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Vortex drift. II: The flow potential surrounding a drifting vortical region

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The potential flowfield surrounding a vortical flow domain is investigated. The vortical region is restricted to a three-dimensional finite domain in an unbounded incompressible viscous fluid that is at rest at infinity. The vector streamfunction of the flow is obtained by the integrals over the given vorticity field, expressing the Biot–Savart law. In the approach proposed here, the asymptotic vector streamfunction in the far field is expressed by integrals that represent the moments of the vorticity vector field. The asymptotic far-field potential is then determined by an integration that is analogous to the procedure well known in two dimensions, where the conjugate potential is found in the case that the streamfunction is known. In three dimensions, no general formula was found but an explicit expression for the asymptotic potential was constructed for the far-field dipole and quadrupole. The integrability conditions that permit the determination of the potential from the vector streamfunction are interpreted as the consequence of the requirement that the underlying vorticity field has to be source free. The asymptotic potentials are discussed first using only the inherent symmetries of the general dipole and the quadrupole fields. The results are then specialized to the case of axial symmetry. It will be shown that (i) the dipole strength is invariant and that (ii) the strength of the flow quadrupole remains constant while it moves together with the dipole with the drift speed already postulated in Part I. The role of the pressure in the flow generating process is also briefly noted.

I. INTRODUCTION

In Part I,[1] (I) the drift motion of a confined vortical region in an unbounded viscous incompressible fluid was treated. Use was made of the fact that the vortical region is surrounded by irrotational flow where a potential exist, so that Bernoulli's equation is valid. Thus, by determining the pressure in the far field from its Poisson equation, the *time rate of change* of the potential was found. However, for the solution of the drift problem, the potential itself also has to be known to a sufficiently high order. The potential of the leading order dipole was found in I using the classical formula of Lamb.[2] The argument given in I for the determination of the drift also calls for the determination of the asymptotic quadrupole potential. The purpose of this paper (Part II) is to present a new method for the determination of the farfield potential that surrounds the vortical region. The closing argument on the drift will be given later.

The problem is solved by determining the vector streamfunction first, with the help of the integrals over the vortex components of the flow, from the Biot–Savart law. This, naturally, is the classical approach. Here, a variation is considered: The asymptotic terms of the streamfunction components are calculated by making the expansion of the integrand kernel first; the integrations are performed second. Then all results are obtained in terms of harmonic functions, with coefficients that are integrals over the vorticity. At the level of the dipole and the quadrupole, the coefficients are given in terms of the first and second moments of the vorticity components, respectively.

In the next step, the potential in the far field is determined. One finds the velocity from the vector streamfunction and sets it equal to the gradient of the potential. In two dimensions this defines the potential as the "conjugate harmonic" of the streamfunction. In three dimensions, it turns out that the three equations will lead to one and the same potential only if the divergence of the streamfunction vector vanishes. This "integrability condition" leads to a set of relations between the vorticity moments; they are interpreted as conditions that have to be fulfilled by the vorticity components to make its divergence vanish. Only a source-free vortical field leads to a solution to the problem considered, and the relations that emerge are considered as expressions of this restriction, obtained *a posteriori*.

The classical way to ensure a source-free vorticity field is to consider it as an assembly of thin closed vortex filaments. Lamb's formula that was used in I is based on this concept. The implementation of this idea calls for integrations over surfaces enclosed by the filaments first and integrations over the volume containing the filaments second. In contrast, the method proposed here uses simple volume integrals over the vorticity that are subjected to certain conditions. Its results will always be checked by the application to the classical example of a single thin filament. It will be seen that this check might lead to some additional constraints that are not obvious from the integrability conditions. Results of practical interest up to the quadrupole level will not be affected by such problems. Particularly encouraging are results that can be expressed by integrals over only two components of the vorticity vector field.

The analysis will be carried out assuming the existence of a preferred direction—that of the impulse—but without further restrictions by initial symmetry. This permits the consideration of cases with a possible "precession" about the impulse axis. Such terms are carried along for completeness as they lead to certain results at the quadrupole level which might be of interest in the future. They are dropped in the final analysis.

