Fig. 1. Gaussian many-to-one IC: K users all causing interference at receiver 0.

I. Introduction

FINDING the capacity region of the two-user Gaussian interference channel is a long-standing open problem. Recently, Etkin, Tse, and Wang [2] made progress on this problem by finding the capacity region to within 1 bit/s/Hz. In light of the difficulty in finding the exact capacity regions of most Gaussian channels, their result introduces a fresh approach towards understanding multiuser Gaussian channels. A natural goal is to apply their approach to the Gaussian interference channel with an arbitrary number of users. This paper makes progress towards this goal by considering two special cases—the many-to-one and one-to-many interference channels (ICs)—where interference is experienced, or is caused, by only one user (see Figs. 1 and 14). The capacity regions of the many-to-one and one-to-many Gaussian ICs are determined to within a constant gap, independent of the channel gains. For the many-to-one IC, the size of the gap is less than $(3K+3)(1+\log(K+1))$ bits per user, where K is the number of users. For the one-to-many IC, the gap is $2K+1$ bits for user 0 and one bit for each of the other users. This result establishes, as a byproduct, the generalized degrees-of-freedom regions of these channels, as defined in [2].

Despite interference occurring only at one user, the capacity regions of the many-to-one and one-to-many ICs exhibit an interesting combinatorial structure, and new outer bounds are required. To elucidate this structure, we make use of a particular deterministic channel model, first introduced in [3]; this model retains the essential features of the Gaussian channel, yet it is significantly simpler. We show that the capacity regions of the deterministic and Gaussian channels are closely related to one another, and in fact, the generalized degrees of freedom region of the Gaussian channel is *equal* to the capacity region of an appropriate deterministic channel.

While the derivation of the outer bound for the many-to-one Gaussian IC parallels that of the deterministic case, the achievable strategy for the Gaussian channel is noteworthy. In order to successfully emulate the strategy for the deterministic channel in the Gaussian setting, it is necessary to use lattice codes. The idea is that since there are multiple interferers, they should align their interference so as to localize the aggregate effect; the impact of the interference is practically as though from one user only. The idea of interference alignment was introduced in a different setting for the multiple-input–multiple-output (MIMO) X-channel by Maddah-Ali, *et al.* [4] and for the many-user interference channel by Cadambe and Jafar [5]. These works aligned signals in the signal space, where parallel channels give rise to a vector space and two signals are aligned if they are transmitted along parallel vectors. In contrast, in this paper, alignment is achieved on the signal *level*. Lattice codes, rather than random codes, are used to achieve this localization. The discrete group structure of the lattice allows alignment within each scalar coordinate, where the signal level, or granularity, of the aggregate interference is the same as in each of the interfering signals. To support the apparent necessity of lattice

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coding, in Section II, we consider an example using a simple generalization (to many users) of the Han–Kobayashi (HK) scheme with Gaussian codebooks. We show that this random coding strategy cannot achieve the degrees-of-freedom of the many-to-one Gaussian IC.

The conference version of this paper [1] is the first to use lattice codes for interference alignment. In particular, since the many-to-one channel is a special case of the many-user interference channel, the results of this paper suggest that lattice codes are necessary for the more general problem as well. Lattice strategies are a natural solution to certain multiuser problems, as originally observed by Körner and Marton [6]; several examples have recently been found for which lattice strategies achieve strictly better performance than any known random codes, including the work of Nazer and Gastpar on computation over multiple access channels [7], and Philosof, *et al.*'s dirty paper coding for multiple access channels [8].

Lattice codes, and more specifically layered lattice strategies, have been subsequently used for interference alignment in several papers. Using the deterministic model and the framework developed in this work, Cadambe *et al.* [9] found a sequence of fully symmetric K-user Gaussian interference channels with arbitrarily close to $K/2$ total degrees of freedom, and Jafar and Vishwanath [10] show that the generalized degrees-of-freedom region of the fully symmetric many-user interference channel (with all the signal-to-noise ratios equal to $\mathtt{SNR}$ and all interference-to-noise ratios equal to $\mathtt{INR} = \mathtt{SNR}^{\alpha}$) is independent of the number of users and is identical to the two-user case except for a singularity at $\alpha = 1$ where the degrees of freedom per user is $\frac{1}{K}$. Generalizing the example of Section II in this paper, Sridharan *et al.* [11] specified a so-called very strong interference parameter regime for the fully symmetric K-user Gaussian channel where the interference does not degrade performance. Sridharan *et al.* [12] also extend the very strong interference regime to a certain class of asymmetric channels.

Other papers subsequently making use of lattice codes for interference alignment in the fully connected interference channel include that of Etkin and Ordentlich [13] and of Motahari *et al.* [14], both of which compute new achievable degrees-of-freedom regions using results from number theory. In [15], Motahari *et al.* use lattice codes to achieve the full $K/2$ degrees-of-freedom for almost all channel gains.

In contrast to the many-to-one IC, the one-to-many IC is simpler, requiring only Gaussian random codebooks. In particular, a generalized HK scheme with Gaussian random codebooks is essentially optimal. As in the many-to-one IC, a deterministic channel model guides the development. Moreover, the deterministic channel model reveals the relationship between the two channels: the capacity regions of the deterministic many-to-one IC and one-to-many IC, obtained by reversing the role of transmitters and receivers, are identical, i.e., the channels are reciprocal. This relationship is veiled in the Gaussian setting, where we can show this reciprocity only in an approximate sense.

While the many-to-one IC is more theoretically interesting, requiring a new achievable scheme using lattices to align interference, the one-to-many IC seems more practically relevant. One easily imagines a scenario with one powerful long-range transmit–receive link and many weak short-range links sharing the wireless medium. Here, to a good approximation, there is no interference except from the single powerful transmitter, and the one-to-many channel model is appropriate.

Independently, Jovicic, Wang, and Viswanath [16] considered the many-to-one and one-to-many interference channels. They found the capacity to within a constant gap for the special case where the direct gains are greater than the cross gains. In this case, Gaussian random codebooks and pairwise constraints for the outer bound are sufficient. As mentioned above, for the many-to-one IC with arbitrary gains, Gaussian random codebooks are suboptimal; also, in general, for both the many-to-one and one-to-many channels, a sum–rate constraint is required for each subset of users.

The paper is organized as follows. Section II introduces the Gaussian many-to-one IC and studies a simple example channel that motivates the entire paper. Section III presents the deterministic channel model. Then, in Section IV, the capacity region of the deterministic many-to-one IC is established. Section V focuses on the Gaussian many-to-one IC and finds the capacity to within a constant gap. Finally, Sections VI and VII consider the one-to-many interference channel, show that the corresponding deterministic model is reciprocal to the many-to-one channel, and approximate the capacity of the Gaussian channel to within a constant gap.

II. Gaussian Interference Channel and Motivating Example

A. Gaussian Interference Channel Model

We first introduce the multiuser Gaussian interference channel. For notational simplicity in the sequel, we assume there are $K+1$ users, labeled $0, 1, \dots, K$. At each time step, the channel outputs are related to the inputs by

$$y_i = \sum_{j=0}^{K} h_{ij} x_j + z_i, \qquad 0 \le i \le K \tag{1}$$

where for $0 \le i \le K$, $x_i, y_i, z_i \in \mathbb{C}$ and the noise processes $z_i \sim \mathcal{CN}(0, N_0)$ are independent identically distributed (i.i.d.) over time. The channel gain between input j and output i is denoted by $h_{ij} \in \mathbb{C}$. The signal-to-noise and interference-to-noise ratios are defined as $\mathtt{SNR}_i = |h_{ii}|^2 P_i / N_0$ for $0 \le i \le K$, and $\mathtt{INR}_{ij} = |h_{ji}|^2 P_i / N_0$ for $0 \le i, j \le K$, $i \ne j$. Each receiver attempts to decode the message from its corresponding receiver.

For a fixed block length N and rate tuple $(r_0, r_1, \dots, r_K)$, transmitter i communicates a message $m_i \in \{1, \dots, 2^{N r_i}\}$, which is assumed to be independent across users. Each transmitter uses an encoding function $x_i : \{1, \dots, 2^{N r_i}\} \to \mathbb{C}^N$, yielding codewords $\{x_i(m_i)\}$. Each codeword must satisfy the average transmit power constraint $\frac{1}{N} \sum_{t=1}^{N} |x_i^N(m_i)[t]|^2 \le P_i$. Receiver i observes the channel outputs $y_i[1], y_i[2], \dots, y_i[N]$, and uses a decoding function $f_{i,N} : \mathbb{C}^N \to \{1, \dots, 2^{N r_i}\}$ to produce an estimate $\hat{m}_i$ of the transmitted message m_i. The average probability of error for user i is $P_{e,i}^{(N)} = E[P(\hat{m}_i \ne m_i)]$, where the expectation is over the choice of message m_i. A rate point $(r_0, r_1, \dots, r_K)$ is said to be achievable if there exists a corresponding sequence of block length N codes with average

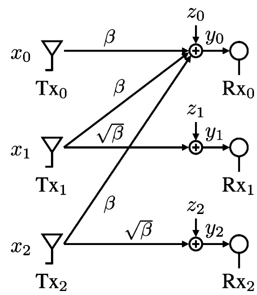

Fig. 2. Example Gaussian channel, with $\mathtt{SNR}_1 = \mathtt{SNR}_2 = \beta$ and $\mathtt{SNR}_0 = \mathtt{INR}_1 = \mathtt{INR}_2 = \beta^2$, for some $\beta > 1$.

decoding error probabilities $P_{e,i}^N$ approaching zero as N goes to infinity.

In this paper, we consider two special cases of the Gaussian IC. In the first half, we study the Gaussian *many-to-one* IC, where all gains are zero except $h_{ii}, 0 \le i \le K$, and $h_{i0}, 1 \le i \le K$. The channel is depicted in Fig. 1. In the second half of the paper, we treat the Gaussian *one-to-many* IC, which is obtained from the many-to-one IC by reversing the roles of transmitters and receivers. The one-to-many IC has all gains equal to zero except for $h_{ii}, 0 \le i \le K$, and $h_{0i}, 1 \le i \le K$. The one-to-many IC is depicted in Fig. 14.

B. Motivating Example

In the two-user Gaussian IC, a simple HK scheme with Gaussian codebooks was shown to be nearly optimal [2]. A natural question is: Is the same type of scheme, a (generalized) HK scheme with Gaussian codebooks, nearly optimal with more than two users? We answer this question by way of an example three-user Gaussian many-to-one channel. This example goes to the heart of the problem and captures the salient features of the many-to-one channel. In particular, the approach used for the two-user interference channel is demonstrated to be inadequate for three or more users, while a simple strategy that aligns interference on the signal level is shown to be asymptotically optimal.

The example channel is depicted in Fig. 2. The power constraints are $P_0 = P_1 = P_2 = 1$ and the complex gains have magnitudes $|h_{00}| = |h_{01}| = |h_{02}| = \beta$ and $|h_{11}| = |h_{22}| = \sqrt{\beta}$. We think of β as being reasonably large, and in particular, we will assume that $\beta \ge 2$.

We first describe the HK scheme with Gaussian codebooks for the three-to-one channel. In the many-to-one channel, each user's signal causes interference only at receiver 0, so the signals from users 1 and 2 are each split into common and private parts as in the two-user scheme (see [2] and [17], for details on the two-user scheme). Each user $i = 1, 2$ employs a superposition of Gaussian codebooks

$$X_i = U_i + W_i$$

with power $P_{U_i} + P_{W_i} = 1$ and rates R_{U_i}, R_{W_i}. The "private" signal U_i is treated as noise by receiver 0, while the "common" signal W_i is decoded by receiver 0. User 0 selects a single random Gaussian codebook with power P_0 and rate R_0.

Note that the interference-to-noise ratios $\mathtt{INR}_1 = \mathtt{INR}_2 = \beta$ are much larger than the signal-to-noise ratios $\mathtt{SNR}_1 = \mathtt{SNR}_2 = \sqrt{\beta}$. Let us momentarily recall a similar situation in the context of the two-user channel: the so-called strong interference regime occurs when the cross gains are stronger than the direct gains. In this situation, after decoding the intended signal, each receiver has a better view of the interfering signal than the intended receiver and can therefore decode the interfering signal assuming a working communication system (this is in spite of each receiver not actually desiring the interfering message); thus, each signal consists entirely of common information. This means the interference is quite damaging, since it contains the full information content of the intended signal. It turns out that similar reasoning applies to the three-user many-to-one channel when using Gaussian codebooks.

Returning to our example channel, let us examine the output at receiver 0, assuming the intended signal at transmitter 0 has already been decoded and subtracted off. The information on x_1 and x_2 at receiver 0 is then $I(y_0; x_1, x_2 | x_0) = h(\beta x_1 + \beta x_2 + z_0) - h(z_0)$. For our example channel, it turns out that when x_1 and x_2 are Gaussian distributed, the entropy $h(\beta x_1 + \beta x_2 + z_0)$ is large enough to allow user 0 to decode both of the signals x_1 and x_2 (assuming the intended receivers can decode). Thus, the signals from users 1 and 2 are entirely common information. When all of the signals are common information, it is easy to bound the sum–rate $r_{\text{sum}}^{HK} = r_0 + r_1 + r_2$, since the rates must lie within the multiple-access channel (MAC) capacity region formed by receiver 0 and the three transmitters. This reasoning yields the following claim.

Claim 1: Let $\beta \ge 2$. An HK-type scheme, with codebooks drawn from the Gaussian distribution, and each of users 1 and 2 splitting their signal into independent private and common information, attains a sum–rate r_{sum}^{HK} of at most $\log(1 + 3\beta^2)$. That is, with this strategy

$$r_{\text{sum}}^{HK} = r_0 + r_1 + r_2 \le \log(1 + 3\beta^2) \approx 2 \log \beta. \tag{2}$$

Proof: The argument bears some resemblance to that of Sato [18] in his treatment of the two-user channel under strong interference. However, here we must show that each of the private and common messages from users 1 and 2 can be decoded by receiver 0. We quickly summarize the argument. Note first that for an achievable rate point, we may assume that each of receivers $0, 1$, and 2 is able to decode their intended signal. Upon decoding signal 0, receiver 0 can subtract it off. The rate tuple $(R_{U_1}, R_{W_1}, R_{U_2}, R_{W_2})$ is then shown to lie within the four-user MAC region (evaluated with Gaussian inputs) at receiver 0 formed by common and private signals from transmitters 1 and 2. It follows that receiver 0 can decode all the signals x_0, x_1, x_2 when using Gaussian inputs, hence the three-user MAC constraint applies. The sum–rate constraint is $\log(1+3\beta^2)$ because the total received power is no more than $3\beta^2$. The calculations are deferred to Appendixes I–III. □

We now propose a different scheme that achieves a rate point within a constant of the optimal sum–rate of approximately $3 \log \beta$ for any $\beta = 2^{2n}$, where n is a positive integer. The restriction of β to even powers of two allows to simplify the analysis of the scheme; the scheme itself, as well as the general scheme presented in Section V, works for arbitrary complex-valued channel gains. Consider first only the real-valued channel (assume that $z_i \sim \mathcal{N}(0,1)$ and the gains and inputs are real valued). Each user generates a random codebook from a *discrete* distribution

$$x_i = \sum_{k=1}^{\log\sqrt{\beta}} x_i(k) 2^{-k}, \qquad i = 0,1,2 \tag{3}$$

where the bits $x_i(k)$ are uniformly random on $\{0,1\}$ and are i.i.d. over time. In order to show an achievable rate, we calculate the single time-step mutual information between input and output for each user

$$I(x_i; y_i), \qquad i = 0,1,2.$$

Let $\tilde{y}_i$ denote the noiseless output

$$\begin{aligned}\tilde{y}_0 &= \sqrt{\beta}x_0 + \beta x_1 + \beta x_2\\ \tilde{y}_1 &= \sqrt{\beta}x_1\\ \tilde{y}_2 &= \sqrt{\beta}x_2.\end{aligned}$$

It is shown in [19, App. A.1] that when using inputs such that the outputs $\tilde{y}_i$ are integer valued, the additive Gaussian noise z_i causes a loss in mutual information of at most 1.5 bits, i.e.,

$$I(x_i;\tilde{y}_i) - 1.5 \le I(x_i; \tilde{y}_i + z_i) = I(x_i; y_i). \tag{4}$$

Informally, this is because $\tilde{y}_i$ can be recovered from y_i by knowing the value of $[z_i]$, where $[\,\cdot\,]$ is the nearest integer function; the estimate $H([z_i]) \le 1.5$ allows to show the inequality (4).

Note the following key observation: it is possible to perfectly recover the signal x_0 from $\tilde{y}_0$. This follows from writing

$$\tilde{y}_0 = \sqrt{\beta}x_0 + \sqrt{\beta}(\sqrt{\beta}(x_1 + x_2)) \in \sqrt{\beta}x_0 + \sqrt{\beta}\mathbb{Z}$$

and the fact that $\sqrt{\beta}x_0 < \sqrt{\beta}$. Hence

$$\tilde{y}_0 \ (\mathrm{mod} \sqrt{\beta}) = \sqrt{\beta}x_0$$

and for $i = 0,1,2$

$$I(x_i; y_i) + 1.5 \ge I(x_i; \tilde{y}_i) = H(x_i) = \frac{1}{2}\log\beta.$$

For the complex-valued channel, each gain has magnitude as given above, with an arbitrary phase. Each transmitter can rotate their signal so that all signals are observed by receiver 0 with zero phase, and the other receivers can also rotate the signals to zero phase. Thus, the same strategy can be used independently in the real and imaginary dimensions, giving an achievable rate of

$$r_i \ge \log\beta - 3, \qquad i = 0,1,2.$$

The sum–rate achieved

$$r_{\text{sum}}^{\text{lattice}} = 3\log\beta - 9 \approx 3\log\beta$$

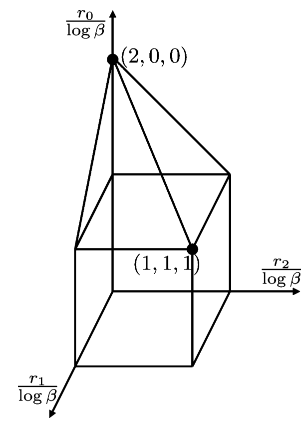

Fig. 3. Approximate capacity region of the example channel considered in this section. The two dominant corner points are emphasized.

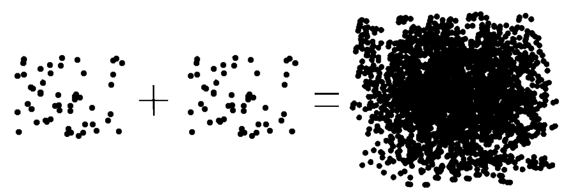

Fig. 4. Sum of two identical random codebooks with 50 points each. The resulting interference covers the entire space, preventing receiver 0 from decoding.

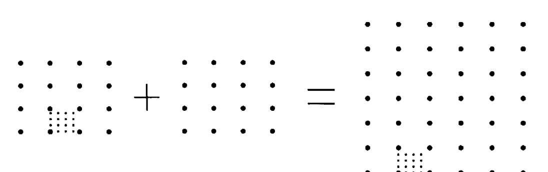

Fig. 5. User 0 can decode the fine signal in the presence of interference from users 1 and 2. The sum of the interference from users 1 and 2 imposes essentially the same cost as from a single interferer.

is therefore arbitrarily larger than the approximately $2\log\beta$ achieved by the strategy employing Gaussian codebooks. The achievable region for large β, normalized by $\log\beta$, is depicted in Fig. 3.

Before proceeding, we reflect on why random Gaussian codebooks are suboptimal for the many-to-one channel. Note that the aggregate interference at receiver 0 has support equal to the sumset of the supports of codebooks 1 and 2. As illustrated in Fig. 4, the sumset[1] of two random (continuously distributed) codebooks fills the space, leaving no room for user 0 to communicate. If each of codebooks 1 and 2 have m points, the sumset can have up to m^2 points. In contrast, as illustrated in Fig. 5, the sum of two codebooks that are subsets of a lattice looks essentially like one of the original codebooks (and, in particular, has cardinality Cm, where C is a constant independent of m). Thus, the cost to user 0 is the same as though due to only one interferer, i.e., the interference is aligned on the signal level. This theme will reappear throughout the paper.

[1]The sumset, or Minkowski sum, of two sets A and B is given by $A + B = \{a + b : a \in A, b \in B\}$.

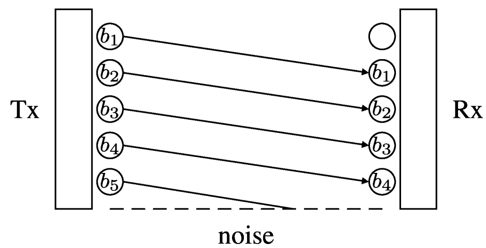

Fig. 6. Deterministic model for the point-to-point Gaussian channel. Each bit of the input occupies a signal level. Bits of lower significance are lost due to noise.