Classical treatments of the streamfunction vector emphasize that the velocity is only determined up to additional gradients of arbitrary harmonic functions. These are the homogeneous solutions of the Poisson equations for the streamfunction vector. However, any such homogeneous solution is a solution of Laplace's equation that is valid in the whole infinite unbounded flow domain and therefore has to have a singularity somewhere in the flow. We have, however, the right to demand regularity everywhere. The applicability of the analysis to viscous flow is an important part of this argument. We will define, before the expansion, the *particular* solution of the Poisson equation that gives the streamfunction vector that fulfills all boundary and regularity conditions and is, therefore, the unique solution. This remains true for the expansion in the asymptotic domain.

II. VORTEX-FIELD KINEMATIC, FIRST ORDER

The complete initial solution of the flow problem, in which at $t=0$ a vorticity vector field $\boldsymbol{\omega}$ is imposed on an incompressible fluid at rest, is given by the Biot–Savart law. Necessary condition for the solubility is, however, that the assumed vorticity field is source free:

$$\operatorname{div} \boldsymbol{\omega}=0. \tag{1}$$

Then the velocity $\mathbf{u}$ follows from a vector streamfunction $\mathbf{A}$

$$\mathbf{u}=\operatorname{rot} \mathbf{A} \tag{2}$$

such that $\mathbf{A}$ fulfills the Poisson equation

$$\nabla^2 \mathbf{A}=-\boldsymbol{\omega} \tag{3}$$

provided that

$$\operatorname{div} \mathbf{A}=0. \tag{4}$$

The necessary condition for the existence of such an $\mathbf{A}$ field is $\operatorname{div} \boldsymbol{\omega}=0$.

The fluid is assumed to be incompressible but not necessarily to be inviscid. The physical effort needed to create a given vorticity in a fluid does depend on its viscosity, but the initial flow kinematic depends only on the assumed $\boldsymbol{\omega}$.

The solution of the Poisson equation (3) in an unbounded infinite domain is

$$\mathbf{A}=\int (4\pi R)^{-1}\boldsymbol{\omega}(x'y'z')dV, \tag{5}$$

where

$$R=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}, \qquad dV=dx'\,dy'\,dz'. \tag{6}$$

It is noted that in (5) there are no provisions to enforce the condition (1) so that there is no *a priori* guarantee for the fulfillment of (4), which is the necessary condition for the validity of (5).

We restrict our attention to unbounded flow fields in which a vortical region is completely surrounded by potential flow. Region(s) of vorticity are then found only inside of a sphere with a certain finite radius. The $\boldsymbol{\omega}$ field has to fulfill the condition

$$\int \boldsymbol{\omega}\, dV=0. \tag{7}$$

It is proposed here to investigate such flows by carrying out the following two steps: (i) to find, from (5), asymptotic functions $\mathbf{A}^{(0)}$, $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$... that represent $\mathbf{A}$ far from the vortical region, and (ii) to determine the potential functions $\phi^{(0)}$, $\phi^{(1)}$, etc., that are valid there.

The expansion of an integral of the type (5) has already been carried out in I. The application to (5) on components of $\mathbf{A}$ will lead to vector streamfunctions in an irrotational flow (!). All components will be represented by linear combinations of harmonic functions, so that

$$\nabla^2 \mathbf{A}^{(i)}=0. \tag{8}$$

However, no provisions were made to assure that the conditions

$$\operatorname{div} \mathbf{A}^{(i)}=0 \tag{9}$$

are met, which relates all three components of the $\mathbf{A}^{(i)}$. It will be seen that (9) is the integrability condition needed to determine the potentials $\phi^{(i)}$ from the relations

$$\operatorname{rot} \mathbf{A}^{(i)}=\operatorname{grad} \phi^{(i)} \tag{10}$$

by actually carrying out the operations for i up to 2.

No direct general proof of the necessary and sufficient role of (9) was found. Besides, there might be a limit on i for the validity of the integration of (10), caused by a restricted region of convergence for the expansion of $(4\pi R)^{-1}$, given by Eqs. (18) and (19) of I. The whole algorithm that emerges for the integration of (10) is based on these equations. Their introduction in (5) is not carried out here in detail as it follows exactly the procedure used in I. The results are given in terms of the normalized source potential φ (with the source strength 1) and its derivatives which are normalized dipoles, quadrupoles, etc. To each such group belongs an asymptotic vector streamfunction indexed with the degree of the differentiation of φ. The leading term is $\mathbf{A}^{(0)}=\varphi\int\boldsymbol{\omega}\, dV$ which has to vanish according to the condition (7). This confirms what we already know. The subsequent terms have coefficients that are volume integrals representing moments of the vorticity vector components.

Before proceeding, a convenient notation is introduced for the coefficients. For the first moments of the vorticity components we write

$$M_x^x=\int \omega_x x'\, dV, \text{ etc.,} \quad M_y^x=\int \omega_x y'\, dV, \text{ etc.,} \tag{11}$$

where the superscript of M indicates the component of $\boldsymbol{\omega}$, and the subscript of M shows the moment that has to be taken. The notation is suitable for the extension to higher moments, e.g.,

$$M_{yz}^x=\int \omega_x y'z'\, dV, \text{ etc.,} \tag{12}$$

to be used later.

The introduction of the basic expansion of (6) in (5) gives the components of $\mathbf{A}^{(1)}$:

$$A_x^{(1)}=M_x^x\varphi_x+M_y^x\varphi_y+M_z^x\varphi_z,$$

$$A_y^{(1)}=M_x^y\varphi_x+M_y^y\varphi_y+M_z^y\varphi_z, \tag{13}$$

$$A_z^{(1)}=M_x^z\varphi_x+M_y^z\varphi_y+M_z^z\varphi_z.$$

The integrability condition div $\mathbf{A}^{(1)}=0$ leads to the relations

$$M_x^x=M_y^y=M_z^z \tag{14}$$

and

$$M_y^x=-M_x^y,\quad M_z^y=-M_y^z,\quad M_x^z=-M_z^x. \tag{15}$$

A nonzero common value of the terms occurring in (14) has no influence on the velocity derived from $\mathbf{A}^{(1)}$ and can be safely ignored; presently it will be argued that this value *is* zero. What remains then is an antisymmetric tensor that can be represented by the vector $\mathbf{A}^{(1)}$.

The conditions (14) and (15) permit indeed the determination of $\phi^{(1)}$. Using the integral of $u^{(1)}$ as the definition of $\phi^{(1)}$ we obtain

$$\phi^{(1)}=M_z^y\varphi_x+M_x^z\varphi_y-M_x^y\varphi_z. \tag{16}$$

The main part of the far field is expected to consist of a dipole aligned with the impulse, which is assumed to be parallel to the x axis. Thus the potential

$$\phi_1^{(1)}=-\tfrac{1}{2}(M_y^z-M_z^y)\varphi_x \tag{17}$$

with a coefficient properly antisymmetric in y and z, is recognized as the part of $\phi^{(1)}$ that represents such a flow field. The same potential was found in I, Eq. (37), and the coefficient of the normalized dipole $-\varphi_x$ is the invariant impulse I.

The last two terms in (16) can be made to vanish by a proper choice of the coordinate system, namely, by aligning the x axis with the resultant direction of the vector $\mathbf{A}^{(1)}$ that represents the antisymmetric tensor (13). However, we have already assumed that the x direction is that of the impulse and (16) shows that in the presence of nonvanishing vorticity components parallel to the x direction there could be a contribution to the potential of the following form [obtained by using the relations (15)]

$$\phi_2^{(1)}=M_y^x\varphi_z-M_z^x\varphi_y. \tag{18}$$

This might, in the absence of axial symmetry, represent a precession about the x axis. There is a "remainder" that still has to be added to give (16), but

$$\phi_3^{(1)}=\tfrac{1}{2}(M_z^y+M_y^z)\varphi_x=0 \tag{19}$$

equals zero according to the conditions (15).

The final analysis will be restricted to the case where $\phi_2^{(1)}$ also vanishes, but it might be of interest to see later that the quadrupole can also contain a term that corresponds to (18).

The moments are now evaluated for the classical example of the thin closed vortex tube. Let $d\mathbf{s}$ be a vector with the components dx', dy', dz'. The integrands in the moments are changed by the transformation

$$\omega\, dV=\kappa\, d\mathbf{s}, \tag{20}$$

where the constant κ is the product of $|\omega|$ and the tube cross-section area. One obtains