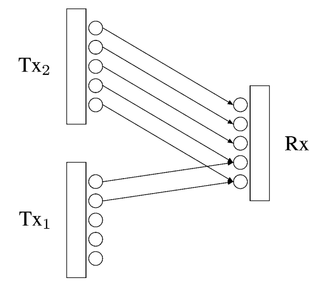

Fig. 7. Deterministic model for the Gaussian MAC. Incoming bits on the same level are added modulo two at the receiver.

In order to generalize the intuition gained from this example and provide the framework for finding the capacity of the many-to-one channel to within a constant gap, we make use of a deterministic channel model, described in the next section.

III. Deterministic Channel Model

We now present a deterministic channel model analogous to the Gaussian channel. This channel was first introduced in [3]. We begin by describing the deterministic channel model for the point-to-point additive white Gaussian noise (AWGN) channel, and then the two-user MAC. After understanding these examples, we present the deterministic interference channel.

Consider the model for the point-to-point channel (see Fig. 6). The real-valued channel input is written in base 2; the signal—a vector of bits—is interpreted as occupying a succession of levels

$$x = 0.b_1 b_2 b_3 b_4 b_5 \ldots. \tag{5}$$

The most significant bit coincides with the highest level, the least significant bit with the lowest level. The levels attempt to capture the notion of *signal level*; a level corresponds to a unit of power in the Gaussian channel, measured on the decibel scale. Noise is modeled in the deterministic channel by truncation. Bits of smaller order than the noise are lost. The channel may be written as

$$y = \lfloor 2^n x \rfloor$$

with the correspondence $n = \lfloor \log \mathtt{SNR} \rfloor$.

Note the similarity of the binary expansion underlying the deterministic model (5) to the discrete inputs (3) in the example channel of the previous section. The discrete inputs (3) are precisely a binary expansion, truncated so that the signal—after scaling by the channel—is integer valued. Evidently, the achievable scheme for the example channel emulates the deterministic model.

The deterministic MAC is constructed similarly to the point-to-point channel (Fig. 7), with n_1 and n_2 bits received above the noise level from users 1 and 2, respectively. To model the superposition of signals at the receiver, the bits received on each level are added *modulo two*. Addition modulo two, rather than normal integer addition, is chosen to make the model more tractable. As a result, the levels do not interact with one another.

If the inputs $x_i(t)$ are written in binary, the channel output can be written as

$$y = \lfloor 2^{n_1} x_1 \rfloor \oplus \lfloor 2^{n_2} x_2 \rfloor \tag{6}$$

where addition is performed on each bit (modulo two) and $\lfloor \cdot \rfloor$ is the integer-part function.

An easy calculation shows that the capacity region of the deterministic MAC is

$$\begin{aligned} r_1 &\le n_1 \\ r_2 &\le n_2 \\ r_1 + r_2 &\le \max(n_1, n_2). \end{aligned} \tag{7}$$

Comparing with the capacity region of the Gaussian MAC

$$\begin{aligned} R_1 &\le \log(1+\mathtt{SNR}_1) \approx \log \mathtt{SNR}_1 \\ R_2 &\le \log(1+\mathtt{SNR}_2) \approx \log \mathtt{SNR}_2 \\ R_1 + R_2 &\le \log(1+\mathtt{SNR}_1+\mathtt{SNR}_2) \\ &\approx \max(\log \mathtt{SNR}_1, \log \mathtt{SNR}_2) \end{aligned} \tag{8}$$

we make the correspondence

$$n_1 = \lfloor \log \mathtt{SNR}_1 \rfloor \quad \text{and} \quad n_2 = \lfloor \log \mathtt{SNR}_2 \rfloor.$$

A. Deterministic Interference Channel

We proceed with the deterministic interference channel model. Note that the model is completely determined by the model for the MAC. There are $K+1$ transmitter–receiver pairs (links), and as in the Gaussian case, each transmitter wants to communicate only with its corresponding receiver. The signal from transmitter j, as observed at receiver i, is scaled by a nonnegative integer gain n_{ij}. The channel may be written as

$$y_i = \lfloor 2^{n_{i0}} x_0 \rfloor \oplus \cdots \oplus \lfloor 2^{n_{iK}} x_K \rfloor \tag{9}$$

where, as before, addition is performed on each bit (modulo two) and $\lfloor \cdot \rfloor$ is the integer-part function. The standard definitions of achievable rates and the associated notions are omitted.

The deterministic interference channel is relatively simple, yet retains two essential features of the Gaussian interference

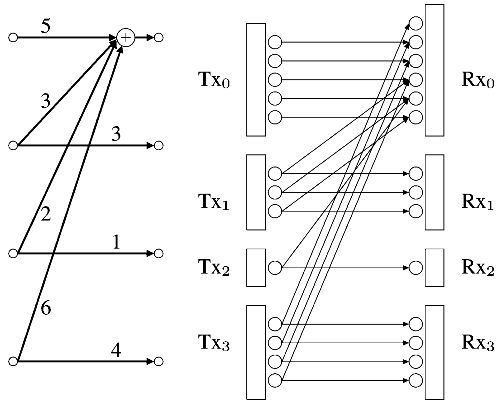

Fig. 8. Both figures depict the same channel. On the left is an example of a deterministic many-to-one interference channel with four users. The right-hand figure shows how the inputs are shifted and added together (modulo two) at each receiver. Each circle on the left-hand side represents an element of the input vector; each circle on the right-hand side represents the received signal at a certain level.

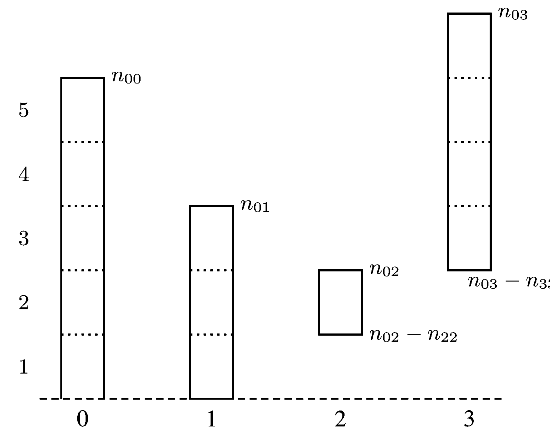

Fig. 9. Interference pattern as observed at receiver 0 for the channel in Fig. 8. Here, $U_1 = \{1\}, U_2 = \{1,2\}, U_3 = \{1,3\}$, and $U_4 = U_5 = \{3\}$. Here $\bar{x}_3$ consists of the $n_{03} - n_{00} = 1$ highest level from signal 3.

channel: the loss of information due to noise, and the superposition of transmitted signals at each receiver. The modeling of noise can be understood through the point-to-point channel above. The superposition of transmitted signals at each receiver is captured by taking the modulo two sum of the incoming signals at each level.

The relevance of the deterministic model is greatest in the high-SNR regime, where communication is interference, rather than noise, limited; however, we will see that even for finite signal-to-noise ratios the deterministic channel model provides significant insight towards the more complicated Gaussian model.

As in the approach for the Gaussian interference channel, we consider only special cases of the deterministic interference channel: the many-to-one and one-to-many ICs. In the many-to-one IC interference occurs only at receiver 0 (see Fig. 8 for an example), and in the one-to-many IC interference is caused by only one user.

IV. Deterministic Many-to-One Interference Channel

In this section, we find the capacity region of the deterministic many-to-one IC. By separately considering each level at receiver 0 together with those signals causing interference to the level, the many-to-one channel is seen to be a *parallel* channel, one subchannel per level at receiver 0. This begs the question: Is the capacity of the many-to-one channel equal to the sum of the capacities of the subchannels? Theorem 4 answers this question in the affirmative.

The channel input–output relationship is given by (9) with all gains equal to zero except the direct gains $n_{ii}, 0 \le i \le K$, and cross gains to receiver 0, $n_{0i}, 1 \le i \le K$. Specifically, we have

$$y_0 = \lfloor 2^{n_{00}} x_0 \rfloor \oplus \cdots \oplus \lfloor 2^{n_{0K}} x_K \rfloor$$

and

$$y_i = \lfloor 2^{n_{ii}} x_i \rfloor, \qquad 1 \le i \le K.$$

Some notation is required. First, we can assume without loss of generality that each input x_i is restricted to the elements that appear in the output y_i, i.e., $x_i \in \mathbb{F}_2^{n_{ii}}$. Denote by $U_k \subseteq \{1, \ldots, K\}, 1 \le k \le n_{00}$, the set of users interfering on level k at receiver 0

$$U_k = \{i : 1 \le i \le K, n_{0i} - n_{ii} < k \le n_{0i}\}.$$

For a set of users $A \subseteq \{0, 1, \ldots, K\}$ and a level $k, 1 \le k \le n_{00}$, denote by $x_{A|k}$ the vector of signals of users in A, restricted to level k *as observed at receiver* 0. See Fig. 9 for an illustration of these definitions.

Let $\underline{x}_i$ be the restriction of the input from transmitter i to the lowest $(n_{ii} - n_{0i})^+$ levels. This is the part of x_i that does not appear as interference at receiver 0, i.e., this part of the interfering signal is below the noise level. Similarly, let $\bar{x}_i$ be the restriction of the input from transmitter i to the highest $(n_{0i} - n_{00})^+$ levels. This is the part of x_i that causes interference above the signal level of user 0 (and therefore does not really interfere). With this notation at our disposal, we are ready to describe the achievable strategy, and then state the capacity region of the deterministic many-to-one IC.

A. Achievable Strategy

The achievable strategy consists of allocating each level separately, by choosing either user 0 to transmit on a given level, or all users interfering with user 0 to transmit on the level. This scheme aligns the interference as observed by receiver 0, so that several users transmitting on a level inflict the same cost to user 0 as one user transmitting on the same level. Because the scheme considers each level separately, the structure of the achievable region is remarkably simple. The region is next described in more detail.

First, note that by transmitting on levels that appear above the signal of user 0 or below the noise level as observed by receiver 0, each user can transmit at rate

$$r_i \le f_{\text{free}}(i)$$

where $f_{\text{free}}(i) = (n_{ii} - n_{0i})^+ + (n_{0i} - n_{00})^+$, without causing any interference to user 0. Thus, the rate region

$$\mathcal{C}_{\text{free}} = \{(r_0, \ldots, r_K) : r_0 = 0, r_i \le f_{\text{free}}(i)\} \tag{10}$$

can be achieved without causing any interference to user 0.

Next, for a subset of users $U_k \subseteq \{1, \ldots, K\}$, let $\mathcal{C}_k$ denote the capacity region of a deterministic many-to-one IC with only one level, and users U_k interfering at receiver 0. Users not in U_k, i.e., $\{1, \ldots, K\} \setminus U_k$, are not present. It is easy to see that $\mathcal{C}_k$ is given by the intersection of the individual rate constraints

$$\begin{aligned} r_i(k) &\le 1, \qquad i \in U_k \cup \{0\} \\ r_i(k) &= 0, \qquad i \notin U_k \end{aligned} \tag{11}$$

and the pairwise rate constraints

$$r_0(k) + r_i(k) \le 1, \qquad i \in U_k. \tag{12}$$

The capacity $\mathcal{C}_k$ is achieved by timesharing between two rate points: 1) user 0 transmits a uniformly random bit, while all other users are silent, or 2) user 0 is silent while each user in U_k transmits a uniformly random bit. This is done for each level $1 \le k \le n_{00}$.

The achievable scheme treats each level separately, so the achievable region is the sum of the regions for each level and the set of points achievable without causing any interference. This is recorded in the following lemma.

Lemma 2: Let $\underline{\mathcal{C}}$ denote the set of rate points achieved by the scheme described above. Then

$$\underline{\mathcal{C}} = \mathcal{C}_{\text{free}} + \sum_{k=1}^{n_{00}} \mathcal{C}_k \tag{13}$$

where $\mathcal{C}_{\text{free}}$ is defined in (10) and $\mathcal{C}_k$ is defined in (11) and (12).

B. Outer Bound

Before turning to the outer bound, let us first reexpress the capacity region $\mathcal{C}_k$ of a single level k in terms of sum–rate constraints. Fixing a level $1 \le k \le n_{00}$, for each set of users $\mathcal{S} \subseteq \{1, \ldots, K\}$, we can form a sum–rate constraint on the users $\mathcal{S} \cup \{0\}$ by adding a single pairwise constraint on $r_0(k) + r_i(k)$ for some $i \in \mathcal{S}$ together with individual rate constraints on users $\mathcal{S} \setminus \{i\}$ (11)

$$r_0(k) + \sum_{i \in (U_k \cap \mathcal{S})} r_i(k) \le f_k(\mathcal{S}) \tag{14}$$

where

$$f_k(\mathcal{S}) = \max(|U_k \cap \mathcal{S}|, 1). \tag{15}$$

These constraints on each single level clearly do not imply any constraints on strategies using all levels jointly. However, perhaps surprisingly, the true outer bound region turns out to be equal to the region obtained by summing these per-level constraints.

The following lemma gives an outer bound $\overline{\mathcal{C}}$ to the capacity region; the expression itself does indeed match what one obtains by summing the per-level constraints (14). The capacity region $\mathcal{C}_D$ of the $K+1$ user deterministic many-to-one IC is therefore bounded as

$$\underline{\mathcal{C}} \subseteq \mathcal{C}_D \subseteq \overline{\mathcal{C}}.$$

Lemma 3: $\mathcal{C}_D$ is contained in $\overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is given by the intersection of the individual rate constraints

$$r_i \le n_{ii}, \qquad 0 \le i \le K \tag{16}$$

and the $2^K - 1$ sum–rate constraints

$$\begin{aligned} r_0 + \sum_{i \in \mathcal{S}} r_i \le f_{\text{free}}(\mathcal{S}) + \sum_{k=1}^{n_{00}} f_k(\mathcal{S}), \\ \mathcal{S} \subseteq \{1, \ldots, K\}, \qquad \mathcal{S} \neq \varnothing \end{aligned} \tag{17}$$

where $f_k(\mathcal{S})$ is defined above in (15) and $f_{\text{free}}(\mathcal{S}) = \sum_{i \in \mathcal{S}} f_{\text{free}}(i)$.

The bound in the lemma is tight, as shown in the following theorem. Thus, the capacity region is equal to the sum of the capacities of the subchannels.

Theorem 4: The achievable region is equal to the outer bound, i.e.,

$$\underline{\mathcal{C}} = \overline{\mathcal{C}} = \mathcal{C}_D.$$

We first prove the constraints in (16) and (17) characterizing $\overline{\mathcal{C}}$, and then show that the region coincides with the achievable region $\underline{\mathcal{C}}$ of (13).

Proof of Lemma 3: Clearly, the rate across each link cannot exceed the point-to-point capacity, implying the constraints in (16).

Next, we prove a sum–rate constraint on an arbitrary set of users $\mathcal{S} \cup \{0\}$, where $\mathcal{S} \subseteq \{1, \ldots, K\}$. We give the following side information to receiver 0: at each level $k, 1 \le k \le n_{00}$, the input signals of all interfering users in $\mathcal{S}$ except for one, and also $\{\bar{x}_i\}_{i \in \mathcal{S}}$ and the inputs of all users not in $\mathcal{S}$. More precisely, for each $k, 1 \le k \le n_{00}$, let Q_k be any set that satisfies $Q_k \subseteq (U_k \cap \mathcal{S})$ and $|Q_k| = (|U_k \cap \mathcal{S}| - 1)^+$. We give the side information

$$s_0 = \left(\{x_{Q_k|k}\}_{k=1}^{n_{00}}, \{x_i\}_{i \notin \mathcal{S}}, \{\bar{x}_i\}_{i \in \mathcal{S}}\right). \tag{18}$$

Recall that $y_{i,k} = x_{i,k}$ for users $i \neq 0$, hence $I(y_i^N; x_i^N) = H(x_i^N)$. Fano's inequality, the data processing inequality, the chain rule for mutual information, independence of x_0 and s_0, and breaking apart the signals according to level gives

$$\begin{aligned} &N\left(r_0 + \sum_{i \in \mathcal{S}} r_i - \epsilon_N\right) \\ &\quad \le I\left(y_0^N, s_0^N; x_0^N\right) + \sum_{i \in \mathcal{S}} I\left(y_i^N; x_i^N\right) \\ &\quad = I(y_0^N; x_0^N | s_0^N) + \sum_{i \in \mathcal{S}} I\left(y_i^N; x_i^N\right) \\ &\quad = H\left(y_0^N \middle| s_0^N\right) - H\left(y_0^N \middle| s_0^N, x_0^N\right) + \sum_{i \in \mathcal{S}} H\left(y_i^N\right) \end{aligned}$$

$$
\begin{aligned}
&= H\left(\left\{x_{0|k}^N + \sum_{i\in U_k} x_{i|k}^N\right\}_{k=1}^{n_{00}}, \sum_{i=1}^{K}\bar{x}_i^N \right.\\
&\qquad\qquad \left.\left|\left\{x_{Q_k|k}^N\right\}_{k=1}^{n_{00}}, \left\{x_i^N\right\}_{i\notin\mathcal{S}}, \left\{\bar{x}_i^N\right\}_{i\in\mathcal{S}}\right.\right)\\
&\quad - H\left(\left\{\sum_{i\in U_k} x_{i|k}^N\right\}_{k=1}^{n_{00}}, \sum_{i=1}^{K}\bar{x}_i^N \right.\\
&\qquad\qquad \left.\left|\left\{x_{Q_k|k}^N\right\}_{k=1}^{n_{00}}, \left\{x_i^N\right\}_{i\notin\mathcal{S}}, \left\{\bar{x}_i^N\right\}_{i\in\mathcal{S}}\right.\right)\\
&\quad + H\left(\left\{x_{Q_k|k}^N\right\}_{k=1}^{n_{00}}, \left\{x_{U_k\cap\mathcal{S}\setminus Q_k|k}^N\right\}_{k=1}^{n_{00}},\right.\\
&\qquad\qquad \left.\left\{\bar{x}_i^N\right\}_{i\in\mathcal{S}}, \left\{\underline{x}_i^N\right\}_{i\in\mathcal{S}}\right).
\end{aligned}
$$

Here $\epsilon_N \to 0$ as $N \to \infty$. Continuing, the fact that s_0 is independent of x_0, removing conditioning, the chain rule for mutual information, and the independence bound on entropy justify the remaining inequalities

$$
\begin{aligned}
&\le H\left(\left\{x_{0|k}^N + x_{U_k\cap\mathcal{S}\setminus Q_k|k}^N\right\}_{k=1}^{n_{00}}\right)\\
&\quad - H\left(\left\{x_{U_k\cap\mathcal{S}\setminus Q_k|k}^N\right\}_{k=1}^{n_{00}} \middle| \left\{\bar{x}_i^N\right\}_{i\in\mathcal{S}}\right)\\
&\quad + H\left(\left\{x_{Q_k|k}^N\right\}_{k=1}^{n_{00}}\right)\\
&\quad + H\left(\left\{x_{U_k\cap\mathcal{S}\setminus Q_k|k}^N\right\}_{k=1}^{n_{00}} \middle| \left\{\bar{x}_i^N\right\}_{i\in\mathcal{S}}\right)\\
&\quad + H\left(\left\{\bar{\mathrm{x}}_i^N\right\}_{i\in\mathcal{S}}\right) + H\left(\left\{\underline{x}_i^N\right\}_{i\in\mathcal{S}}\right)\\
&\le N n_{00} + N\sum_{k=1}^{n_{00}}(|U_k\cap\mathcal{S}| - 1)^+\\
&\quad + N\sum_{i\in\mathcal{S}}\left((n_{0i} - n_{00})^+ + (n_{ii} - n_{0i})^+\right)\\
&= N\left(f_{\text{free}}(\mathcal{S}) + \sum_{k=1}^{n_{00}} f_k(\mathcal{S})\right). \qquad (19)
\end{aligned}
$$

Taking $N \to \infty$ proves the sum–rate constraint. □

C. Proof of Theorem 4

Lemmas 2 and 3 give algebraic characterizations of $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$; therefore, the result of Theorem 4 is essentially an algebraic property. The proof can be summarized as follows. It is easy to see that the achievable region contains at least a single point on each constraint in Lemma 3; in order to achieve such a point the achievable strategy allocates each signal level either to user 0 or to the interfering users according to a simple rule. Two constraints, however, might have conflicting rules for the allocation of the signal levels. The compatibility of the allocation rules for achieving a point simultaneously on a collection of constraints is understood through a bipartite graph. Turning to the outer bound, the corner points of the outer bound region are identified with the set of active constraints; sets of constraints actually corresponding to a corner point will be called consistent. The proof follows by showing that all outer bound corner points, i.e., all consistent sets of constraints, do not require conflicting allocation of levels in the achievable strategy, and hence all corner points can be achieved.

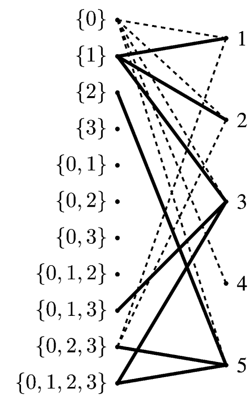

Fig. 10. Bipartite graph associated with the interference pattern in Fig. 9. For clarity, only the edges adjacent to the top three and bottom three vertices on the left-hand side were included.

We begin the proof by taking another look at the achievable region. As outlined above, the achievable strategy consists of allocating each level $k, 1 \le k \le n_{00}$, entirely to user 0 or to all users in U_k (recall that U_k is the set of users potentially causing interference to user 0 on level k). Now, the sum–rate constraint (17) on a single set of users $\mathcal{S}$ can be met with equality by: 1) having each user in $\mathcal{S}$ transmit on levels not causing interference to user 0, and 2) if $|U_k \cap \mathcal{S}| \ge 2$, then users in $\mathcal{S}$ transmit on level k while user 0 is silent, if $U_k \cap \mathcal{S} = \varnothing$, then user 0 transmits on level k while all other users are silent, and if $|U_k \cap \mathcal{S}| = 1$, then either user 0 or the interfering user transmits on level k.

These rules for allocating levels can be encoded in a bipartite graph (see Fig. 10). The left-hand set of vertices is indexed by the subsets $\mathcal{S} \cup \{0\}$ for each nonempty $\mathcal{S} \subseteq \{1, \dots, K\}$ and also a vertex for each user $0, 1, \dots, K$; for each constraint in Lemma 3 there is a vertex labeled by the users participating in the constraint. The right-hand set of vertices is indexed by the levels $k, 1 \le k \le n_{00}$. There is a solid edge between $\mathcal{S} \cup \{0\}$ and k if $|U_k \cap \mathcal{S}| \ge 2$, signifying that for our scheme to achieve the constraint on $\mathcal{S} \cup \{0\}$ with equality, it is required that all users in $\mathcal{S}$ other than user 0 transmit on level k. There is a dashed edge between $\mathcal{S} \cup \{0\}$ and k if $U_k \cap \mathcal{S} = \varnothing$, signifying that no users other than user 0 may transmit on level k. There is no edge if $|U_k \cap \mathcal{S}| = 1$. Finally, for each of the $K + 1$ individual constraints (on user $i, 0 \le i \le K$), the vertex labeled i has solid edges to those k with $i \in U_k$ (and no other edges), signifying that user i must fully use all available levels.

For a vertex v on the left-hand side, let $N_{\text{solid}}(v)$ be the set of (right-hand side) vertices connected by solid edges to v, and similarly, let $N_{\text{dashed}}(v)$ be the set of (right-hand side) vertices connected by dashed edges to v.

The bipartite graph, encoding the rules for allocating signal levels in the achievable strategy, allows to see visually when the rules are in agreement for achieving with equality a given collection of constraints.

Definition 5: Given a set of constraints with bipartite graph as described above, a collection of constraints is said to be *compatible* if for any two of the constraints on sets $\mathcal{S}, \mathcal{S}'$, it holds that

$N_{\text{solid}}(\mathcal{S}) \cap N_{\text{dashed}}(\mathcal{S}') = \varnothing$ and $N_{\text{solid}}(\mathcal{S}') \cap N_{\text{dashed}}(\mathcal{S}) = \varnothing$.

The next lemma shows that, indeed, compatible constraints induce compatible requirements on the allocation of levels in the achievable strategy, and is immediate from the definitions.

Lemma 6: It is possible to achieve at least one point in the intersection of the hyperplanes defining any collection of compatible constraints.

Proof: It is necessary to check that the assignment for achieving each constraint individually works for the collection of compatible constraints simultaneously. To see this, note that if a set A of constraints (indexed by the sets of users) are compatible, it must be that

$$\left(\bigcup_{\mathcal{S}\in A} N_{\text{solid}}(\mathcal{S})\right) \bigcap \left(\bigcup_{\mathcal{S}\in A} N_{\text{dashed}}(\mathcal{S})\right) = \varnothing.$$

Thus, in the graph induced by constraints in A, each vertex on the right-hand side of the graph has only dashed edges or only solid edges (or no edges), i.e., the assignments agree and all constraints can be achieved simultaneously. This proves the lemma. □

Turning now to the outer bound region, each corner point may be identified with the set of active constraints; note that not all possible subsets of constraints are represented. Those sets of constraints actually corresponding to a corner point will be called consistent.

Definition 7: Given a set of sum–rate constraints from Lemma 3, a subset A of these constraints is said to be *consistent* if there is a point that lies on each constraint in A simultaneously, and the point does not violate any of the other constraints.

To finish the proof of Theorem 4, we show in the next lemma that if a collection of constraints is consistent, then it is also compatible. In other words, the corner points of the outer bound polyhedron are compatible, and hence by Lemma 6 achievable.

Lemma 8: If a collection of constraints from Theorem 4 is consistent, then it is also compatible.

Proof: The proof is deferred to Appendix II. □

The proof of Theorem 4, which gives the capacity region of the many-to-one deterministic IC, now requires only a straightforward application of the previous lemmas.

Consider any corner point of the outer bound polyhedron. It is located at the intersection of $K+1$ consistent constraints, and this point is achievable by the previous two lemmas. Hence, all corner points of the outer bound polyhedron are achievable, and because it is convex, the polyhedron defined by all the constraints is the capacity region of the channel. □

Remark 9: It is a pleasing feature of this channel that all corner points of the capacity region can be achieved with zero probability of error using uncoded transmission of symbols.

Remark 10: There is a natural generalization of the HK scheme from the two-user interference channel to the many-user interference channel. The capacity-achieving strategy for the deterministic many-to-one IC presented in this section falls within this generalized class, with each user's signal consisting entirely of private information.

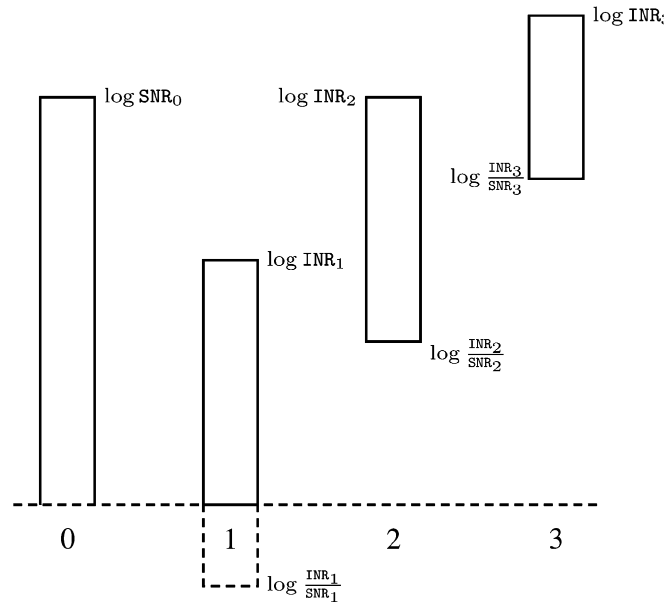

Fig. 11. This figure is analogous to Fig. 9, and shows the interference pattern as observed by receiver 0 (for a different choice of channel gains).

V. Approximate Capacity Region of the Gaussian Many-To-One Interference Channel

In this section, we present inner and outer bounds to the capacity region of the Gaussian many-to-one IC, analogous to those proved for the deterministic case. However, unlike in the deterministic case, the inner and outer bounds do not match: there is a gap of approximately $(3K+3)(1+\log(K+1))$ bits per user (there are $K+1$ users). In comparing the inner and outer bounds, we make use of the deterministic capacity result from the previous section. The achievable region and outer bound, in turn, are shown to lie within $(2K+1)(1+\log(K+1))$ and $(K+2)\log(K+1)$, respectively, bits per user of the capacity region of an appropriately chosen deterministic channel.

In order to harness the understanding gained from the deterministic channel toward the Gaussian case, we construct a similar diagram as that used earlier to describe the signal observed at receiver 0 (Fig. 11). Recall the notation $\mathtt{SNR}_i = |h_{ii}|^2 P_i/N_0$ for $0 \le i \le K$, and $\mathtt{INR}_{ij} = |h_{ji}|^2 P_i/N_0$ for $0 \le i,j \le K$. Since interference occurs only at receiver 0, we will write $\mathtt{INR}_i$ instead of $\mathtt{INR}_{i0}$. For convenience, assume without loss of generality (w.l.o.g.) that the users are ordered so that $\mathtt{INR}_i/\mathtt{SNR}_i \le \mathtt{INR}_{i+1}/\mathtt{SNR}_{i+1}$ for $1 \le i \le K-1$. We also assume that $\mathtt{SNR}_i \ge 1$, since users with $\mathtt{SNR}_i < 1$ may simply be silent, resulting in a loss of at most 1 bit for that user.

A. Achievable Region

The achievable strategy mimics the strategy for the deterministic channel, generalizing the scheme proposed for the example channel in Section II. It can be summarized in a few key steps. First, the range of power-to-noise ratios at receiver

0 is partitioned into intervals to form levels, like in the deterministic channel. There is an independent lattice code for each level, chosen in such a way that the levels do not interact. The scheme then reduces to the achievable scheme for the deterministic channel (with different rates on each level).

Remark 11: In using a random lattice instead of the binary expansion, the construction is seemingly different from the one used for the example channel; yet the binary expansion is also a lattice, and both schemes partition the power-to-noise ratios into levels. A direct generalization of the example scheme using binary inputs is also possible; such an approach is not pursued here because it leads to a larger gap from the outer bound and also requires a more technical development (see [19], where a direct approach is taken for the two-user interference channel).

We now describe the achievable scheme in detail.

Partitioning of power range into levels. The power range as observed at receiver 0 is partitioned according to the values $\mathtt{INR}_i$ and $\frac{\mathtt{INR}_i}{\mathtt{SNR}_i}$ for all users $i, 1 \leq i \leq K$. More precisely, let $\mathtt{SNR}_0 = v_1$ and for $1 \leq i \leq K$ let $v_{2i} = \mathtt{INR}_i$ and $v_{2i+1} = \frac{\mathtt{INR}_i}{\mathtt{SNR}_i}$. Next, remove elements of $\{v_1, \dots, v_{2K+1}\}$ of magnitude less than 1, i.e., let $\{u_1, \dots, u_M\} = \{v_i : v_i > 1\}$. Denote by q_k the kth smallest value among $\{u_1, \dots, u_M\}$, for $k \geq 1$, and let $q_{-1} = 0, q_0 = 1$. The highest endpoint is $q_M = \max(\mathtt{SNR}_0, \max_k \mathtt{INR}_k)$. The resulting intervals are $[q_{k-1}, q_k], 0 \leq k \leq M$. The partition of power ranges into intervals plays the role of levels in the deterministic channel. The associated definitions such as that of $U_j, 1 \leq j \leq n_{00}$, etc., are the same as for the deterministic channel in the beginning of Section IV.

A signal power θ_k, as observed by receiver 0, is associated with each level

$$\theta_k = (q_k - q_{k-1})N_0. \tag{20}$$

The signal powers θ_k are chosen to allow each user to satisfy the transmit power constraint [see (21) and (22)]. Each user $i, 0 \leq i \leq K$, decomposes the transmitted signal at each time step into a sum of independent components

$$x_i = \sum_{k=0}^{M} X_i(k)$$

component $X_i(k)$ being user i's input to the kth level. The signal $X_i(k)$ has power $\theta_k/|h_{0i}|^2$, so is observed by receiver 0 to be of power θ_k. Of course, each user must satisfy an average power constraint, so does not transmit on higher levels than the power constraint allows: $X_i(k) \equiv 0$ for $k > k_{\max}(i)$, where $q_{k_{\max}(i)} = \mathtt{INR}_i$ for $1 \leq i \leq K$ and $q_{k_{\max}(0)} = \mathtt{SNR}_0$. Also, user 0 does not transmit on level 0, losing at most 1 bit. The total power used by transmitter 0 is bounded as

$$E|x_0|^2 \leq \sum_{k=1}^{k_{\max}(0)} \frac{\theta_k}{|h_{00}|^2} < \frac{\mathtt{SNR}_0}{|h_{00}|^2} = P_0. \tag{21}$$

Similarly, the total power used by transmitter i is

$$E|x_i|^2 \leq \sum_{k=0}^{k_{\max}(i)} \frac{\theta_k}{|h_{0i}|^2} = \frac{\mathtt{INR}_i}{|h_{0i}|^2} = P_0. \tag{22}$$

Lattice code for each level. For each interval $[q_{k-1}, q_k]$, a lattice code is selected, as described in [20]: the spherical shaping region has average power per dimension θ_k and the lattice is good for channel coding. The rate R_k of the lattice is chosen to allow decoding, and will be specified later. All users transmitting on a given level use the same code (with independent dithers). As in the deterministic channel, for each level, either user 0 transmits or all of the interfering users transmit.

Decoding procedure. We next describe the decoding procedure at receiver 0. Decoding occurs from the top level downwards, treating the signals from lower levels as Gaussian noise. When the signal on a level is decoded, it is subtracted off completely, and decoding proceeds with the next highest level. Therefore, in describing the decoding procedure, we inductively assume all higher levels have been correctly decoded. On levels where user 0 is silent and interfering users transmit, only the aggregate interfering signal on the level is decoded. This is accomplished by lattice decoding, i.e., decoding to the nearest lattice point (regardless of whether it is a codeword).

The probability of error analysis is simple, because the sum of subsets of an infinite lattice constellation results in a subset of the same infinite constellation. Furthermore, the probability of decoding error when using lattice decoding does not depend on the transmitted codeword. Thus, because each user transmitting on a level uses a subset of the same infinite lattice, it suffices to consider the decoding of an arbitrary codeword from the lattice. Theorem 7 of [20] shows that if the rate (density of lattice points) is not too high relative to the noise entropy, then receiver 0 is able to decode the sum. Treating all lower signals as Gaussian noise results in the desired rates since the Gaussian maximizes entropy subject to a given covariance constraint and Theorem 7 is in terms of the noise entropy. The following is a special case discussed immediately following the more general result of Theorem 7.

Theorem 12 [20]: Arbitrarily reliable transmission at rate R is possible with lattice codes of the form $(v + \Lambda) \cap S$, provided

$$R < \log\left(\frac{P}{\sigma^2}\right).$$

Here $\Lambda \subset \mathbb{R}^N$ is a lattice, $v \in \mathbb{R}^N$ is a dither (i.e., shift), S is a spherical shaping region with power P per dimension, and σ^2 is the noise variance per dimension.

Specifying rate for each level. It remains to specify the rates R_k for each level, and to verify that all users can decode. This is first done for levels $k \geq 1$, where R_k is chosen to allow receiver 0 to decode; we return later to the rates of level 0 codebooks, which user 0 does not decode, as they will be different for each user. It will turn out that user 0 acts as the bottleneck in decoding levels $k \geq 1$. Denote by $N_0(k), 1 \leq k \leq M$, the variance of all signals on levels $0, \dots, k-1$ plus the additive Gaussian noise, as observed at receiver 0. Using Theorem 12, choosing R_k amounts to estimating the effective noise $N_0(k)$

$$R_k = \log\left(\frac{\theta_k}{N_0(k)}\right)^+. \tag{23}$$

Since user 0 does not transmit on level 0, we may write

$$\begin{aligned}N_0(k) &= N_0 + \sum_{i=1}^{K} \mathbf{E}|X_i(0)|^2|h_{0i}|^2 \\ &\quad + \sum_{i=0}^{K}\sum_{j=1}^{k-1} \mathbf{E}|X_i(j)|^2|h_{0i}|^2 \\ &\le (K+1)N_0 + \sum_{i=0}^{K}\sum_{j=1}^{k-1}\theta_j \\ &= (K+1)q_{k-1}N_0. \end{aligned} \tag{24}$$

From the choice of powers θ_k (20) and the effective noise estimate (24), we have that the rate of the codebook for level $k \ge 1$ is

$$\begin{aligned}R_k &= \log\left(\frac{\theta_k}{N_0(k)}\right)^+ \\ &\ge \log\left(\frac{q_k - q_{k-1}}{(K+1)q_{k-1}}\right)^+ \\ &\ge \log\left(1 + \frac{q_k - q_{k-1}}{q_{k-1}}\right) - 1 - \log(K+1) \\ &= \log q_k - \log q_{k-1} - 1 - \log(K+1). \end{aligned} \tag{25}$$

The rates R_k are chosen so that receiver 0 can decode each level $k \ge 1$, but it must be verified that decoding is possible at each receiver $1 \le i \le K$. First, the received power of signal level k at receiver i is $\theta_k |h_{ii}|^2/|h_{0i}|^2 = \theta_k \mathtt{SNR}_i/\mathtt{INR}_i$. User i does not transmit on levels k with $q_k \le \mathtt{INR}_i/\mathtt{SNR}_i$, since these signal levels are received below the noise level at receiver i. Next, the effective noise from levels $k-1$ and weaker is $q_{k-1}N_0|h_{ii}|^2/|h_{0i}|^2 = q_{k-1}N_0\mathtt{SNR}_i/\mathtt{INR}_i$, and there is also Gaussian noise at power $N_0 \le q_{k-1}N_0\mathtt{SNR}_i/\mathtt{INR}_i$ (the inequality follows since $q_{k-1} \ge \mathtt{INR}_i/\mathtt{SNR}_i$). Thus, the signal-to-noise ratio for decoding level k at receiver i is at least $\theta_k/(2q_{k-1}N_0)$, which is greater than the signal-to-noise ratio for decoding at receiver 0, and hence all receivers can decode.

Now, for level $k=0$, the signal from each user is received at the noise level at receiver 0 and is treated as noise, so decoding is performed only at the intended receiver. The transmitted power of the signal at level 0 for user i is $\frac{N_0}{|h_{0i}|^2} = \frac{\mathtt{SNR}_i}{|h_{ii}|^2\mathtt{INR}_i}N_0$. When receiver i attempts to decode this level all other levels have been decoded, so the only remaining noise is the Gaussian noise with variance N_0, and the rate achieved is therefore

$$\log\left(\frac{\mathtt{SNR}_i}{\mathtt{INR}_i}\right)^+. \tag{26}$$

Assigning levels and comparison with deterministic channel. We can now finish describing the achievable strategy for the Gaussian channel by assigning levels exactly as in the strategy for the deterministic channel. To allow comparison of the achievable region with the region for the deterministic channel, we make the correspondence

$$n_{ii} = \log \mathtt{SNR}_i, \qquad 0 \le i \le K$$

and

$$n_{0i} = \log \mathtt{INR}_i, \qquad 1 \le i \le K.$$

On levels without user 0 present, i.e., $k=0$ or $k > k_{\max}(0)$, all users use the full available rate as given in (26) and (25): for $k=0$ user i gets rate

$$\log\left(\frac{\mathtt{SNR}_i}{\mathtt{INR}_i}\right)^+ = (n_{ii} - n_{0i})^+ \tag{27}$$

and on the levels $k > k_{\max}(0)$ user i gets rate

$$\begin{aligned}&\sum_{k=k_{\max}(0)+1}^{k_{\max}(i)} (\log q_k - \log q_{k-1} - 1 - \log(K+1)) \\ &\quad\le (\log q_{k_{\max}(i)} - \log q_{k_{\max}(0)})^+ \\ &\qquad - (M - k_{\max}(0))(1+\log(K+1)) \\ &\quad= \log\left(\frac{\mathtt{INR}_i}{\mathtt{SNR}_0}\right)^+ - (M - k_{\max}(0))(1+\log(K+1)) \\ &\quad= (n_{0i} - n_{00})^+ - (M - k_{\max}(0))(1+\log(K+1)). \end{aligned} \tag{28}$$

Adding the rates in (27) and (28), we see that the total rate of the codebooks on level 0 and levels $k > k_{\max}(0)$ may be chosen to achieve the rates

$$(n_{ii}-n_{0i})^+ + (n_{0i}-n_{00})^+ - (M-k_{\max}(0))(1+\log(K+1)), \quad \text{for each user } 1 \le i \le K. \tag{29}$$

In other words, precisely $\mathcal{C}_{\text{free}}$ from (10) is achievable, with a loss of at most $(M - k_{\max}(0))(1+\log(K+1))$ bits per user, without any further constraints on the rates of codebooks on levels $1 \le k \le k_{\max}(0)$.