$$M_x^x=\int\omega_x x'\, dV=\kappa\int x'\, dx'=0,\ \text{etc.}, \tag{21}$$

$$M_z^y=\kappa\int z'\, dy'=-\kappa\int y'\, dz'=-M_y^z,\ \text{etc.} \tag{22}$$

The result (21) goes beyond the requirement (14), but as noted before, this does not affect the usable part of the solution. The integrals (22) are the projection areas of the directed space curve, which is the spine of the vortex tube, on the three planes containing the coordinate axes. The areas have to be counted positive or negative depending on the sense of the contours of the projected areas, while the sense on the space curve is fixed. The tensor character of the moments proves that an orientation of the axis x (say) can be found such that the projected area of the closed space curve on the yz plane reaches the highest possible value, namely, the invariant length of the vector whose three components are the three projected areas. Then the projected areas are zero for all parallel projections in directions perpendicular to the x axis. In all cases actually studied thus far, this is also the axis of the impulse.

III. VORTEX-FIELD KINEMATIC, SECOND ORDER

To write down the components of $\mathbf{A}^{(2)}$ is tedious but straightforward using the notation introduced in (12). Here only the relations are given that the triple-indexed M have to fulfill so that div $\mathbf{A}^{(2)}=0$:

$$M_{xx}^x=2M_{yx}^y+M_{yy}^x=2M_{zx}^z+M_{zz}^x, \tag{23}$$

$$M_{yy}^y=2M_{zy}^z+M_{zz}^y=2M_{xy}^x+M_{xx}^y, \tag{24}$$

$$M_{zz}^z=2M_{yz}^y+M_{yy}^z=2M_{xz}^x+M_{xx}^z, \tag{25}$$

$$M_{yz}^x+M_{zx}^y+M_{yx}^z=0. \tag{26}$$

The starting point for the discussion of $\phi^{(2)}$ is again the form that is obtained by integrating the expression for the x component of the velocity. In this expression, there are no moments of ω_x among the coefficients:

$$\phi^{(2)}=M_{yx}^z\varphi_{yy}-M_{zx}^y\varphi_{zz}+(M_{zx}^z-M_{yx}^y)\varphi_{yz}$$
$$+\tfrac{1}{2}(M_{xx}^z-M_{yy}^z)\varphi_{xy}+\tfrac{1}{2}(M_{zz}^y-M_{xx}^y)\varphi_{xz}. \tag{27}$$

The first two terms are rearranged with the help of Laplace's equation,

$$M_{yx}^z\varphi_{yy}-M_{zx}^y\varphi_{zz}=\tfrac{1}{2}(M_{zx}^y-M_{yx}^z)\varphi_{xx}$$
$$+\tfrac{1}{2}(M_{zx}^y+M_{yx}^z)(\varphi_{yy}-\varphi_{zz}). \tag{28}$$

This corresponds to the first step in the rearranging of the expression for $H^{(2)}$, Eq. (23) in I. It permits to identify in the solution (27) a first part that represents the axisymmetric quadrupole:

$$\phi_1^{(2)} = -\tfrac{1}{2}(M_{yx}^z - M_{zx}^y)\varphi_{xx}. \tag{29}$$

This contribution to the far-field potential can be made to vanish, by the simple strategem of shifting the plane $x=0$ such that it contains the centroid of the coefficient of the quadrupole (29). The expression for the velocity of this centroid gives the "kinematic" definition of the drift, to be discussed presently.

First, however, the question is discussed whether the coefficients of φ_{xy} and φ_{xz} in (27) can be brought into a form that shows their relationship to the terms in the particular dipole potential (18). This can be done by introducing to the relations (23)–(26) two more constraints, of a dynamic nature. They express that the two components of the flow angular momentum perpendicular to the x axis vanish. Such a constraint puts the x axis on the line of action of the impulse. (A momentum about the x axis is not affected.) These additional relations are, according to Lamb[3]

$$M_{xx}^y + M_{zz}^y = 0, \quad M_{xx}^z + M_{yy}^z = 0. \tag{30}$$

There are four second-order moments that are affected by the shift of the origin in the directions y and z. They are the moments in which only y and z appear, both in superscripts and subscripts. They are found in the middle of the expressions of Eqs. (24) and (25). Evaluation of these expressions for a thin closed vortex tube gives the additional information that

$$2M_{zy}^z + M_{zz}^y = 0, \quad 2M_{yz}^y + M_{yy}^z = 0 \tag{31}$$

so that the three expressions that are equal to each other in (24) and (25) are actually equal to zero. With the help of (24), (25), (30), and (31) one can put the last two terms of (27) into the form

$$\phi_2^{(2)} = -2M_{xz}^x\varphi_{xy} + 2M_{xy}^x\varphi_{xz} \tag{32}$$

which shows the expected relationship with $\phi_2^{(1)}$ given by (18).