Now, each level k with $1 \le k \le k_{\max}(0)$ (user 0 is present on these levels) can support the rate points $(R_k, 0, \ldots, 0)$ and $\{(0, r_1, \ldots, r_K) : r_i = R_k \text{ if } i \in U_k, r_i = 0 \text{ if } i \notin U_k\}$, i.e., restricting attention to level k, the region

$$R_k C_{\log q_k}$$

is achievable, where C_j is the capacity of a deterministic many-to-one IC with a single level, restricted to users $\{0\} \cup U_j$, given in (11) and (12). Note that the regions C_j are the same for $\log q_{k-1} < j \le \log q_k$, since the sets U_j themselves are the same for j in this range. Thus, $\sum_{j=\log q_{k-1}+1}^{\log q_k} C_j$ counts C_j with multiplicity exactly $R_j + 1 + \log(K+1)$, and hence, the achievable region restricted to levels $1 \le k \le k_{\max}(0)$ is within $k_{\max}(0)(1+\log(K+1))$ bits per user of

$$\begin{aligned}&\sum_{k=1}^{k_{\max}(0)} (R_k + 1 + \log(K+1)) C_{\log q_k} \\ &\quad\supseteq \sum_{k=1}^{k_{\max}(0)} \sum_{j=\log q_{k-1}+1}^{\log q_k} C_j \\ &\quad= \sum_{j=1}^{n_{00}} C_j. \end{aligned} \tag{30}$$

Adding the region (30) to the region from (29), we obtain the following lemma.

Lemma 13: The achievable region, as described above, is within $(2K+1)(1+\log(K+1))$ bits per user of

$$\underline{\mathcal{C}} = \mathcal{C}_{\text{free}} + \sum_{j=1}^{n_{00}} C_j \tag{31}$$

which is exactly the deterministic capacity region (13).

The gap between (31) and the sum of (30) to the region from (29) is at most $M(1+\log(K+1))$ per user; the lemma follows by noting that $M \leq 2K+1$, since there are $2K+2$ total endpoints including those of user 0's signal.

Remark 14: The fact that the gains n_{ij} are restricted to be integer valued in the deterministic channel has been disregarded in the above argument. However, this does not pose a problem: instead of putting $n_{ij} = |h_{ij}|^2 P_j/N_0$, one may scale by a sufficiently large integer T and set $n_{ij} = \lfloor T|h_{ij}|^2 P_j/N_0 \rfloor$, and normalize by T. The result is that (31) is simply replaced by the same expression minus ϵ, where ϵ is an arbitrary constant greater than zero. An important point is that the achievable region itself has been set; in this section, the capacity of the deterministic channel is only used to relate two algebraic quantities.

We now turn to the outer bound.

B. Outer Bound

We attempt to emulate the proof of the outer bound for the deterministic case, where we gave receiver 0 side information consisting of all but one of the interfering signals at each level. Continuing with the analogy that additive Gaussian noise corresponds to truncation in the deterministic channel, we introduce independent Gaussian noise with appropriate variance in order to properly restrict the side information given to receiver 0. For example, if $\mathtt{INR}_i = p$ and $\mathtt{INR}_{i-1}/\mathtt{SNR}_{i-1} = q$, then giving the part of the signal x_i above q as side information to receiver 0 calls for $s = x_i + w_i$ where $w_i \sim \mathcal{CN}(0, qN_0)$. Use of this idea leads to the outer bound of the following lemma.

Lemma 15: The capacity region of the Gaussian many-to-one IC is bounded by each of the individual constraints

$$r_i \leq \log(1+\mathtt{SNR}_i), \qquad 0 \leq i \leq K.$$

Moreover, for each $\mathcal{S} \subseteq \{1,\ldots,K\}$ with the property that a relabeling of the indices of $\mathcal{S}$ allows $\mathcal{S} = \{1,\ldots,m\}$ (where $m = |\mathcal{S}|$) such that

$$\begin{gathered}\mathtt{SNR}_0 > 1, \quad \frac{\mathtt{INR}_m}{\mathtt{SNR}_m} \leq \mathtt{SNR}_0, \quad \mathtt{INR}_i > 1, \qquad 1 \leq i \leq m \\ \frac{\mathtt{INR}_i}{\mathtt{SNR}_i} \leq \frac{\mathtt{INR}_{i+1}}{\mathtt{SNR}_{i+1}}, \quad \mathtt{INR}_i < \mathtt{INR}_{i+1}, \qquad 1 \leq i \leq m-1\end{gathered} \tag{32}$$

the following sum–rate constraint holds:

$$\begin{aligned} &r_0 + r_1 + \cdots + r_m \\ &\leq \sum_{i=1}^{m} \log\left(\frac{\mathtt{SNR}_i}{\mathtt{INR}_i}\right)^+ + \sum_{i=1}^{m-1}\left(\log(\mathtt{INR}_i) - \log\left(\frac{\mathtt{INR}_{i+1}}{\mathtt{SNR}_{i+1}}\right)^+\right)^+ \\ &\quad + \max(\log(\mathtt{INR}_m), \log(\mathtt{SNR}_0)) + (m+2)\log(m+1). \end{aligned} \tag{33}$$

Proof: The proof is deferred to Appendix III. □

Remark 16: The conditions (32) do not nullify any useful constraints. If $\mathtt{SNR}_0 \leq 1$, then $r_0 \leq 1$ (from the point-to-point constraint), and the capacity region is essentially (within 1 bit per user) given by the intersection of the individual rate constraints. The other conditions ensure that a user causes meaningful interference to receiver 0, and should therefore be included in the constraint: if $\frac{\mathtt{INR}_m}{\mathtt{SNR}_m} > \mathtt{SNR}_0$, then the signal from user m may be subtracted off by receiver 0 before attempting to decode the intended signal (user m must reduce the rate by at most $\log K$ bits for this to be true); if the signal from transmitter i has $\mathtt{INR}_i \leq 1$, then transmitter i may just transmit at the full available power, causing essentially (again up to 1 bit) no interference to user 0. The choice $\frac{\mathtt{INR}_i}{\mathtt{SNR}_i} \leq \frac{\mathtt{INR}_{i+1}}{\mathtt{SNR}_{i+1}}$ is simply a relabeling of the users; with this labeling, if $\mathtt{INR}_i \geq \mathtt{INR}_{i+1}$, then user $i+1$ may be removed from the sum–rate constraint (the sum–rate constraint on $\{0,1,\ldots,m\}$ is implied by the sum–rate constraint on $\{0,1,\ldots,i,i+2,\ldots,m\}$ together with the individual constraint on user $i+1$). This is most easily understood by checking the equivalent condition for the deterministic channel.

This region (33) may be compared to the capacity region of a deterministic channel by making the correspondence $n_{ii} = \log \mathtt{SNR}_i, 0 \leq i \leq K$, and $n_{0i} = \log \mathtt{INR}_i, 1 \leq i \leq K$, as before. With this choice, (33) gives for each $\mathcal{S} \subseteq \{1,\ldots,K\}$ such that a relabeling of the indices allows $\mathcal{S} = \{1,\ldots,m\}$ with $n_{0m} - n_{mm} \leq n_{00}, n_{0i} > 0, 0 \leq i \leq K$, and also $n_{0i} - n_{ii} \leq n_{0,i+1} - n_{i+1,i+1}$ and $n_{0i} \leq n_{0,i+1}$ for $1 \leq i \leq m-1$, the sum–rate constraint

$$\begin{aligned} &r_0 + r_1 + \cdots + r_m \\ &\quad \leq \sum_{i=1}^{m}(n_{ii} - n_{0i})^+ + \sum_{i=1}^{m-1}\left(n_{0i} - (n_{0,i+1} - n_{i+1,i+1})^+\right)^+ \\ &\qquad + \max(n_{0m}, n_{00}) + (m+2)\log(m+1) \end{aligned} \tag{34}$$

$$\begin{aligned} &\quad = \sum_{i=1}^{m}\left((n_{0i} - n_{00})^+ + (n_{ii} - n_{0i})^+\right) \\ &\qquad + \sum_{k=1}^{n_{00}}(|U_k \cap \mathcal{S}| - 1)^+ + n_{00} + (m+2)\log(m+1) \end{aligned} \tag{35}$$

$$\quad = f_{\text{free}}(\mathcal{S}) + \sum_{k=1}^{n_{00}} f_k(\mathcal{S}) + (m+2)\log(m+1). \tag{36}$$

The step leading from (34) and (35) can be understood with the help of Fig. 11. The first sum of (34) counts the signals of each user received below the noise level at receiver 0. Each term in the second sum in (34) counts the overlap of rectangle i with rectangle $i+1$. By the conditions (32) the signal of each user that interferes above user 0's signal (for user i this is the top $(n_{0i} - n_{00})^+$ levels) also overlaps with the signal from user m, so it is counted in this sum; this accounts for the first sum in (35), except for the expression $(n_{0m} - n_{00})^+$ in the $i = m$ term, which is dealt with later. Next, consider a level $k, 1 \leq k \leq n_{00}$. If there are t interfering users at this level, i.e., $|U_k \cap \mathcal{S}| = t$, then the second sum of (34) counts a contribution from each of the t interfering users except the last, since this last user does not overlap with an additional user at level k. Thus, adding over levels $1 \leq k \leq n_{00}$, this gives rise to the term $\sum_{k=1}^{n_{00}}(|U_k \cap \mathcal{S}| -$

$1)^+$ in (35). Finally, note that $\max(n_{0m}, n_{00}) = n_{00} + (n_{0m} - n_{00})^+$ precisely contributing the quantity neglected earlier and also the term n_{00}.

Since $m \leq K$, (36) implies a constant gap between the outer bound of Lemma 15 and the outer bound of Lemma 3 for the deterministic channel.

Lemma 17: The Gaussian many-to-one IC outer bound (Lemma 15) lies within $(K+2)\log(K+1)$ bits per user of the outer bound $\overline{\mathcal{C}}$ for the corresponding deterministic channel.

All the ingredients are in place for the main result of the paper.

Theorem 18: The capacity region of the Gaussian many-to-one interference channel lies within $(3K+3)(1+\log(K+1))$ bits per user of the region given in Lemma 15.

Proof: Directly comparing the outer bound with the achievable region would require proving a counterpart to Lemma 8. Fortunately, the outer bound and the achievable region have each already been compared in Lemmas 13 and 17 to the capacity region $\mathcal{C}_D = \overline{\mathcal{C}} = \underline{\mathcal{C}}$ of a corresponding deterministic channel, expressed in two different ways. This upper bounds the gap between the achievable region and outer bound, proving the theorem. □

The notion of the generalized degrees-of-freedom region, defined in [2], gives insight towards the behavior at high SNR and INR. The generalized degrees-of-freedom region for the many-to-one IC is found by putting $\mathtt{SNR}_i = s^{\alpha_i}$ for $0 \leq i \leq K$ and $\mathtt{INR}_i = s^{\beta_i}$ and taking the limit $s \to \infty$. The constants α_i and β_i are proportional to $\mathtt{SNR}_i$ and $\mathtt{INR}_i$ in the decibel scale. Let $C(s, \overrightarrow{\alpha}, \overrightarrow{\beta})$ be the capacity region of a many-to-one IC with $\{\mathtt{SNR}_i\}, \{\mathtt{INR}_i\}$ thus defined. The resulting degrees-of-freedom region is

$$D(\overrightarrow{\alpha}, \overrightarrow{\beta}) = \lim_{s\to\infty} \frac{C(s, \overrightarrow{\alpha}, \overrightarrow{\beta})}{\log s}.$$

To evaluate this limit, note that Theorem 18 allows to directly calculate the degrees-of-freedom from the outer bound of Lemma 15.

Corollary 19: The generalized degrees-of-freedom region of the Gaussian many-to-one channel is given by the set of points $(d_0, d_1, \ldots, d_K)$ satisfying each of the individual constraints

$$d_i \leq \alpha_i, \qquad 0 \leq i \leq K$$

and for each $\mathcal{S} \subseteq \{1, \ldots, K\}$ with the property that a relabeling of the indices of $\mathcal{S}$ allows $\mathcal{S} = \{1, \ldots, m\}$ (where $m = |\mathcal{S}|$) such that

$$\alpha_0 > 0, \quad \beta_m - \alpha_m \leq \alpha_0, \quad \beta_i > 0, \quad 1 \leq i \leq m. \qquad (37)$$

and $\beta_i - \alpha_i \leq \beta_{i+1} - \alpha_{i+1}$ for $1 \leq i \leq m-1$, the following sum–rate constraint holds:

$$\begin{aligned} d_0 + d_1 + \cdots + d_m &\\ \leq \sum_{i=1}^{m} (\alpha_i - \beta_i)^+ &+ \sum_{i=1}^{m-1} (\beta_i - (\beta_{i+1} - \alpha_{i+1})^+)^+ \\ &+ \max(\beta_m, \alpha_0). \end{aligned}$$

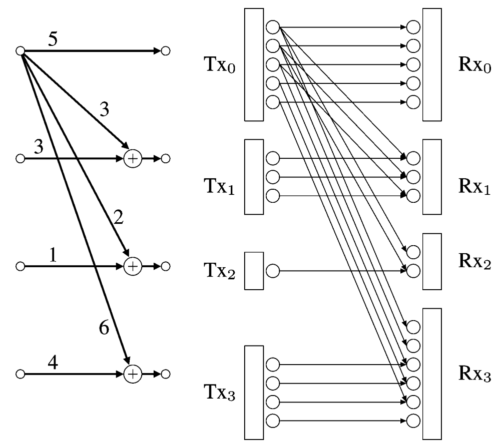

Fig. 12. One-to-many interference channel in this figure is obtained by reversing the roles of transmitters and receivers in the many-to-one channel in Fig. 8.

Remark 20: This is exactly the scaled capacity region of a particular deterministic channel, assuming $\{\alpha_i\}, \{\beta_i\}$ are rational numbers. The first sum accounts for the part of each signal that is received below the noise level at user 0. The second sum corresponds to the number of users minus one on levels with multiple interferers, and the final term is the rate that is achieved with each level used exactly once up to β_m or α_0, whichever is larger. The constraint may be compared to (17), recalling the conditions (37).

This concludes the treatment of the many-to-one IC. The second half of the paper tackles the one-to-many IC.

VI. Deterministic One-to-Many Interference Channel

Consider the channel obtained by reversing the roles of the transmitters and receivers in the deterministic many-to-one IC of Section IV. More precisely, if the original channel has gains $\tilde{n}_{ii}, 0 \leq i \leq K$ and $\tilde{n}_{0i}, 1 \leq i \leq K$, let the reversed channel have gains $n_{ii} = \tilde{n}_{ii}, 0 \leq i \leq K$ and $n_{i0} = \tilde{n}_{0i}, 1 \leq i \leq K$ (see Fig. 12).

The channel input–output relationship is given by (9) with all gains equal to zero except the direct gains $n_{ii}, 0 \leq i \leq K$, and cross gains from transmitter $0, n_{i0}, 1 \leq i \leq K$. Specifically, we have

$$y_0 = \lfloor 2^{n_{00}} x_0 \rfloor$$

and

$$y_i = \lfloor 2^{n_{ii}} x_i \rfloor \oplus \lfloor 2^{n_{i0}} x_0 \rfloor, \qquad 1 \leq i \leq K.$$

Recall the simple capacity achieving scheme for the deterministic many-to-one IC: each level as observed at receiver 0 is allocated entirely to user 0 or to all users causing interference on the level. The corresponding achievable scheme for the deterministic one-to-many IC allocates each level as observed at *transmitter* 0 either to user 0 or to all other users *experiencing* interference from this level. A little thought reveals that the two achievable regions are the same, and one suspects that the capacity regions are the same as well. This is confirmed by the following theorem. Thus, the many-to-one and one-to-many chan-

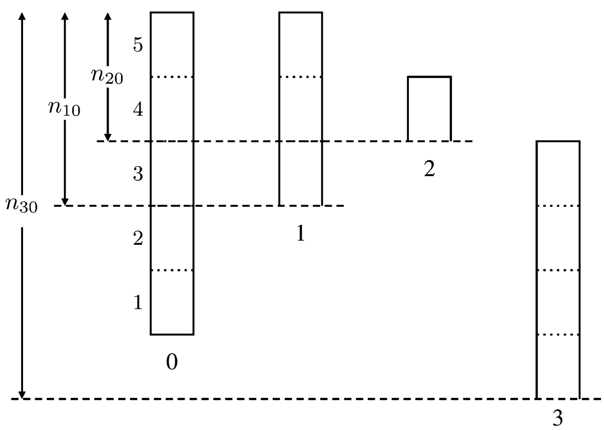

Fig. 13. Interference pattern as *created* by transmitter 0 for the channel in Fig. 12. Here, $U_1 = \{3\}, U_2 = \{3\}, U_3 = \{1,3\}, U_4 = \{1,2\}$, and $U_5 = \{1\}$. Here $\underline{\mathrm{x}}_3$ consists of the $n_{03} - n_{00} = 1$ lowest levels from signal 3.

nels are reciprocal (see [21] for a discussion of reciprocal channels).

Theorem 21: The capacity region of a deterministic one-to-many IC with channel gains $n_{ii}, 0 \le i \le K$ and $n_{0i}, 1 \le i \le K$, is equal to the capacity region of a deterministic many-to-one IC (as given in Lemma 3) with gains $\tilde{n}_{ii} = n_{ii}, 0 \le i \le K$ and $\tilde{n}_{0i} = n_{i0}, 1 \le i \le K$.

The notation in this section is very similar to that used for the many-to-one deterministic interference channel of Section IV (see Fig. 13). Assume without loss of generality that x_0 is restricted to the elements that appear in the output y_0, i.e., $x_0 \in \mathbb{F}_2^{n_{00}}$. Denote by $U_k \subseteq \{1,\dots,K\}, 1 \le k \le n_{00}$, the set of users potentially experiencing interference from the kth level at transmitter 0

$$U_k = \{i : 1 \le i \le K, n_{00} - n_{i0} < k \le n_{ii} - n_{i0} + n_{00}\}.$$

This definition for U_k follows by considering how receiver i observes the incoming signal levels: the kth level at transmitter 0 is received at level $k - n_{00} + n_{i0}$, the bottom of signal i is at 0, and the top of signal i is at n_{ii}. Thus, level k at transmitter 0 causes interference to user i if $0 < k - n_{00} + n_{i0} \le n_{ii}$, giving the above definition. For a user $i, 1 \le i \le K$, and a level $k, 1 \le k \le n_{00}$, denote by $x_{i|k}$ the signal of user i, restricted to the level that overlaps with level k of user 0's signal. Finally, let $\underline{\mathrm{x}}_i$ be the restriction of the input from transmitter i to the lowest $(n_{i0} - n_{00})^+$ levels. This is the part of x_i that appears below the interference from user 0. Similarly, let $\bar{\mathrm{x}}_i$ be the restriction of the input from transmitter i to the highest $(n_{ii} - n_{i0})^+$ levels. This is the part of x_i that lies above the interference from user 0.

Let us quickly relate the sets U_k for the one-to-many channel to the analogous sets in the many-to-one channel. As in Theorem 21, consider a many-to-one channel with gains $\tilde{n}_{ii} = n_{ii}, 0 \le i \le K$ and $\tilde{n}_{0i} = n_{i0}, 1 \le i \le K$. Denote the set of users causing interference to the kth level of user 0's signal at receiver 0 by $\widetilde{U}_k = \{i : 1 \le i \le K, \tilde{n}_{0i} - \tilde{n}_{ii} < k \le \tilde{n}_{0i}\}$ (this is the definition of U_k for the many-to-one channel; see Section IV). It holds that

$$\begin{aligned} &\tilde{n}_{0i} - \tilde{n}_{ii} < k \le \tilde{n}_{0i} \\ &\quad \Leftrightarrow n_{i0} - n_{ii} < k \le n_{i0} \\ &\quad \Leftrightarrow n_{i0} - n_{ii} - n_{00} < k - n_{00} \le n_{i0} - n_{00} \\ &\quad \Leftrightarrow n_{00} - n_{i0} < 1 + n_{00} - k \le n_{ii} - n_{i0} + n_{00} \end{aligned}$$

whence $\widetilde{U}_k = U_{1+n_{00}-k}$. In particular, for any $\mathcal{S} \subseteq \{1,\dots,K\}$, we have

$$\sum_{k=1}^{n_{00}} |U_k \cap \mathcal{S}| = \sum_{k=1}^{n_{00}} |\widetilde{U}_k \cap \mathcal{S}|. \tag{38}$$

Using this last equation, we may explicitly write the capacity region of the deterministic many-to-one channel referred to in Theorem 21 in terms of the gains n_{ij} and sets U_k defined for the one-to-many channel. The deterministic many-to-one capacity region (see Lemma 3) is given by those rate points satisfying the individual rate constraints

$$r_i \le \tilde{n}_{ii} = n_{ii}, \qquad 0 \le i \le K \tag{39}$$

and the $2^K - 1$ sum–rate constraints, one for each nonempty $\mathcal{S} \subseteq \{1,\dots,K\}$

$$\begin{aligned} r_0 + \sum_{i\in\mathcal{S}} r_i &\le \tilde{n}_{00} + \sum_{k=1}^{\tilde{n}_{00}} (|\widetilde{U}_k \cap \mathcal{S}| - 1)^+ \\ &\quad + \left(\sum_{i\in\mathcal{S}} (\tilde{n}_{0i} - \tilde{n}_{ii})^+ + (\tilde{n}_{0i} - \tilde{n}_{00})^+\right) \\ &\le n_{00} + \sum_{k=1}^{n_{00}} (|U_k \cap \mathcal{S}| - 1)^+ \\ &\quad + \left(\sum_{i\in\mathcal{S}} (n_{i0} - n_{ii})^+ + (n_{i0} - n_{00})^+\right). \end{aligned} \tag{40}$$

It may appear that the deterministic many-to-one capacity region is simply being repeated here, but the point is that the capacity region has been reexpressed in terms of the parameters for the corresponding one-to-many channel (note that the parameters for the many-to-one channel have been relabeled to be of the form $\tilde{n}_{ij}$ in this section). Thus, proving Theorem 21 amounts to showing that (39) and (40) determine the capacity region for the one-to-many channel. We first show that this region is an outer bound to the capacity, and then show achievability. Afterward, an alternative achievable strategy, a generalized HK strategy, is shown to be optimal as well.

A. Proof of Outer Bound

We may (without loss of generality) order the users so that $n_{i0} \le n_{i+1,0}, 1 \le i \le K-1$. As before, the rate across each link cannot exceed the point-to-point capacity, hence

$$r_i \le n_{ii}, \qquad 0 \le i \le K. \tag{41}$$

Next, we prove the claimed sum–rate constraint (40) on a set of users $\mathcal{S} \cup \{0\}$, where $\mathcal{S} \subseteq \{1, \ldots, K\}$. Unlike the deterministic many-to-one channel, no side information is required to prove the constraint. For each $1 \le i \le K$, let $\sigma_i = \{x_{0|k} : (n_{00} - n_{i0} + n_{ii})^+ < k \le n_{00}\}$ be the part of signal 0 that appears above the intended signal at receiver i. Note that by the definition σ_i is determined by y_i, and also σ_i is independent of x_i, hence

$$\begin{aligned} I\left(x_i^N; y_i^N\right) &= H\left(x_i^N\right) - H\left(x_i^N \middle| y_i^N\right) \\ &= H\left(x_i^N \middle| \sigma_i^N\right) - H\left(x_i^N \middle| \sigma_i^N, y_i^N\right) \\ &= I\left(x_i^N; y_i^N \middle| \sigma_i^N\right). \end{aligned}$$

Now, Fano's inequality and the data processing inequality give

$$\begin{aligned} &N\left(r_0 + \sum_{i\in\mathcal{S}} r_i - \epsilon_N\right) \\ &\quad\le I\left(x_0^N; y_0^N\right) + \sum_{i\in\mathcal{S}} I\left(y_i^N; x_i^N\right) \\ &\quad= I\left(x_0^N; y_0^N\right) + \sum_{i\in\mathcal{S}} I\left(y_i^N; x_i^N \middle| \sigma_i^N\right) \\ &\quad= H\left(x_0^N\right) + \sum_{i\in\mathcal{S}} \left(H(y_i^N | \sigma_i^N) - H\left(y_i^N \middle| x_i^N, \sigma_i^N\right)\right). \end{aligned}$$

Breaking the signals apart by level, using the independence bound on entropy, the chain rule for entropy, and removing conditioning, we may rewrite the above as

$$\begin{aligned} &H\left(x_0^N\right) + \sum_{i\in\mathcal{S}} \left(H(y_i^N | \sigma_i^N) - H\left(y_i^N \middle| x_i^N, \sigma_i^N\right)\right) \\ &\quad= \sum_{i\in\mathcal{S}} \Big(H\left(\underline{x}_i^N, \left\{x_{i|k}^N + x_{0,k}^N : k \text{ s.t. } i \in U_k\right\}, \bar{x}_i^N\right) \\ &\qquad - H\left(\{x_{0,k}^N : k \text{ s.t. } i \in U_k\}\right)\Big) + H\left(\left\{x_{0|k}^N\right\}_{k=1}^{n_{00}}\right) \\ &\quad\le \sum_{i\in\mathcal{S}} \Big(H\left(\underline{x}_i^N\right) \\ &\qquad + H\left(\bar{x}_i^N\right) + H\left(\left\{x_{i|k}^N + x_{0,k}^N : k \text{ s.t. } i \in U_k\right\}\right)\Big) \\ &\qquad + H\left(\left\{x_{0|k}^N : k \text{ s.t. } \mathcal{S} \cap U_k = \varnothing\right\}\right) \\ &\quad\le N\Bigg(\sum_{i\in\mathcal{S}} ((n_{i0} - n_{00})^+ + (n_{ii} - n_{i0})^+) \\ &\qquad + \sum_{k=1}^{n_{00}} \max(|U_k \cap \mathcal{S}|, 1)\Bigg). \end{aligned}$$

The first sum in the last expression comes from the independence bound on $H(\underline{x}_i^N)$ and $H(\bar{x}_i^N)$. For the second sum, $\sum_{k=1}^{n_{00}} \max(|U_k \cap \mathcal{S}|, 1)$, note that in the next-to-last expression a level $k, 1 \le k \le n_{00}$ is counted once for each user in $U_k \cap \mathcal{S}$, and if $\mathcal{S} \cap U_k = \varnothing$, then it is counted exactly once. Taking $N \to \infty$ proves the constraint. □

B. Achievability of Outer Bound

As mentioned before, the achievable scheme is nearly the same as that of the deterministic many-to-one IC, with either user 0 or all other users transmitting on a level. Each level $1 \le k \le n_{00}$ viewed individually has capacity C_k, where C_k is given by (11) and (12). By transmitting on levels above and below the interference from user 0, the region C_{free} is achievable without affecting the remaining levels. Thus, the achievable region

$$\mathcal{C}_{\text{free}} + \sum_{k=1}^{n_{00}} \mathcal{C}_k$$

is exactly the same as for the deterministic many-to-one channel (13). Also, the outer bound is the same as for the many-to-one channel, and since they match by Theorem 4, this completes the proof of Theorem 21. □

C. Generalized Han–Kobayashi Scheme

The achievable scheme of the previous section treats each level separately. In the Gaussian one-to-many IC, however, instead of decomposing the channel into independent subchannels by level, it will turn out to be more natural to consider a generalized HK scheme. Comparing the achievable region of the HK scheme to the outer bound is most readily performed in the deterministic setting, where the two regions are equal. Therefore, we give an HK scheme for the deterministic channel.

Assume without loss of generality that the users are ordered by increasing interference from user 0, i.e., $n_{i0} \ge n_{i-1,0}$ for $2 \le i \le K$, and that $n_{10} \ge 1$ and $n_{K0} - n_{00} \le n_{KK}$ (so that all users actually experience interference from user 0). User 0 will split its signal into levels, with each receiver decoding the interference from user 0 that is received above the noise level.

To simplify the subsequent definitions, we put $n'_{00} = 0$ and $n'_{i0} = n_{i0}$ for $1 \le i \le K$. With this definition, the truncation of signal 0 at each receiver $1 \le i \le K$ occurs at level $(n_{00} - n_{i0})^+$, i.e., this is the highest level that is truncated, and all higher levels are observed by receiver i. The signal from user 0 now decomposes naturally according to which users can observe each level: let the ith signal, $1 \le i \le K$, from user 0 be

$$X_0(i) = \{x_{0|k} : (n_{00} - n_{i0})^+ < k \le n_{00} - n'_{i-1,0}\}$$

and let

$$X_0(K+1) = \{x_{0|k} : 1 \le k \le (n_{00} - n_{K0})^+\}.$$

The signals $X_0(1), \ldots, X_0(i)$ are received by user i above the noise level and are decoded, i.e., signal $X_0(i)$ is common information to users $i, \ldots, K$. The signal $X_0(K+1)$ (possibly vacuous in the case $n_{K0} \ge n_{00}$, i.e., if receiver K observes all of user 0's signal) is received below the noise level of all users except user 0, and is therefore private information.

Each user $i, 1 \le i \le K$, jointly decodes the intended signal x_i together with $X_0(1), \ldots, X_0(i)$. Thus, the achievable rate region is given by the intersection of a collection of MAC, one at each receiver. Denote the rate of signal $X_0(i)$ by $R_0(i)$. The MAC constraints at receiver 0 (on the rates $R_0(1), \ldots, R_0(K+1)$) are implied by the "individual" rate constraints

$$\begin{aligned} R_0(k) &\le n_{k0} - n_{k-1,0}, \qquad 2 \le k \le K \\ R_0(1) &\le \min(n_{10}, n_{00}) \\ R_0(K+1) &\le (n_{00} - n_{K0})^+. \end{aligned} \tag{42}$$

Some notation is necessary to cleanly express the constraints at the other receivers. Let $\lambda(i) \in \{1, \dots, i\}$ be such that the signal $X_0(\lambda(i))$ interferes at the top level of x_i at receiver i, i.e.,

$$n'_{\lambda(i),0} - n_{00} + n_{i0} < n_{ii} \leq n_{\lambda(i)-1,0} - n_{00} + n_{i0}.$$

Rearranging, we have

$$n'_{\lambda(i),0} < n_{ii} + n_{00} - n_{i0} \leq n_{\lambda(i)-1,0}$$

if there is such a λ, and otherwise set $\lambda(i) = 0$. For example, in Fig. 15, we have $\lambda(1) = 0$, $\lambda(2) = 1$, and $\lambda(3) = 3$.

Now, the signals $X_0(1), \dots, X_0(\lambda(i) - 1)$ appear above the signal x_i, and are therefore observed cleanly. Thus, the MAC constraints at receiver $i, 1 \leq i \leq K$ are implied by the following subset of constraints: the above individual constraints (42) on $R_0(1), \dots, R_0(i)$ and the individual constraint

$$r_i \leq n_{ii} \tag{43}$$

together with the sum–rate constraints

$$r_i + \sum_{k=\lambda(i)}^{i} R_0(k) \leq n_{i0} - n'_{\lambda(i)-1,0} \tag{44}$$

and

$$r_i + \sum_{k=\lambda(i)+1}^{i} R_0(k) \leq n_{ii}. \tag{45}$$

We now check that the achievable region for the HK scheme contains the achievable region from the previous section obtained by considering each level separately. First, the sum–rate constraints (44) and (45) at each user are easily seen to result from adding the pairwise constraints (12) on users i and 0 on the relevant levels. Similarly, the individual constraints on the rates $R_0(1), \dots, R_0(K+1)$ are implied by adding the individual constraints (11) on the relevant levels. Thus, the constraints defining the HK achievable region are looser than those defining the capacity-achieving scheme, and hence the HK scheme achieves capacity. These conclusions are recorded in the following proposition.

Proposition 22: The capacity region of the deterministic one-to-many IC is achieved using a generalized HK scheme as described above and can be expressed by the constraints (42), (43), (44), and (45).

VII. Approximate Capacity Region of the One-to-Many Gaussian Interference Channel

Define the signal to noise ratios $\mathtt{SNR}_i = |h_{ii}|^2 P_i / N_0, 0 \leq i \leq K$ and $\mathtt{INR}_i = |h_{i0}|^2 P_0 / N_0, 1 \leq i \leq K$. We assume the users are ordered by increasing values of $\mathtt{INR}$, i.e., $\mathtt{INR}_{i+1} > \mathtt{INR}_i$ for $1 \leq i \leq K-1$. Moreover, we assume as in the many-to-one IC that $\mathtt{INR}_1 > 1$: any user with $\mathtt{INR}_i \leq 1$ can simply treat the interference as noise and lose at most 1 b relative to the point to point AWGN channel. Fig. 14 depicts the one-to-many Gaussian interference channel.

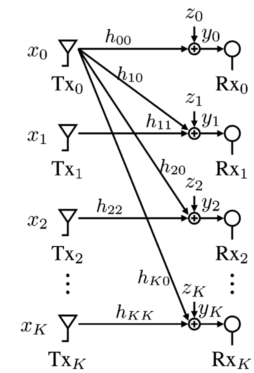

Fig. 14. One-to-many Gaussian interference channel has one user causing interference to K other users.

Theorem 23: The one-to-many Gaussian IC with power-to-noise ratios $\mathtt{SNR}_i, 0 \leq i \leq K$, and $\mathtt{INR}_i, 1 \leq i \leq K$, has capacity region within $(2K+1, 1, \dots, 1)$ bits of the region defined by: the individual rate constraints

$$r_i \leq \log(1 + \mathtt{SNR}_i), \qquad 0 \leq i \leq K \tag{46}$$

and for each subset of users $\mathcal{S} \subseteq \{1, \dots, K\}$ with the property that a relabeling of the indices of $\mathcal{S}$ allows $\mathcal{S} = \{1, \dots, m\}$ (where $|\mathcal{S}| = m$) such that

$$\mathtt{INR}_{i+1} > \mathtt{INR}_i, \qquad \text{for } 1 \leq i \leq K-1$$

and

$$\mathtt{INR}_1 > 1$$

the following sum–rate constraint holds:

$$r_0 + \sum_{i=1}^{m} r_i \leq \log\left(1 + \frac{\mathtt{SNR}_0}{1 + \mathtt{INR}_m}\right) + \log(1 + \mathtt{SNR}_1 + \mathtt{INR}_1) + \sum_{i=2}^{m} \log\left(1 + \mathtt{SNR}_i + \frac{\mathtt{INR}_i}{1 + \mathtt{INR}_{i-1}}\right). \tag{47}$$

A. Outer Bound

In contrast with the deterministic case, side information is required to prove the sum–rate constraint. Let $\mathcal{S} \subseteq \{1, \dots, K\}$, and by relabeling, assume $\mathcal{S} = \{1, \dots, m\}$ where $m = |\mathcal{S}|$. Furthermore, assume $\frac{\mathtt{INR}_m}{\mathtt{SNR}_0} \leq \mathtt{SNR}_m$; otherwise receiver m can cleanly decode the interference from user 0 while treating its own signal as noise, and the constraint is redundant.

We give as side information to receiver 0 the interfering signal as observed at receiver m (the receiver experiencing the greatest interference), and we give as side information to each receiver $i, 2 \leq i \leq m$, the interfering signal x_0 as observed at receiver $i - 1$

$$\begin{aligned} s_0 &= h_{m0} x_0 + z_m \\ s_1 &= \varnothing \\ s_i &= h_{i-1,0} x_0 + z_{i-1}, \qquad 2 \leq i \leq m. \end{aligned} \tag{48}$$

Now, Fano's inequality and the data processing inequality give

$$\begin{aligned} N&\left(r_0+\sum_{i=1}^m r_i-\epsilon_N\right) \\ &\le I\left(y_0^N,s_0^N;x_0^N\right)+\sum_{i=1}^m I\left(y_i^N,s_i^N;x_i^N\right) \\ &= h\left(y_0^N\middle| s_0^N\right)+h\left(s_0^N\right)-h\left(z_0^N,z_m^N\right) \\ &\quad +\sum_{i=1}^m\left(h\left(y_i^N|s_i^N\right)+h\left(s_i^N\right)-h\left(y_i^N,s_i^N|x_i^N\right)\right). \end{aligned} \quad (49)$$

The fact that conditioning reduces entropy implies that for $1\le i\le m-1$

$$\begin{aligned} h\left(y_i^N,s_i^N|x_i^N\right) &= h\left(h_{i0}x_0^N+z_i^N,s_i^N\right) \\ &\ge h\left(h_{i0}x_0^N+z_i^N\right)=h\left(s_{i+1}^N\right) \end{aligned}$$

and

$$h\left(y_m^N,s_m^N\middle| x_m^N\right)\ge h\left(y_m^N\middle| x_m^N\right)=h\left(s_0^N\right).$$

Plugging this into (49), the sum telescopes, producing

$$\begin{aligned} N&\left(r_0+\sum_{i=1}^m r_i-\epsilon_N\right) \\ &\le h\left(y_0^N\middle| s_0^N\right)-h\left(z_0^N,z_m^N\right)+h(y_1)+\sum_{i=2}^m h\left(y_i^N\middle| s_i^N\right). \end{aligned} \quad (50)$$

We bound each term using the fact that the Gaussian distribution maximizes entropy for a fixed conditional variance

$$\begin{aligned} &h\left(y_0^N\middle| s_0^N\right) \\ &\quad = h\left(h_{00}x_0^N+z_0^N\middle| h_{m0}x_0^N+z_m^N\right) \\ &\quad \le N\log\left(1+\frac{\mathtt{SNR}_0}{1+\mathtt{INR}_m}\right)+N\log(\pi e N_0) \end{aligned}$$

and for $2\le i\le m$

$$\begin{aligned} &h\left(y_i^N\middle| s_i^N\right) \\ &\quad = h\left(h_{i0}x_0^N+h_{ii}x_i^N+z_i^N\middle| h_{i-1,0}x_0^N+z_{i-1}^N\right) \\ &\quad \le N\log\left(1+\mathtt{SNR}_i+\frac{\mathtt{INR}_i}{1+\mathtt{INR}_{i-1}}\right)+N\log(\pi e N_0). \end{aligned}$$

Also

$$h\left(y_1^N\right)\le N\log(1+\mathtt{SNR}_1+\mathtt{INR}_1)+N\log(\pi e N_0).$$

Combining these calculations and taking $N\to\infty$, we have the sum–rate constraint

$$\begin{aligned} r_0+\sum_{i=1}^m r_i &\le \log\left(1+\frac{\mathtt{SNR}_0}{1+\mathtt{INR}_m}\right)+\log(1+\mathtt{SNR}_1+\mathtt{INR}_1) \\ &\quad +\sum_{i=2}^m\log\left(1+\mathtt{SNR}_i+\frac{\mathtt{INR}_i}{1+\mathtt{INR}_{i-1}}\right). \end{aligned} \quad (51)$$

□

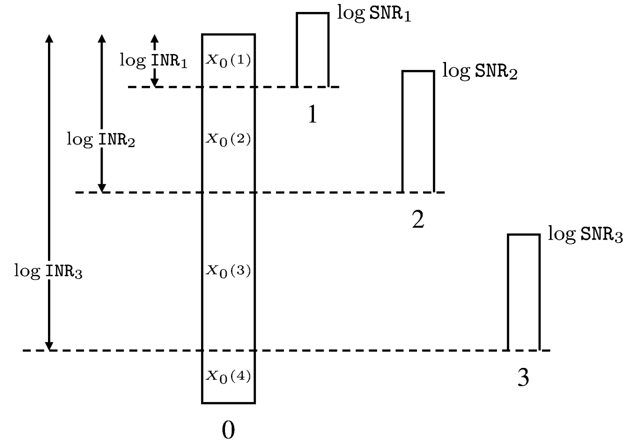

Fig. 15. Superposition of the received signal levels at each user for an example channel. The dashed lines indicate the noise floor for each receiver. User 0 employs a superposition code with codebooks $\{X_0(k)\}_{k=1}^i$ intended for receiver i.

Before proceeding with the achievable scheme, let us first rewrite this region in a form that will allow to easily compare with the deterministic channel region (42), (43), (44), and (45). Let $n_{ii}=\log\mathtt{SNR}_i, 0\le i\le K$, and $n_{i0}=\log\mathtt{INR}_i, 1\le i\le K$. The region given by (46) and (47) may be enlarged to give the region defined by the constraints

$$r_i\le 1+n_{ii}, \qquad 0\le i\le K \quad (52)$$

and for each subset of users $\mathcal{S}\subseteq\{1,\dots,K\}$ as above

$$\begin{aligned} r_0+\sum_{i=1}^m r_i &\le (m+1)+(n_{00}-n_{m0})+\max(n_{11},n_{10}) \\ &\quad +\sum_{i=2}^m\max(n_{ii},n_{i0}-n_{i-1,0}). \end{aligned} \quad (53)$$

Summing over levels instead of users yields exactly the sum–rate constraint (40) of the deterministic channel with an added gap of $m+1$. Viewing the gap as coming entirely from the rate of user 0, the deterministic achievable region is within $K+1$ b/s/Hz at user 0 of the outer bound.