Finally, the "remainder" of the potential $\phi^{(2)}$ is gathered in the expression

$$2\phi_3^{(2)} = -M_{yz}^x(\varphi_{yy} - \varphi_{zz}) + (M_{yy}^x - M_{zz}^x)\varphi_{yz}. \tag{33}$$

Here, both coefficients have been changed by the use of the relations (24) and (27) such that only moments extended over ω_x occur, just as in (32). There seems to be no simple way to judge the possible significance of these terms. It is recalled that $\int \omega_x\, dV = 0$ has been presupposed, but the formulas should remain valid for a nonvanishing angular momentum about the x axis, even for vanishing x impulse. The terms might also occur when the x component of the angular momentum vanishes but the axial symmetry is disturbed.

The continuation of the investigation of higher-order poles at infinity would involve considerable effort but could shed more light on the problems that arise in the absence of initial symmetries.

IV. THE IMPULSE AND THE DRIFT

The only results needed to give the extended information on the far-field pressure, to be used for the closing argument for the dynamic drift definition given in I, are the coefficients of the axisymmetric dipole and quadrupole. The essence of the argument was already given in commenting on the result (29), and the evaluation of the integrals that give the coefficients was already performed in I in cylindrical coordinates. Nevertheless, a brief evaluation of the integrals in Cartesian coordinates is given here, which makes Part II self-contained. The purpose of this presentation is to illustrate the following general conclusion: The time rate-of-change of the far-field potential is obtained in terms of the moments of the vector $\mathbf{q} = \mathbf{u} \times \boldsymbol{\omega}$, while the farfield potential itself is expressed by analogous terms using the moments of the vorticity vector $\boldsymbol{\omega}$. The analogy of the operations on $\boldsymbol{\omega}$ to those performed on $\mathbf{q}$ in I will become evident.

The coefficient of the normalized dipole potential (17) was already recognized as the impulse:

$$2I = \int (y\omega_z - z\omega_y)\, dV \tag{34}$$

and the value of this integral is independent of the choice of the origin of the Cartesian coordinate system.

The time independence of I follows from the dynamic vorticity equation

$$\partial\boldsymbol{\omega}/\partial t - \nu\nabla^2\, \boldsymbol{\omega} = \mathrm{rot}\, \mathbf{q} = \mathrm{rot}[\mathbf{u} \times \boldsymbol{\omega}]. \tag{35}$$

The time derivative of $2I$ is

$$\begin{aligned} &\int \left[y\left(\frac{\partial\omega_z}{\partial t}\right) - z\left(\frac{\partial\omega_y}{\partial t}\right)\right] dV \\ &\quad = \int \left[y\left(\frac{\partial q_y}{\partial x}\right) + z\left(\frac{\partial q_z}{\partial x}\right) - y\left(\frac{q_x}{\partial y}\right) - z\left(\frac{\partial q_x}{\partial z}\right)\right] dV \\ &\qquad + \nu \int (y\, \nabla^2\, \omega_z - z\, \nabla^2\, \omega_y)\, dV = 0. \end{aligned} \tag{36}$$

The vanishing of the first right-hand side (rhs) term follows upon partial integration [see I, Eq. (21)], as $y(\partial q_x/\partial y) = \partial(yq_x)/\partial y - q_x$, etc., and $\int q_x\, dV = 0$, etc. The viscous term also vanishes as $y\,\nabla^2\omega_z = \nabla^2(y\omega_z) - 2\partial\omega_z/\partial y$, etc.

The integral that gives the coefficient of the quadrupole (29) differs from the integral in (34) by a factor x in the integrand. Thus it is possible to make this coefficient vanish by shifting the origin to x_C (say) that is defined by the equation

$$\int (y\omega_z - z\omega_y)x\, dV = x_C \int (y\omega_z - z\omega_y)\, dV. \tag{37}$$

Taking the time derivative on both sides leads to the determination of dx_C/dt. Now the same kind of replacements by (35) and subsequent partial integrations are used as above. But $xy(q_y/\partial x) = \partial(xyq_y)/\partial x - yq_y$, etc., and the integrals over the first moments of the q components do not vanish. The viscous terms, however, give zero upon integration as before. The result of the integration is, therefore,

$$2I\frac{dx_C}{dt}=\int(-yq_y-zqz+2xq_x)dV=2S, \qquad (38)$$

where the second equal sign indicates the identity of the integral with S, defined in I by Eq. (30). Thus the centroid speed is S/I, the same as the drift speed of the dipole origin that was postulated in I.