Remark 24: The constraint can be interpreted using Fig. 15. In the figure, the received signal power occupancy at each receiver is superimposed in the appropriate position relative to the signal of user 0. The noise floor of user i is $\log\mathtt{INR}_i$ levels from the top of user 0's signal. As in the deterministic channel, on each level either user 0 transmits or all other users transmit. The sum–rate constraint (51) counts each level once if user 0 causes interference to one or no users, and equal to the number of users if more than two users are interfered by a level.

B. Achievable Region

As in the many-to-one channel, the achievable strategy emulates the approach used for the deterministic case. In the many-to-one channel lattice codes are used to align the interference at receiver 0; in contrast, since there are only two signals at each receiver in the one-to-many channel, it suffices to adopt a rate-splitting approach using a superposition of random Gaussian codebooks. The scheme is completely analogous to the HK scheme for the deterministic channel of the previous section.

We now describe the achievable scheme. In constructing the scheme, we temporarily assume that $\mathtt{INR}_K \leq \mathtt{SNR}_K$. Transmitter 0 uses a superposition of independent Gaussian codebooks (see Fig. 15)

$$X_0 = \sum_{k=1}^{K+1} X_0(k). \tag{54}$$

Each codebook corresponds to a level; the power range $[\frac{P_0}{\mathtt{SNR}_0}, P_0]$ used by transmitter 0 is divided into intervals (levels) according to the interference caused, just as in the deterministic case. More precisely, the values $P_0/\mathtt{INR}_i$ for $1 \leq i \leq K$ partition the interval $[\frac{P_0}{\mathtt{SNR}_0}, P_0]$ into power levels. The power and rate associated with each level is that which would be assigned to user 0 using rate splitting, with a small reduction in rate to be described later (see, e.g., [22] for more detail on rate splitting). Each user $i, 1 \leq i \leq K$, uses a random Gaussian codebook at full power, i.e., received at power $\mathtt{SNR}_i$ by receiver i. Receiver i first decodes those codebooks from user 0 that are received above the intended signal (while treating all other signals as noise), and then jointly decodes the signal from transmitter i and the remaining signals from user 0 which are received above the noise level (treating interference received below the noise level as noise).

Recall the assumptions $\mathtt{SNR}_0 > 1, \mathtt{INR}_1 > 1$ and $\mathtt{INR}_{i+1} \geq \mathtt{INR}_i$. The power of each of user 0's codebooks in (54) is chosen in such a way that the sum of codebooks $i+1$ through $K+1, \sum_{k=i+1}^{K+1} X_0(k)$, is observed by receiver i to be at the noise level (assuming all the codebooks are used). More precisely, letting ρ_i denote the power transmitted in codebooks $i+1$ through $K+1$

$$\rho_i = \sum_{k=i+1}^{K+1} |X_0(k)|^2.$$

We require

$$\rho_i |h_{i,0}|^2 = N_0$$

or, equivalently

$$\rho_i = \frac{P_0}{\mathtt{INR}_i}, \qquad 1 \leq i \leq K. \tag{55}$$

The ith power interval is given by $[\rho_i, \rho_{i-1}], 1 \leq i \leq K+1$, where $\rho_0 = P_0, \rho_{K+1} = \frac{P_0}{\mathtt{SNR}_0}$, and $\rho_i = \frac{P_0}{\mathtt{INR}_i}$ for $1 \leq i \leq K$. The power used by transmitter 0 on level $i, 1 \leq i \leq K$ is

$$\theta_i = \mathbf{E}[|X_0(i)|^2] = \rho_{i-1} - \rho_i$$

so that user 0 satisfies the power constraint (assuming user 0 transmits on all levels)

$$\mathbf{E}[|X_0|^2] = \sum_{i=1}^{K+1} \mathbf{E}[|X_0(i)|^2] = \sum_{i=1}^{K+1} \theta_i = \rho_0 - \rho_{K+1} \leq P_0.$$

Receiver 0 decodes the signals sequentially from the highest level (lowest index) downwards, treating the weaker signals as noise and subtracting off the decoded signal at each step. Thus, when decoding level i, receiver 0 experiences an effective noise variance of at most

$$N_0(i) \leq N_0 + |h_{00}|^2 \sum_{k>i} \theta_k = |h_{00}|^2 \rho_i = N_0 \frac{\mathtt{SNR}_0}{\mathtt{INR}_i}.$$

The rates of user 0's codebooks are chosen to satisfy the inequalities

$$R_0(i) \leq \log\left(1 + \frac{\theta_i |h_{i0}|^2}{3N_0}\right), \qquad 1 \leq i \leq K+1. \tag{56}$$

Note that user 0 can decode its own signals since

$$R_0(i) \leq \log\left(1 + \frac{\theta_i |h_{i0}|^2}{3N_0}\right) = \log\left(1 + \frac{\theta_i |h_{00}|^2}{3N_0 \frac{\mathtt{SNR}_0}{\mathtt{INR}_i}}\right) \leq \log\left(1 + \frac{\theta_i |h_{00}|^2}{N_0(i)}\right).$$

The quantity $\theta_i |h_{00}|^2 / N_0(i)$ is the SINR of the ith signal from user 0. One observes from the preceding equation that the factor 3 in the rate $R_0(i)$ for user 0's level i could be removed for the purposes of decoding at receiver 0; however, the rate is slightly reduced to allow decoding of user 0's signal level i at each of the interfered users as computed in (61).

We now account for decoding at receiver $i, 1 \leq i \leq K$. A natural procedure is for receiver i to jointly decode the i strongest levels from user 0, i.e., $X_0(1), \ldots, X_0(i)$, along with its own signal X_i. Since Gaussian codebooks are used, which is optimal for the MAC, it follows that the achievable region is determined by the MAC region at each receiver. Instead of this natural scheme, in order to ease the analysis, we describe a slight variation as used for the deterministic channel: receiver i first decodes those interfering signals from user 0 that appear above the intended signal x_i, and only then jointly decodes x_i together with the remaining interfering signals from $X_0(1), \ldots, X_0(i)$. Decoding first the interference received above the intended signal x_i corresponds to the fact that in the deterministic channel such interference does not actually interact with the intended signal.

We quickly bound the effective noise when receiver i decodes the very strongly interfering signal levels from user 0 that appear above the intended signal x_i. When receiver i is decoding signal $X_0(k)$ for $k \leq i$, assuming the stronger signals $X_0(1), \ldots, X_0(k-1)$ have already been decoded and subtracted off, receiver i treats as noise all levels from user 0 weaker than k (index $l > k$) and also its own signal x_i, giving an effective noise power of at most

$$\begin{aligned} N_i(k) &= N_0 + P_i |h_{ii}|^2 + |h_{i0}|^2 \sum_{l>k} \theta_l \\ &= N_0(1 + \mathtt{SNR}_i) + |h_{i0}|^2 \rho_k \\ &= N_0\left(1 + \mathtt{SNR}_i + \frac{\mathtt{INR}_i}{\mathtt{INR}_k}\right). \end{aligned} \tag{57}$$

Similarly to the deterministic case, let $\lambda(i) \in \{1, \ldots, i\}$ be such that

$$\rho_{\lambda(i)}|h_{i0}|^2 < P_i|h_{ii}|^2 \le \rho_{\lambda(i)-1}|h_{i0}|^2 \tag{58}$$

or, equivalently

$$\frac{\mathtt{INR}_i}{\mathtt{INR}_{\lambda(i)}} < \mathtt{SNR}_i \le \frac{\mathtt{INR}_i}{\mathtt{INR}_{\lambda(i)-1}}. \tag{59}$$

Thus, using (57) and (59), when receiver i is decoding the signal $X_0(k), k < \lambda(i)$, the effective noise is

$$N_i(k) \le N_0\left(1 + 2\frac{\mathtt{INR}_i}{\mathtt{INR}_k}\right) \tag{60}$$

and hence, the effective SNR is at least

$$\frac{\theta_k|h_{i0}|^2}{N_0\left(1+2\frac{\mathtt{INR}_i}{\mathtt{INR}_k}\right)} = \frac{\theta_k|h_{i0}|^2}{N_0\left(1+2\frac{|h_{i0}|^2}{|h_{k0}|^2}\right)} \ge \frac{\theta_k|h_{k0}|^2}{3N_0} \tag{61}$$

where the inequality follows from the fact that $|h_{k0}| \le |h_{i0}|$ for $k \le i$. Since this SNR can support the rates of user 0's codebooks given in (56), receiver i can decode all the signals $X_0(1), \ldots, X_0(\lambda(i) - 1)$ while treating the signals x_i and $X_0(\lambda(i)), \ldots, X_0(K+1)$ as noise.

It remains to check which rates allow for joint decoding of signals $X_0(\lambda(i)), \ldots, X_0(i)$ and x_i by receiver i. The MAC constraints at receiver i are

$$\sum_{k\in\Lambda} R_0(k) \le \log\left(1 + \frac{\sum_{k\in A}\theta_k|h_{i0}|^2}{N_0}\right) \tag{62}$$

and

$$\sum_{k\in\Lambda} R_0(k) \le \log\left(1 + \frac{N_0\mathtt{SNR}_i + \sum_{k\in A}\theta_k|h_{i0}|^2}{N_0}\right), \quad A \subseteq \{\lambda(i), \ldots, i\}. \tag{63}$$

We may ignore the first set of constraints (62): they are readily seen to be satisfied by the choice of rates $R_0(k)$ in (56). The second set of constraints (63) can also be simplified: it turns out that just as in the deterministic channel, the two constraints for $A = \{\lambda(i), \ldots, i\}$ and $A = \{\lambda(i)+1, \ldots, i\}$ imply the others (up to a small gap). To see this, note that because θ_k is decreasing in k and by the definition (58) of $\lambda(i)$, for $k > \lambda(i)$, it holds that

$$\theta_k|h_{i0}|^2 \le \theta_{\lambda(i)}|h_{i0}|^2 \le N_0\mathtt{SNR}_i.$$

Thus, for any $A \subseteq \{\lambda(i), \ldots, i\}$ with $\lambda(i) \in A$

$$\begin{aligned} r_i + \sum_{k\in A} R_0(k) &\le r_i + \sum_{k\in\{\lambda(i),\ldots,i\}} R_0(k) \\ &\le \log\left(1 + \frac{2\rho_{\lambda(i)-1}|h_{i0}|^2}{N_0}\right) \\ &\le 1 + \log\left(1 + \frac{N_0\mathtt{SNR}_i + \sum_{k\in A}\theta_k|h_{i0}|^2}{N_0}\right) \end{aligned}$$

and similarly, for any $A \subseteq \{\lambda(i), \ldots, i\}$ with $\lambda(i) \notin A$

$$\begin{aligned} r_i + \sum_{k\in A} R_0(k) &\le r_i + \sum_{k\in\{\lambda(i)+1,\ldots,i\}} R_0(k) \\ &\le \log\left(1 + \frac{2N_0\mathtt{SNR}_i}{N_0}\right) \\ &\le 1 + \log\left(1 + \frac{N_0\mathtt{SNR}_i + \sum_{k\in A}\theta_k|h_{i0}|^2}{N_0}\right). \end{aligned}$$

Thus, up to a gap of 1 bit per user (and dropping the one in the logarithms, which only reduces the achievable rate), it is possible to achieve any point in the region determined by the sum–rate constraints

$$r_i + \sum_{k\in\{\lambda(i),\ldots,i\}} R_0(k) \le \log\left(\frac{\rho_{\lambda(i)-1}|h_{i0}|^2}{N_0}\right), \quad 1 \le i \le K \tag{64}$$

and

$$r_i + \sum_{k\in\{\lambda(i)+1,\ldots,i\}} R_0(k) \le \log(\mathtt{SNR}_i), \qquad 1 \le i \le K \tag{65}$$

together with the individual rate constraints

$$r_i \le \log(\mathtt{SNR}_i), \qquad 1 \le i \le K \tag{66}$$

and

$$R_0(i) \le \log\left(1 + \frac{\theta_i|h_{i0}|^2}{3N_0}\right), \qquad 1 \le i \le K+1. \tag{67}$$

We now compare the achievable region to the capacity region of the deterministic one-to-many IC. Let $n_{ii} = \mathtt{SNR}_i, 0 \le i \le K$, and $n_{i0} = \mathtt{INR}_i, 1 \le i \le K$. Then, the achievable region given by (64), (65), (66), and (67) contains the region given by

$$r_i + \sum_{k\in\{\lambda(i),\ldots,i\}} R_0(k) \le n_{i0} - n_{\lambda(i)-1,0}, \qquad 1 \le i \le K$$

and

$$r_i + \sum_{k\in\{\lambda(i)+1,\ldots,i\}} R_0(k) \le n_{ii}, \qquad 1 \le i \le K$$

together with the individual rate constraints

$$r_i \le n_{ii}, \qquad 1 \le i \le K$$

and

$$R_0(i) \le (n_{i0} - n_{i-1,0} - 1)^+, \qquad 1 \le i \le K+1.$$

Comparing this with the deterministic channel region (43), (44), and (45), evidently the regions are the same except that user 0 loses up to 1 bit per signal level, for a total loss of at most K bits. Since the outer bound has a gap from the deterministic channel of $K+1$ bits at user 0, we have determined the capacity region of the one-to-many IC to within a gap of $(2K+1, 1, \ldots, 1)$. This completes the proof of Theorem 23.

Remark 25: Instead of the HK scheme used here, it is possible to use an achievable scheme that creates independent levels, and then to emulate the first scheme presented for the deterministic one-to-many channel. However, such an approach yields a larger gap between the inner and outer bounds.

As with the many-to-one channel, the generalized degrees of freedom can now be computed.

Theorem 26: Put $\mathtt{SNR}_i = s^{\alpha_i}$ for $0 \leq i \leq K$ and $\mathtt{INR}_i = s^{\beta_i}$ for $1 \leq i \leq K$. The degrees-of-freedom region of the one-to-many Gaussian IC is the set of points satisfying the individual constraints

$$d_i \leq \alpha_i, \qquad 0 \leq i \leq K$$

and the sum–rate constraints (for each set of users relabeled as $\{1, \dots, m\}$, with $\beta_{i+1} > \beta_i, \beta_1 > 0$, and $\beta_m - \beta_0 \leq \alpha_m$)

$$\sum_{i=0}^{m} d_i \leq (\alpha_0 - \beta_m)^+ + \max(\alpha_1, \beta_1) + \sum_{i=2}^{m} \max(\alpha_i, \beta_i - \beta_{i-1}).$$

VIII. Conclusion

In finding the capacity of the many-to-one and one-to-many Gaussian interference channels, two main themes emerge: the power of the deterministic model approach, and the use of lattice codes for interference alignment. Throughout the entire development, the deterministic model serves as a faithful guide to the Gaussian channel. The structure of the outer bound, namely the existence of sum–rate constraints for every subset of users, is most easily observed in the deterministic channel. Moreover, the proofs of the Gaussian outer bounds closely follow those for the deterministic channels, with the side information used to prove outer bounds in the Gaussian case translated directly from the deterministic case.

The capacity achieving schemes are very simple in the deterministic channels. The interference alignment phenomenon emerges in the deterministic many-to-one channel, but in order to translate the scheme to the Gaussian channel, lattices are necessary to provide an alignment in the signal level. Yet another success of the deterministic model is that the reciprocity between the many-to-one and one-to-many channels is evident in the deterministic setting; this basic relationship between the two channels is veiled in the Gaussian case.

The approach used here should be contrasted with the direct approach of [19], where the problem of finding the capacity of the two-user Gaussian IC to within a constant gap was *reduced* to that of finding the capacity of a corresponding deterministic channel. More generally, the limitations and potential of the deterministic approach beg to be studied.

The gap of $(3K+3)(1+\log(K+1))$ bits per user between the achievable region and outer bound in the many-to-one Gaussian IC (Theorem 18) is somewhat loose. One way that the bound could be improved is to account for the combinatorial structure of the interference pattern (see Fig. 11) in evaluating the achievable strategy. There is a balance between the number of intervals formed by the interfering signals and the loss required by each user due to addition of signals from lower levels. The outer bound can probably also be tightened by more carefully performing the estimate in (104).

Appendix I
Gaussian HK Achieves Sum–Rate of at Most $\log(1+3\beta^2)$

This section contains a proof of Claim 1, showing that an HK scheme with Gaussian codebooks cannot achieve a sum–rate greater than $\log(1 + 3\beta^2)$.

At an achievable rate point, receiver 0 is assumed to be able to decode message 0. After decoding, receiver 0 may subtract away signal 0; since users 1 and 2 use a superposition codebook with private and common messages in the HK scheme [17], there are four messages which we will show receiver 0 is able to decode. This will give the desired outer bound on the sum–rate. Let the four (Gaussian) codebooks have rates $R_{U_1}, R_{W_1}, R_{U_2}, R_{W_2}$ ($r_1 = R_{U_1} + R_{W_1}$ and $r_2 = R_{U_2} + R_{W_2}$) and received power-to-noise ratios $\beta\gamma_1 S_1, \beta\gamma_1(1 - S_1), \beta\gamma_2 S_2, \beta\gamma_2(1 - S_2)$, respectively, at the intended receivers (we have parameterized the power-to-noise ratios $P_{U_1}, P_{W_1}, P_{U_2}, P_{W_2}$ in terms of $\gamma_1, \gamma_2, S_1, S_2 \in [0, 1]$). Receivers 1 and 2 are assumed to be able to decode their own signals, so the MAC constraints at receiver $i = 1, 2$ hold

$$\begin{aligned} R_{U_i} &\leq \log(1 + \beta\gamma_i S_i) \\ R_{W_i} &\leq \log(1 + \beta\gamma_i(1 - S_i)) \\ R_{U_i} + R_{W_i} &\leq \log(1 + \beta\gamma_i). \end{aligned} \tag{68}$$

Now, consider the MAC constraints at receiver 0. It is assumed throughout that $\beta \geq 2$. The received power-to-noise ratios at receiver 0 are each scaled by β, since the gains are $h_{01} = h_{02} = \beta$ as compared to the gains $h_{11} = h_{22} = \sqrt{\beta}$ on links 1 and 2. In order to show that receiver 0 can decode all the signals, we therefore wish to show that the rates $R_{U_1}, R_{W_1}, R_{U_2}, R_{W_2}$ lie within the MAC region at receiver 0 given by

$$R_{U_i} \leq \log(1 + \beta^2\gamma_i S_i), \qquad i = 1, 2 \tag{69}$$
$$R_{W_i} \leq \log(1 + \beta^2\gamma_i(1 - S_i)), \qquad i = 1, 2 \tag{70}$$
$$R_{U_i} + R_{W_i} \leq \log(1 + \beta^2\gamma_i), \qquad i = 1, 2 \tag{71}$$
$$R_{U_1} + R_{U_2} \leq \log(1 + \beta^2\gamma_1 S_1 + \beta^2\gamma_2 S_2) \tag{72}$$
$$R_{W_1} + R_{W_2} \leq \log(1 + \beta^2\gamma_1(1 - S_1) + \beta^2\gamma_2(1 - S_2)) \tag{73}$$
$$R_{U_1} + R_{W_2} \leq \log(1 + \beta^2\gamma_1 S_1 + \beta^2\gamma_2(1 - S_2)) \tag{74}$$
$$R_{W_1} + R_{U_2} \leq \log(1 + \beta^2\gamma_1(1 - S_1) + \beta^2\gamma_2 S_2) \tag{75}$$
$$R_{U_1} + R_{W_1} + R_{U_2} \leq \log(1 + \beta^2\gamma_1 + \beta^2\gamma_2 S_2) \tag{76}$$
$$R_{U_1} + R_{U_2} + R_{W_2} \leq \log(1 + \beta^2\gamma_1 S_1 + \beta^2\gamma_2) \tag{77}$$
$$R_{U_1} + R_{W_1} + R_{W_2} \leq \log(1 + \beta^2\gamma_1 + \beta^2\gamma_2(1 - S_2)) \tag{78}$$
$$R_{W_1} + R_{U_2} + R_{W_2} \leq \log(1 + \beta^2\gamma_1(1 - S_1) + \beta^2\gamma_2) \tag{79}$$
$$R_{W_1} + R_{U_2} + + R_{U_2} + R_{W_2} \leq \log(1 + \beta^2(\gamma_1 + \gamma_2)). \tag{80}$$

To begin, the constraints (69), (70), and (71) are obviously satisfied at receiver 0. To check the remaining constraints we make use of the following simple inequality.

Lemma 27: Let $\alpha_1, \alpha_2 \in [0, 1]$, and $\beta \geq 2$. Then

$$\log(1 + \beta\alpha_1) + \log(1 + \beta\alpha_2) \leq \log\left(1 + \beta^2(\alpha_1 + \alpha_2)\right).$$

Proof: Since

$$\log(1+\beta\alpha_1)+\log(1+\beta\alpha_2) = \log(1+\beta\alpha_1+\beta\alpha_2+\beta^2\alpha_1\alpha_2)$$

we must show that

$$\beta\alpha_1 + \beta\alpha_2 + \beta^2\alpha_1\alpha_2 \leq \beta^2(\alpha_1 + \alpha_2).$$

But this follows from the inequalities

$$\frac{\alpha_1 + \alpha_2}{2} \geq \alpha_1\alpha_2$$

for $\alpha_1, \alpha_2 \in [0, 1]$ and $\frac{\beta^2}{2} \geq \beta$ for $\beta \geq 2$, whence

$$\frac{\beta^2}{2}(\alpha_1 + \alpha_2) \geq \beta(\alpha_1 + \alpha_2).$$ □

Now, to check that the constraint (72) on $R_{U_1} + R_{U_2}$ is satisfied, note that by the constraints at receivers 1 and 2 (68)

$$\begin{aligned} R_{U_1} + R_{U_2} &\leq \log(1 + \beta\gamma_1 S_1) + \log(1 + \beta\gamma_2 S_2) \\ &= \log(1 + \beta\gamma_1 S_1 + \beta\gamma_2 S_2 + \beta^2\gamma_1\gamma_2 S_1 S_2) \\ &\leq \log\left(1 + \beta^2(\gamma_1 S_1 + \gamma_2 S_2)\right) \end{aligned} \quad (81)$$

where the last step follows from Lemma 27 with $\alpha_1 = \gamma_1 S_1$ and $\alpha_2 = \gamma_2 S_2$. Defining $S_i' = 1 - S_i$, (81) shows that the constraint (73) on $R_{W_1} + R_{W_2}$ is also satisfied at receiver 0.

The remaining inequalities follow in a straightforward way, similarly to (81), from Lemma 27. Hence, the rates $R_{U_1}, R_{W_1}, R_{U_2}, R_{W_2}$ lie within the MAC formed by transmitters 1 and 2 and receiver 0, and in a working communication system receiver 0 is able to decode the messages from transmitters 1 and 2 (despite not desiring to decode these messages). It follows that receiver 0 can decode all three messages from users 0, 1, and 2, and the MAC constraints from the three transmitters to receiver 0 apply. In particular, the sum–rate constraint on the three users applies, and since the total received power is no more than $3\beta^2$, the sum–rate achieved by a Gaussian HK scheme is upper bounded as

$$r_{\text{sum}}^{HK} = r_1 + r_2 + r_3 \leq \log(1 + 3\beta^2).$$

Appendix II
Proof of Lemma 6

We prove the contrapositive of the statement of Lemma 6: Suppose a set of constraints A is not compatible, i.e., there are two incompatible constraints $\mathcal{T}, \mathcal{T}'$ in A. Let $\vec{r}$ be some rate point lying on the two incompatible constraints $\mathcal{T}, \mathcal{T}'$ from Theorem 4. Then, there exists another constraint from Theorem 4 that is violated by $\vec{r}$.

Recall that a constraint $\mathcal{T}$ is identified with the set of users to which the constraint applies. The Lemma is claiming that if two constraints are incompatible, and hence not simultaneously achievable by the proposed strategy, then these constraints are inconsistent. Inconsistent constraints do not define corner points of the outer bound region, and therefore the fact that they are not achievable does not discredit the proposed achievable scheme. The proof proceeds by assuming there is a rate point $\vec{r}$ lying on the constraints $\mathcal{T}, \mathcal{T}'$, and furthermore assuming that all constraints in Theorem 4 are satisfied; from these assumptions a contradiction is derived.

From the perspective of the bipartite graph associated to the constraints, where $\mathcal{T}, \mathcal{T}'$ correspond to vertices on the left-hand side, incompatibility of $\mathcal{T}, \mathcal{T}'$ implies some right-hand side vertex k^* is a solid neighbor of $\mathcal{T}'$ and a dashed neighbor of $\mathcal{T}$ (or *vice versa*). Recall that vertices corresponding to individual rate constraints have only solid edges. Thus, $\mathcal{T}$ must correspond to a sum–rate constraint, as it is assumed to have at least one dashed edge (to vertex k^*). If the vertex $\mathcal{T}'$ corresponds to an individual rate constraint, it is straightforward to show that the given rate point $\vec{r}$ violates the constraint on users $\mathcal{T} \cup \mathcal{T}'$: the rate point $\vec{r}$ allows user 0 to transmit on level k since it achieves the constraint $\mathcal{T}$, while $\vec{r}$ also allows the individual user $\{i\} \in \mathcal{T}'$ to transmit since $\vec{r}$ achieves the individual constraint on user i. Thus, we consider sets of users $\mathcal{T} = \mathcal{S} \cup \{0\}$ and $\mathcal{T}' = \mathcal{S}' \cup \{0\}$, where $\mathcal{S}, \mathcal{S}' \subseteq \{1, \ldots, K\}$ and both $\mathcal{S}$ and $\mathcal{S}'$ are nonempty. We continue to assume throughout that $\mathcal{T} = \mathcal{S} \cup \{0\}$ has a dashed edge to k^* and $\mathcal{T}' = \mathcal{S}' \cup \{0\}$ has a solid edge to k^*.

The following claim simplifies the interference patterns from users in $\mathcal{S}$ and $\mathcal{S}'$ which must be considered.

Claim 28: We may assume the following.

1) At most two users from each of $\mathcal{S}, \mathcal{S}'$ interfere on a given level.
2) Exactly two users a and b in $\mathcal{S}'$ interfere on level k^*, i.e., $U_k^* \cap \mathcal{S}' = \{a, b\}$.
3) Users a and b satisfy $n_{0b} > n_{0a}$ and $n_{0a} - n_{aa} < n_{0b} - n_{bb}$.

Proof: A user i is said to be *occluded* by a set of users I if for each k with $i \in U_k, |U_k \cap (I \setminus \{i\})| \geq 1$. This means each level (at receiver 0) at which user i can interfere is occupied by at least one user in $I \setminus \{i\}$. The proof of the claim proceeds by studying the implications of a user being occluded.

If the sum–rate constraint on a set of users I is met with equality, and $i \in I$ is occluded by I, it is straightforward to check that the sum rate constraint on $I \setminus \{i\}$ is also met with equality: each level where i interferes has some user from $I \setminus \{i\}$ also transmitting, so these levels are fully utilized even when i is removed.

Next, note that if there are three or more users interfering on a level, then the user interfering at the highest level among the three together with the user interfering at the lowest level among the three occlude the remaining user. Together with the previous paragraph this proves part 1) of the claim, since occluded users may simply be removed from the sets $\mathcal{S}, \mathcal{S}'$ under consideration. Since $\mathcal{S}'$ has a solid edge to vertex k^* in the bipartite graph, it must have at least two users interfering on level k^*, proving part 2) of the claim.

Turning to part 3), note that the choice $n_{0b} > n_{0a}$ is arbitrary (this means the top of signal b is above the top of signal a at receiver 0); given this choice, if $n_{0a} - n_{aa} \geq n_{0b} - n_{bb}$, then b occludes a (this would mean the bottom of a is higher than the bottom of b). Now, if there is an occluded user $i \in \mathcal{S}'$ interfering on level k^*, then, as in the case where $\mathcal{S}'$ corresponds to an individual rate constraint, the constraint on the set $\{i\} \cup \mathcal{S}$ is violated (the level k^* is double counted in $\vec{r}$, but counted only once in the sum–rate constraint on $\{i\} \cup \mathcal{S}$). This proves the claim. □

With the simplified structure of $\mathcal{S}, \mathcal{S}'$ given by the claim above, we can partition each of the sets of users in a useful way. Let $\mathcal{S}_a = \{i \in \mathcal{S} : n_{0i} < k^*\}$ be the set of users in $\mathcal{S}$

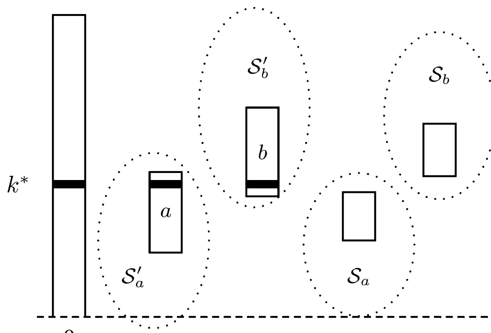

Fig. 16. This figure illustrates the choice of the sets of users S_a, S_b, S_a', S_b'. Notice that both users a and b in $\mathcal{S}' = \mathcal{S}_a' \cup \mathcal{S}_b'$ interfere on level k^*, while no user in $\mathcal{S} = \mathcal{S}_a \cup \mathcal{S}_b$ interferes on level k^*.

whose interference at receiver 0 occurs at levels below k^*, and let $\mathcal{S}_b = \mathcal{S} \setminus \mathcal{S}_a$ be the remaining users in $\mathcal{S}$. Note that $\mathcal{S}_a$ and $\mathcal{S}_b$ occupy disjoint sets of levels, because level k^* separates $\mathcal{S}_a$ and $\mathcal{S}_b$. Similarly, let $\mathcal{S}_a' = \{a\} \cup \{i \in \mathcal{S}' : n_{0i} \leq k^*\}$ be the set of users in $\mathcal{S}'$ whose interference at receiver zero occurs at or below user a, and let $\mathcal{S}_b' = \mathcal{S}' \setminus \mathcal{S}_a'$. The sets $\mathcal{S}_a'$ and $\mathcal{S}_b'$ interact only through level k^* (and possibly levels adjacent to k^*). Fig. 16 depicts the relationship between the various sets.

Denote by $f(\mathcal{S})$ the value of the sum–rate constraint on $\mathcal{S}$, i.e.,

$$f(\mathcal{S}) = f_{\text{free}}(\mathcal{S}) + \sum_{k=1}^{n_{00}} f_k(\mathcal{S}).$$

It is given in the statement of the lemma that the sum–rate constraints on $\mathcal{S} \cup \{0\}$ and $\mathcal{S}' \cup \{0\}$ are met with equality

$$r_0 + \sum_{i \in \mathcal{S}_a} r_i + \sum_{i \in \mathcal{S}_b} r_i = f(\mathcal{S}) \tag{82}$$

and

$$r_0 + \sum_{i \in \mathcal{S}_a'} r_i + \sum_{i \in \mathcal{S}_b'} r_i = f(\mathcal{S}'). \tag{83}$$

Next, we may assume that the constraints on $\mathcal{S}_b \cup \mathcal{S}_a' \cup \{0\}$ and $\mathcal{S}_a \cup \mathcal{S}_b' \cup \{0\}$ are satisfied, i.e.,

$$\sum_{i \in \mathcal{S}_b} r_i + \sum_{i \in \mathcal{S}_a'} r_i + r_0 \leq f(\mathcal{S}_b \cup \mathcal{S}_a') \tag{84a}$$

and

$$\sum_{i \in \mathcal{S}_a} r_i + \sum_{i \in \mathcal{S}_b'} r_i + r_0 \leq f(\mathcal{S}_a \cup \mathcal{S}_b') \tag{84b}$$

otherwise, we have the desired violated constraint. Plugging these two inequalities into (82) gives

$$r_0 + f(\mathcal{S}_b \cup \mathcal{S}_a') + f(\mathcal{S}_a \cup \mathcal{S}_b') - r_0 - \sum_{i \in \mathcal{S}_a'} r_i - r_0 - \sum_{i \in \mathcal{S}_b'} r_i \geq f(\mathcal{S})$$

or, upon rearranging

$$f(\mathcal{S}_b \cup \mathcal{S}_a') + f(\mathcal{S}_a \cup \mathcal{S}_b') - f(\mathcal{S}) \geq r_0 + \sum_{i \in \mathcal{S}_a'} r_i + \sum_{i \in \mathcal{S}_b'} r_i.$$

This, with (83), implies that

$$f(\mathcal{S}_b \cup \mathcal{S}_a') + f(\mathcal{S}_a \cup \mathcal{S}_b') \geq f(\mathcal{S}) + f(\mathcal{S}') \tag{85}$$

which, as we will show, is a contradiction. Before showing this precisely, we can argue informally as follows, restricting attention to level k^* (the other levels will turn out not to matter). On the left-hand side, the first term, $f(\mathcal{S}_b \cup \mathcal{S}_a')$, counts only one interfering user on level $k^*, a \in \mathcal{S}_a'$, and the second term, $f(\mathcal{S}_a \cup \mathcal{S}_b')$, has only one interfering user on level $k^*, b \in \mathcal{S}_b'$, giving a value of 2 for the left-hand side. However, the right-hand side has a contribution of one from $f(\mathcal{S})$ and a contribution of two (a and b) from $f(\mathcal{S}')$, resulting in $2 \geq 3$, the desired contradiction.

We now proceed with the proof. The definition of $f(\cdot)$, the fact that $\mathcal{S}_a \cap \mathcal{S}_b = \mathcal{S}_a \cap \mathcal{S}_b' = \mathcal{S}_b \cap \mathcal{S}_a' = \varnothing$, and the fact that $\mathcal{S}_a$ and $\mathcal{S}_b$ occupy disjoint sets of levels gives

$$\begin{aligned}
&(f(\mathcal{S}_b \cup \mathcal{S}_a') + f(\mathcal{S}_a \cup \mathcal{S}_b') - f(\mathcal{S})) \\
&\quad = \sum_{i \in \mathcal{S}_b \cup \mathcal{S}_a'} \left((n_{i0} - n_{ii})^+ + (n_{i0} - n_{00})^+\right) + n_{00} \\
&\qquad + \sum_{k=1}^{n_{00}} (|U_k \cap (\mathcal{S}_b \cup \mathcal{S}_a')| - 1)^+ \\
&\qquad + \sum_{i \in \mathcal{S}_a \cup \mathcal{S}_b'} \left((n_{i0} - n_{ii})^+ + (n_{i0} - n_{00})^+\right) + n_{00} \\
&\qquad + \sum_{k=1}^{n_{00}} (|U_k \cap (\mathcal{S}_a \cup \mathcal{S}_b')| - 1)^+ \\
&\qquad - \sum_{i \in (\mathcal{S}_a \cup \mathcal{S}_b)} \left((n_{i0} - n_{ii})^+ + (n_{i0} - n_{00})^+\right) - n_{00} \\
&\qquad - \sum_{k=1}^{n_{00}} (|U_k \cap (\mathcal{S}_a \cup \mathcal{S}_b)| - 1)^+ \\
&\quad = \sum_{i \in \mathcal{S}_a' \cup \mathcal{S}_b'} \left((n_{i0} - n_{ii})^+ + (n_{i0} - n_{00})^+\right) + n_{00} \\
&\qquad + \sum_{k=1}^{n_{00}} (|U_k \cap (\mathcal{S}_b \cup \mathcal{S}_a')| - 1)^+ \\
&\qquad + \sum_{k=1}^{n_{00}} (|U_k \cap (\mathcal{S}_a \cup \mathcal{S}_b')| - 1)^+ \\
&\qquad - \sum_{k=1}^{n_{00}} (|U_k \cap \mathcal{S}_a| - 1)^+ - \sum_{k=1}^{n_{00}} (|U_k \cap \mathcal{S}_b| - 1)^+.
\end{aligned} \tag{86}$$

Note that only one of $|U_k \cap \mathcal{S}_a|, |U_k \cap \mathcal{S}_b|$ is positive for each k, allowing to write the two (instead of one) sums in the last line of (86). To continue, we rewrite the term in the third-from-last line of (86)

$$\begin{aligned}
&\sum_{k=1}^{n_{00}} (|U_k \cap (\mathcal{S}_b \cup \mathcal{S}_a')| - 1)^+ \\
&\quad = \sum_{k: |U_k \cap \mathcal{S}_b| \neq 0} (|U_k \cap \mathcal{S}_b| - 1) + \sum_{k: |U_k \cap \mathcal{S}_b| \neq 0} |U_k \cap \mathcal{S}_a'| \\
&\qquad + \sum_{k: |U_k \cap \mathcal{S}_b| = 0} (|U_k \cap \mathcal{S}_a'| - 1)^+
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n_{00}} (|U_k \cap \mathcal{S}_b| - 1)^+ + \sum_{k:|U_k \cap \mathcal{S}_b| \neq 0} |U_k \cap \mathcal{S}'_a| \\
&\quad + \sum_{k:b \notin U_k} (|U_k \cap \mathcal{S}'_a| - 1)^+
\end{aligned}
\tag{87}
$$

where the last step changing the conditions under the third summation from $\{k : |U_k \cap \mathcal{S}_b| = 0\}$ to $\{k : b \notin U_k\}$ follows from the observations: 1) for k such that $|U_k \cap \mathcal{S}'_a| \neq 0$, $|U_k \cap \mathcal{S}_b| \neq 0$ implies $b \in U_k$ so we are not summing over any extra levels in the latter sum; 2) conversely for levels potentially left out, i.e., levels with $b \in U_k$, it holds that $|U_k \cap \mathcal{S}'_a| \leq 1$ (otherwise level k contains two users from $\mathcal{S}'_a$ in addition to b, contradicting Claim 28), so $(|U_k \cap \mathcal{S}'_a| - 1)^+ = 0$ and these levels do not contribute, showing that the two sums are equal. Similarly, for the second term in the second-from-last line of (86), we have

$$
\begin{aligned}
&\sum_{k=1}^{n_{00}} (|U_k \cap (\mathcal{S}_a \cup \mathcal{S}'_b)| - 1)^+ \\
&\quad = \sum_{k:|U_k \cap \mathcal{S}_a| \neq 0} (|U_k \cap \mathcal{S}_a| - 1) + \sum_{k:|U_k \cap \mathcal{S}_a| \neq 0} |U_k \cap \mathcal{S}'_b| \\
&\qquad + \sum_{k:|U_k \cap \mathcal{S}_a| = 0} (|U_k \cap (\mathcal{S}_a \cup \mathcal{S}'_b)| - 1)^+ \\
&\quad = \sum_{k=1}^{n_{00}} (|U_k \cap \mathcal{S}_a| - 1)^+ + \sum_{k:|U_k \cap \mathcal{S}_a| \neq 0} |U_k \cap \mathcal{S}'_b| \\
&\qquad + \sum_{k:a \notin U_k} (|U_k \cap \mathcal{S}'_b| - 1)^+.
\end{aligned}
\tag{88}
$$

Plugging (87) and (88) into (86) and canceling terms results in the expression

$$
\begin{aligned}
&\sum_{i \in \mathcal{S}'_a \cup \mathcal{S}'_b} \left((n_{i0} - n_{ii})^+ + (n_{i0} - n_{00})^+ \right) + n_{00} \\
&\quad + \sum_{k:|U_k \cap \mathcal{S}_a| \neq 0} |U_k \cap \mathcal{S}'_b| + \sum_{k:a \notin U_k} (|U_k \cap \mathcal{S}'_b| - 1)^+ \\
&\quad + \sum_{k:|U_k \cap \mathcal{S}_b| \neq 0} |U_k \cap \mathcal{S}'_a| + \sum_{k:b \notin U_k} (|U_k \cap \mathcal{S}'_a| - 1)^+.
\end{aligned}
\tag{89}
$$

By performing manipulations in the same style as above, it is possible to write

$$
\begin{aligned}
f(\mathcal{S}') = &\sum_{i \in \mathcal{S}'_a \cup \mathcal{S}'_b} \left((n_{i0} - n_{ii})^+ + (n_{i0} - n_{00})^+ \right) + n_{00} \\
&+ |\{k : \{a, b\} \subseteq U_k\}| + \sum_{k:b \notin U_k} (|U_k \cap \mathcal{S}'_a| - 1)^+ \\
&+ \sum_{k:a \notin U_k} (|U_k \cap \mathcal{S}'_b| - 1)^+.
\end{aligned}
\tag{90}
$$

Here $|\{k : \{a, b\} \subseteq U_k\}|$ accounts for the contribution of levels where both a and b interfere (the latter two sums exclude these levels). Comparing (90) with the previous expression (89), we conclude that

$$
\begin{aligned}
&f(\mathcal{S}_b \cup \mathcal{S}'_a) + f(\mathcal{S}_a \cup \mathcal{S}'_b) - f(\mathcal{S}) \\
&\quad = f(\mathcal{S}') - |\{k : \{a, b\} \subseteq U_k\}| \\
&\qquad + \sum_{k:|U_k \cap \mathcal{S}_a| \neq 0} |U_k \cap \mathcal{S}'_b| + \sum_{k:|U_k \cap \mathcal{S}_b| \neq 0} |U_k \cap \mathcal{S}'_a| \\
&\quad = f(\mathcal{S}') - |\{k : \{a, b\} \subseteq U_k\}| \\
&\qquad + |\{k : b \in U_k, |U_k \cap \mathcal{S}_a| \neq 0\}| \\
&\qquad + |\{k : a \in U_k, |U_k \cap \mathcal{S}_b| \neq 0\}| \\
&\quad \leq f(\mathcal{S}') - 1.
\end{aligned}
$$

The last step follows from Claim 28, since a covers those levels belonging to both b and $\mathcal{S}_a$ (otherwise the bottom of signal a is above the bottom of signal b, contradicting the Claim), and similarly b covers those levels belonging to both a and $\mathcal{S}_b$. But this contradicts (85), proving the lemma.