The flow picture at infinity that finally emerges is very simple. The leading term in the flow potential is a drifting dipole. The next term is a quadrupole with constant strength that is drifting *with the same speed*. Any constant, time-independent difference in the "origin" of these singularities is asymptotically irrelevant. In other words, the velocity quadrupole strength vanishes when its origin is properly positioned. Thus the leading term of the asymptotic pressure is a quadrupole that is caused by the drift of the dipole flow field, as found in I.

V. FLOW GENERATION

The quantity I, generally written in the form

$$I=\frac{1}{2}\int(\mathbf{r}\times\omega)dV, \qquad (39)$$

is called the impulse of the vortex system. It is important, however, to realize that this is the *generating* impulse that is needed to create the flow by a force field distributed in space and acting locally to generate impulsively the local speed. The impulse is the integral of this force in space *and* over the time of its duration.

The linear momentum of the flow in an infinite domain is asymptotically equal to the linear momentum associated with its farfield dipole ϕ_∞ with the potential (17):

$$\phi_\infty=-I\varphi_x=Ix/4\pi(x^2+r^2)^{3/2}. \qquad (40)$$

Its value is calculated as the momentum of a sphere of fluid in uniform motion that has the far-field potential (40), with the radius $R=(x^2+r^2)^{1/2}$ of the sphere tending to infinity. The radius and the velocity are found by superposing a uniform parallel stream to the dipole field. The linear momentum obtained in this way, however, is only two-thirds of I.

In the interpretation of Cantwell,[4] the reason that only a fraction of the generating impulse becomes flow impulse is found in the fact that a pressure field is created during the generation process that resists the flow-creating forces. The impulse in the creation phase is a function of time, and the pressure in the far field is obtained by taking the time-derivative of I in (40) and ignoring the drift of the dipole. The resultant instantaneous force resisting the acceleration is now calculated over the same infinite sphere that contains the flow impulse. We set in (40) $x=R\cos\alpha$, $r=R\sin\alpha$, and the surface element on the sphere projected onto the yz plane equal to $2\pi R^2\sin\alpha\cos\alpha\, d\alpha$, and obtain for the resisting force

$$\frac{1}{2}\frac{dI}{dt}\int_0^\pi\cos^2\alpha\sin\alpha\, d\alpha=\frac{1}{3}\frac{dI}{dt} \qquad (41)$$

so that only (2/3) dI/dt is left to accelerate the fluid.

Once the flow is established, the dipole pressure field disappears and the quadrupole pressure due to the drift becomes the dominant term. The impulse in the pressure pulse associated with (1/3) dI/dt is radiated away with the speed of sound ($=\infty$, say) while the dipole flow field is established inside the sphere covered by the pulse.

In two dimensions it has to be noted that I, defined again as the generating impulse, has no factor (1/2) in front of the integral when given in the form (39). The resultant resisting pressure force is found to be (1/2) dI/dt in plane flow for a cylindrical pulse, so that (1/2) I is the flow-field impulse.

The generating impulse gives the farfield dipole strength both in three and in two dimensions, in case that the dipole potential is normalized as $-\varphi_x$, where φ equals $1/4\pi R$ in three dimensions, and equals $-(1/2\pi)\log r$ in the plane case.

VI. SUMMARY

The potential field surrounding a restricted domain of vortical flow in an infinite unbounded fluid turned out to be a boundary region that can be gainfully studied, as it reflects essential properties of the whole flow. Absence of solid boundaries permits the simplest possible representation of the vector streamfunction by the vorticity vector, and the regularity requirement excludes the occurrence of additional harmonic functions. A scheme is proposed to determine the far-field potential and pressure from the vector streamfunction. The question remains open whether such a method can be useful in the absence of axial symmetry.

Results further restricted to simple rings and tori were not considered here. Several important examples in this area are noted in a recent survey article by Shariff and Leonard.[5]

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[3] See Ref. 2, p. 215.
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