Appendix III
Proof of the Sum–Rate Constraint For the Many-to-One Gaussian Channel

The proof of the sum–rate constraint uses a genie-aided channel, or in other words, allows the receivers access to side information. The main difficulty of the proof lies in choosing this side information. The crucial insight is provided by the deterministic channel model. Recall the side information given to receiver 0 in the many-to-one deterministic IC (18); there, on each level receiver 0 was given the signals of all interfering users except for one. From Fig. 17, we see that this side information corresponds exactly to giving the top portion of each interfering signal. Informed by the analogy that additive Gaussian noise corresponds to truncation in the deterministic channel (see Fig. 6), we give side information (here m is the number of users appearing in the sum–rate constraint)

$$
\begin{aligned}
s_0 &= \sigma_m \\
s_k &= \left(\sum_{i=1}^{k} h_{0i} x_i + z_0, \sigma_k \right), \qquad 1 \leq k \leq m
\end{aligned}
$$

where for each $k, 1 \leq k \leq m$, we have

$$
\sigma_k = (h_{01} x_1 + w_1 + z_0, h_{02} x_2 + w_2, \ldots, h_{0k} x_k + w_k)
$$

with $\sigma_0 = 0$ and

$$
\begin{aligned}
w_i &\sim \mathcal{CN}(0, N_0 \max(\mathtt{INR}_{i+1}/\mathtt{SNR}_{i+1}, 1)), \qquad 1 \leq i \leq m-1 \\
w_m &\sim \mathcal{CN}(0, N_0 \mathtt{SNR}_0).
\end{aligned}
$$

With this choice of side information the proof is straightforward albeit slightly lengthy. Fano's inequality and the data processing inequality imply that

$$
N(r_0 + r_1 + \cdots + r_m - \epsilon_N) \leq \sum_{i=0}^{m} I\left(x_i^N; y_i^N, s_i^N\right) \tag{91}
$$

where $\epsilon_N \to 0$ as $N \to \infty$. Each term in the sum can be expanded as

$$
I\left(x_i^N; y_i^N, s_i^N\right) = h\left(y_i^N \middle| s_i^N\right) + h\left(s_i^N\right) - h\left(y_i^N, s_i^N \middle| x_i^N\right).
$$

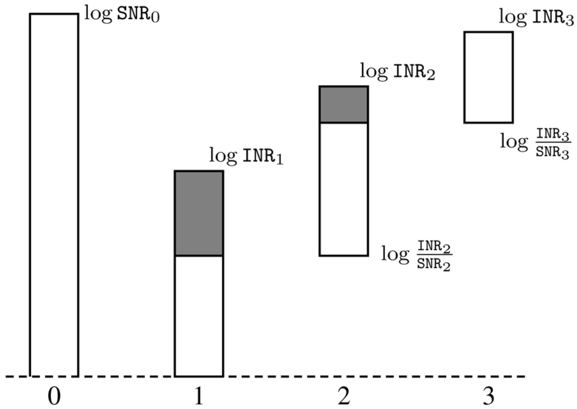

Fig. 17. Side information given to user 0 is shaded. The side information is precisely the portion of each signal overlapping with the next signal.

Using the fact that x_0 is independent of σ_m and x_k is independent of σ_{k-1} for $1 \le k \le m$, the negative terms evaluate to

$$\begin{aligned} &h\left(y_0^N, s_0^N \middle| x_0^N\right)\\ &\quad= h\left(h_{00}x_0^N + \sum_{i=1}^m h_{0i}x_i^N + z_0^N, \sigma_m^N \middle| x_0^N\right)\\ &\quad= h\left(\sum_{i=1}^m h_{0i}x_i^N + z_0^N, \sigma_m^N\right)\\ &\quad= h\left(s_m^N\right)\\ &h\left(y_1^N, s_1^N \middle| x_1^N\right)\\ &\quad= h\left(h_{11}x_1^N + z_1^N, h_{01}x_1^N + z_0^N, h_{01}x_1^N + w_1^N + z_0^N \middle| x_1^N\right)\\ &\quad= h\left(z_1^N, z_0^N, w_1^N\right)\\ &\quad= h\left(z_1^N\right) + h\left(z_0^N\right) + h\left(w_1^N\right) \end{aligned}$$

and for $2 \le k \le m$

$$\begin{aligned} &h\left(y_k^N, s_k^N \middle| x_k^N\right)\\ &\quad= h\left(h_{kk}x_k^N + z_k^N, \sum_{i=1}^k h_{0i}x_i^N + z_0^N, \sigma_k^N \middle| x_k^N\right)\\ &\quad= h\left(z_k^N\right) + h\left(\sum_{i=1}^{k-1} h_{i0}x_i^N + z_0^N, \sigma_{k-1}^N, w_k^N\right)\\ &\quad= h\left(z_k^N\right) + h\left(s_{k-1}^N\right) + h\left(w_k^N\right). \end{aligned}$$

Now, the sum in (91) telescopes, giving

$$\begin{aligned} &N(r_0 + r_1 + \cdots + r_m - \epsilon_N)\\ &\quad\le \sum_{k=0}^m h\left(y_k^N \middle| s_k^N\right) + h(s_0^N) - h\left(y_1^N, s_1^N \middle| x_1^N\right)\\ &\qquad + h\left(s_m^N\right) - h\left(y_0^N, s_0^N | x_0^N\right)\\ &\qquad + \sum_{k=1}^{m-1}\left[h\left(s_k^N\right) - h\left(y_{k+1}^N, s_{k+1}^N | x_{k+1}^N\right)\right]\\ &\quad= \sum_{k=0}^m \left[h\left(y_k^N \middle| s_k^N\right) - h\left(z_k^N\right)\right] - \sum_{k=1}^m h\left(w_k^N\right) + h\left(s_0^N\right). \end{aligned} \tag{92}$$

Next, we bound each term using the independence bound on entropy, and the fact that the Gaussian distribution maximizes differential entropy for a fixed (conditional) variance (see [23]).

Fact 29 (Worst Case Conditional Entropy): Let $z_1 \sim \mathcal{CN}(0, \sigma_1^2), z_2 \sim \mathcal{CN}(0, \sigma_2^2)$, and x be mutually independent with $E(|x|^2) \le P$. Then

$$h(x + z_1 | x + z_2) \le \log\left[\pi e\left(\sigma_1^2 + \frac{P\sigma_2^2}{P + \sigma_2^2}\right)\right]. \tag{93}$$

Proof:

$$\begin{aligned} &h(x + z_1 | x + z_2)\\ &\quad= h(x + z_1 - \alpha(x + z_2) | x + z_2)\\ &\quad\le h(z_1 + x(1 - \alpha) - \alpha z_2)\\ &\quad\le \log\left[\pi e\left(\sigma_1^2 + P(1 - \alpha)^2 + \alpha^2\sigma_2^2\right)\right]\\ &\quad= \log\left[\pi e\left(\sigma_1^2 + P\frac{\sigma_2^2}{(P + \sigma_2^2)^2} + \frac{P^2}{(P + \sigma_2^2)^2}\sigma_2^2\right)\right]\\ &\quad= \log\left[\pi e\left(\sigma_1^2 + \frac{P\sigma_2^2}{P + \sigma_2^2}\right)\right] \end{aligned}$$

where the second to last equality follows by choosing $\alpha = P/(P + \sigma_2^2)$. ☐

We have

$$\begin{aligned} &h\left(s_0^N\right)\\ &\quad\le \sum_{j=1}^N\left[\sum_{k=2}^m h(h_{0k}x_{k,j} + w_{k,j}) + h(h_{01}x_{1,j} + z_{0j} + w_{1,j})\right]\\ &\quad\le \sum_{j=1}^N\left[\sum_{k=2}^m \log\left[\pi e\left(|h_{0k}|^2 P_{k,j} + P_{w_k}\right)\right]\right.\\ &\qquad\qquad \left. + \log\left[\pi e\left(|h_{01}|^2 P_{1,j} + N_0 + P_{w_1}\right)\right]\right]\\ &\quad\le N\sum_{k=2}^m\left\{\log\left[\pi e\left(|h_{0k}|^2\frac{1}{N}\sum_{j=1}^N P_{k,j} + P_{w_k}\right)\right]\right.\\ &\qquad\qquad \left. + \log\left[\pi e\left(|h_{01}|^2\frac{1}{N}\sum_{j=1}^N P_{1,j} + N_0 + P_{w_1}\right)\right]\right\} \end{aligned}$$

where $P_{k,j} = E|x_{k,j}|^2$. Jensen's inequality, the power constraint $\frac{1}{N}\sum_j P_{k,j} \le P_k$, and the fact that $\log x$ is an increasing function justify the remaining steps, continuing from above

$$\begin{aligned} &h\left(s_0^N\right)\\ &\quad\le N\sum_{k=2}^m \log\left[\pi e\left(|h_{0k}|^2 P_k + P_{w_k}\right)\right]\\ &\qquad + N\log\left[\pi e\left(|h_{01}|^2 P_1 + N_0 + P_{w_1}\right)\right]\\ &\quad= N\left[\sum_{k=2}^{m-1}\log\left(\mathtt{INR}_k + \max\left(\frac{\mathtt{INR}_{k+1}}{\mathtt{SNR}_{k+1}}, 1\right)\right)\right.\\ &\qquad\qquad \left. + \log\left(1 + \mathtt{INR}_1 + \max\left(\frac{\mathtt{INR}_2}{\mathtt{SNR}_2}, 1\right)\right)\right]\\ &\qquad + N\left[\log(\mathtt{INR}_m + \mathtt{SNR}_0) + m\log(\pi e N_0)\right] \end{aligned}$$

$$\leq N\left[1+\sum_{k=1}^{m-1}\log\left(\mathtt{INR}_k+\max\left(\frac{\mathtt{INR}_{k+1}}{\mathtt{SNR}_{k+1}},1\right)\right)\right.$$
$$\left.+\ \log(\mathtt{INR}_m+\mathtt{SNR}_0)+m\log(\pi e N_0)\right] \tag{94}$$

and similarly for $2\leq k\leq m$, we have (95), shown at the bottom of the page. Likewise

$$h\left(y_1^N\middle|\, s_1^N\right)$$
$$\leq N\log\left(1+\frac{\mathtt{SNR}_1\cdot\left(\frac{\mathtt{SNR}_1}{\mathtt{INR}_1}+1\right)}{\mathtt{SNR}_1+\frac{\mathtt{SNR}_1}{\mathtt{INR}_1}+1}\right)+N\log(\pi e N_0) \tag{96}$$

and

$$h\left(y_0^N\middle|\, s_0^N\right)$$
$$\leq N\log\left[2\mathtt{SNR}_0+\sum_{i=1}^{m-1}\max\left(\frac{\mathtt{INR}_{i+1}}{\mathtt{SNR}_{i+1}},1\right)\right]+N\log(\pi e N_0). \tag{97}$$

Finally, by the definition of w_i

$$h(w_i)=\log\left[\pi e N_0\max\left(\frac{\mathtt{INR}_{i+1}}{\mathtt{SNR}_{i+1}},1\right)\right],\qquad 1\leq i\leq m-1$$
$$h(w_m)=\log\left(\pi e N_0\mathtt{SNR}_0\right). \tag{98}$$

Plugging (94)–(98) into (92) and taking $N\rightarrow\infty$, we have the desired sum–rate bound

$$r_0+r_1+\cdots+r_m$$
$$\leq\log\left[2\mathtt{SNR}_0+\sum_{i=1}^{m-1}\max\left(\frac{\mathtt{INR}_{i+1}}{\mathtt{SNR}_{i+1}},1\right)\right] \tag{99}$$
$$+\log\left(1+\frac{\mathtt{SNR}_1\cdot\left(\frac{\mathtt{SNR}_1}{\mathtt{INR}_1}+1\right)}{\mathtt{SNR}_1+\frac{\mathtt{SNR}_1}{\mathtt{INR}_1}+1}\right) \tag{100}$$
$$+\sum_{k=2}^{m}\log\left[1+\frac{\mathtt{SNR}_k\left(\frac{\mathtt{SNR}_k}{\mathtt{INR}_k}\sum_{i=2}^{k}\max\left(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i},1\right)\right)}{\mathtt{SNR}_k+\left(\frac{\mathtt{SNR}_k}{\mathtt{INR}_k}\sum_{i=2}^{k}\max\left(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i},1\right)\right)}\right] \tag{101}$$

$$-\sum_{k=1}^{m-1}\log\left[\max\left(\frac{\mathtt{INR}_{k+1}}{\mathtt{SNR}_{k+1}},1\right)\right]-\log(\mathtt{SNR}_0) \tag{102}$$
$$+1+\sum_{k=1}^{m-1}\log\left[\mathtt{INR}_k+\max\left(\frac{\mathtt{INR}_{k+1}}{\mathtt{SNR}_{k+1}},1\right)\right]$$
$$+\log\left(\mathtt{INR}_m+\mathtt{SNR}_0\right). \tag{103}$$

The structure of the outer bound is not clear from the preceding equation; therefore, we loosen the constraints in order that their form resemble the deterministic channel constraints. We will use the assumed ordering on the users, $\mathtt{INR}_i/\mathtt{SNR}_i\leq\mathtt{INR}_{i+1}/\mathtt{SNR}_{i+1}$ for $1\leq i\leq m-1$, and the assumption that

$$\begin{aligned}
h\left(y_k^N\middle|\, s_k^N\right) &\leq\sum_{j=1}^{N}h(y_{k,j}|s_{k,j})\\
&=\sum_{j=1}^{N}h\left(h_{kk}x_{k,j}+z_{k,j}\middle|\sum_{i=1}^{k}h_{i0}x_{i,j}+z_0,\sigma_{k,j}\right)\\
&\leq\sum_{j=1}^{N}h\left(h_{kk}x_{k,j}+z_{k,j}\middle|h_{0k}x_{k,j}-\sum_{i=1}^{k-1}w_{i,j}\right)\\
&=\sum_{j=1}^{N}h\left(h_{kk}x_{k,j}+z_{k,j}\middle|h_{kk}x_{k,j}-\sum_{i=1}^{k-1}\frac{h_{kk}}{h_{0k}}w_{i,j}\right)\\
&\leq\sum_{j=1}^{N}\log\left[\pi e\left(N_0+\frac{P_{k,j}|h_{kk}|^2\left(N_0\frac{|h_{kk}|^2}{|h_{0k}|^2}\right)\sum_{i=2}^{k}\max\left(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i},1\right)}{P_{k,j}|h_{kk}|^2+\left(N_0\frac{|h_{kk}|^2}{|h_{0k}|^2}\right)\sum_{i=2}^{k}\max(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i},1)}\right)\right]\\
&\leq N\log\left[\pi e\left(N_0+\frac{P_k|h_{kk}|^2(N_0|h_{kk}|^2/|h_{0k}|^2)\sum_{i=2}^{k}\max\left(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i},1\right)}{P_k|h_{kk}|^2+(N_0|h_{kk}|^2/|h_{0k}|^2)\sum_{i=2}^{k}\max\left(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i},1\right)}\right)\right]\\
&=N\log\left(1+\frac{\mathtt{SNR}_k\left(\frac{\mathtt{SNR}_k}{\mathtt{INR}_k}\sum_{i=2}^{k}\max(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i},1)\right)}{\mathtt{SNR}_k+\left(\frac{\mathtt{SNR}_k}{\mathtt{INR}_k}\sum_{i=2}^{k}\max\left(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i},1\right)\right)}\right)+N\log(\pi e N_0).
\end{aligned} \tag{95}$$

$\mathtt{INR}_m/\mathtt{SNR}_m \leq \mathtt{SNR}_0$. Beginning with the first term (99), we have

$$\log\left[2\mathtt{SNR}_0 + \sum_{i=1}^{m-1} \max\left(\frac{\mathtt{INR}_{i+1}}{\mathtt{SNR}_{i+1}}, 1\right)\right] \leq \log(m+1) + \log(\mathtt{SNR}_0).$$

Next, we simplify (100) as

$$\begin{aligned}\log\left(1 + \frac{\mathtt{SNR}_1 \cdot \left(\frac{\mathtt{SNR}_1}{\mathtt{INR}_1}+1\right)}{\mathtt{SNR}_1 + \frac{\mathtt{SNR}_1}{\mathtt{INR}_1}+1}\right) &\leq 1 + \min\left(\log \mathtt{SNR}_1, 1 + \log\left(\frac{\mathtt{SNR}_1}{\mathtt{INR}_1}\right)^+\right) \\ &\leq 2 + \log\left(\frac{\mathtt{SNR}_1}{\mathtt{INR}_1}\right)^+.\end{aligned}$$

Now for (101), consider two cases: $\frac{\mathtt{INR}_k}{\mathtt{SNR}_k} \geq 1$, or $\frac{\mathtt{INR}_k}{\mathtt{SNR}_k} < 1$. In the first case, we have

$$\frac{\mathtt{SNR}_k}{\mathtt{INR}_k} \sum_{i=2}^{k} \max\left(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i}, 1\right) \leq k-1.$$

In the second case

$$\frac{\mathtt{SNR}_k}{\mathtt{INR}_k} \sum_{i=2}^{k} \max\left(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i}, 1\right) = \frac{\mathtt{SNR}_k}{\mathtt{INR}_k}(k-1).$$

The sum (101) can therefore be bounded as

$$\begin{aligned}&\sum_{k=2}^{m} \log\left[1 + \frac{\mathtt{SNR}_k\left(\frac{\mathtt{SNR}_k}{\mathtt{INR}_k}\sum_{i=2}^{k}\max\left(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i},1\right)\right)}{\mathtt{SNR}_k + \left(\frac{\mathtt{SNR}_k}{\mathtt{INR}_k}\sum_{i=2}^{k}\max\left(\frac{\mathtt{INR}_i}{\mathtt{SNR}_i},1\right)\right)}\right] \\ &\leq \sum_{k=2}^{m}\left(\log k + \log\left(\frac{\mathtt{SNR}_k}{\mathtt{INR}_k}\right)^+\right) \\ &\leq \int_2^{m+1} \log x dx + \sum_{k=2}^{m} \log\left(\frac{\mathtt{SNR}_k}{\mathtt{INR}_k}\right)^+ \\ &= -(m+1) + (m+1)\log(m+1) + \sum_{k=2}^{m}\log\left(\frac{\mathtt{SNR}_k}{\mathtt{INR}_k}\right)^+.\end{aligned}$$

Finally, the sums in (102) and (103) can be simplified using the inequality $\log(a+b) \leq 1 + \max(\log a, \log b)$

$$\begin{aligned}&\sum_{k=1}^{m-1} \log\left(\mathtt{INR}_k + \max\left(\frac{\mathtt{INR}_{k+1}}{\mathtt{SNR}_{k+1}}, 1\right)\right) \\ &- \sum_{k=1}^{m-1} \log\left(\frac{\mathtt{INR}_{k+1}}{\mathtt{SNR}_{k+1}}\right)^+ \\ &\leq (m-1) + \sum_{k=1}^{m-1} \max\left(\log(\mathtt{INR}_k), \log\left(\frac{\mathtt{INR}_{k+1}}{\mathtt{SNR}_{k+1}}\right)^+\right) \\ &\quad - \log\left(\frac{\mathtt{INR}_{k+1}}{\mathtt{SNR}_{k+1}}\right)^+ \\ &= (m-1) + \sum_{k=1}^{m-1}\left(\log(\mathtt{INR}_k) - \log\left(\frac{\mathtt{INR}_{k+1}}{\mathtt{SNR}_{k+1}}\right)^+\right)^+\end{aligned}$$

resulting in the cruder, yet simpler, sum–rate bound

$$\begin{aligned}&r_0 + r_1 + \cdots + r_m \\ &\leq \sum_{k=1}^{m} \log\left(\frac{\mathtt{SNR}_k}{\mathtt{INR}_k}\right)^+ + \sum_{k=1}^{m-1}\left(\log(\mathtt{INR}_k) - \log\left(\frac{\mathtt{INR}_{k+1}}{\mathtt{SNR}_{k+1}}\right)^+\right)^+ \\ &\quad + \max\left(\log(\mathtt{INR}_m), \log(\mathtt{SNR}_0)\right) \\ &\quad + (m+2)\log(m+1) + 1.\end{aligned}$$

This completes the proof of the desired sum–rate constraint.

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