Vortex drift. I: Dynamic interpretation

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Vortical flow, restricted to a finite domain (in three dimensions) in an unbounded incompressible viscous fluid that is at rest at infinity, is investigated by the consideration of the dynamics in the potential flow region that surrounds the vortical domain. The evolution equations are considered for a flow that is given at an initial time t. The potential change in the far field is connected to the pressure, which in turn is expressed as the solution of a Poisson equation with sources distributed over the whole flow field. The leading term of the pressure at infinity is a quadrupole, which is caused by a drifting dipole field with a constant strength that is given by the impulse. This "dynamic" value of the drift is then identified with the classical "kinematic" definition as the speed of the impulse centroid. The main new result obtained by this method is the solution of the asymptotic drift problem in three dimensions, complementing the corresponding solution of Cantwell and Rott [Phys. Fluids **31**, 3213 (1988)] for plane flow. The connection to the solution of the classical drift problem for a vortex ring is also established.

I. INTRODUCTION

In a recent paper, Cantwell and Rott[1] discussed the time decay of a vortex pair in plane viscous flow by use of a "heuristic model." From the initial state, given by a pair of concentrated point vortices, to the final stage, which is a dipole in slow (Stokes) flow, the flow was modeled for all times by a superposition of two decaying Oseen vortices. The total circulation in the half-planes separated by the symmetry line was kept constant first but was joined smoothly for later times by the law valid for the Stokes dipole, which is a decay with $t^{-1/2}$. The distance between the two Oseen vortex centers was determined by the requirement of a constant impulse. The drift was then found with the help of a pressure requirement. The fact was used that the materialized symmetry plane does not sustain a resultant force when the vortices are moving free, i.e., when they are not subject to *any* force. This is the corollary to the statement that if the vortices sustain a force (like a bound vortex creating the lift of an airplane), then the pressure integral over the symmetry plane is equal to the lift. The heuristic model provided an approximation to the pressure on the symmetry line, and the results obtained for the drift were found to be satisfactory.

For the final stage of the flow, Cantwell and Rott[1] intended to find an exact asymptotic value for the drift, with the help of the second order pressure in Stokes flow which follows from the velocities as a solution of a Poisson equation. However, this method determines the pressure correctly in any moving system, a point that will be discussed again later. Thus the resultant of the pressure on the symmetry line came out correctly, namely, equal to zero. Nevertheless, the effort to find the pressure from the Poisson equation was not in vain. The asymptotic pressure obtained in the far-field region, where there is no vorticity, has to be equal to the pressure obtained from Bernoulli's equation. This determined the exact value of the velocity of the vortex system.

The advantage of this conceptually simple idea becomes fully evident only after realizing that a "shortcut" is possible in the classical situation where a confined vortical region moves in an unbounded fluid that is at rest at infinity. From the representation of the pressure as a solution of the Poisson equation (which is valid both in inviscid and in viscous incompressible flow) it suffices to find the leading term for the asymptotic expression in the far-field potential flow region that surrounds the vortical domain. The leading pressure terms are solutions of Laplace's equation; they consist of quadrupoles, with different orientation of the axes. The coefficients for these harmonic functions are given by volume integrals extended over the components of the cross product between velocity and vorticity. The drift follows by identifying the pressure quadrupole with the pressure caused by the *motion* of a dipole whose coefficient is an invariant—the impulse.

The need for a definition of the drift which goes beyond the classical terms was first encountered by Cantwell and Rott[1] in their effort to improve the asymptotic Stokes flow solution that emerges at the final stage of the viscous decay of vortical flow. The correction needed to account for the nonlinear terms of the Navier–Stokes equation is obtained by solving consecutively two second-order linear inhomogeneous differential equations: a diffusion equation for the vorticity and a Poisson equation for the streamfunction. The drift emerges as the necessary condition for the existence of regular inhomogeneous solutions. A method for the determination of the drift in any phase of the flow will be based on a variation of this approach, to be discussed presently. In the asymptotic Stokes flow region, the drift problem has different character in two and in three dimensions. The discussion of this interesting point will be given at the end of the paper.

The proposed generalization is based on a strategem used by Cantwell and Rott[1] which permits to reverse the order in which the inhomogeneous problems are solved. A new intermediate variable, to be called the rate streamfunction (short for time-rate-of-change-vector streamfunction, or the evolution streamfunction) is introduced; it is to be

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determined as the solution of a Poisson equation. The vector streamfunction itself follows then as a solution of a diffusion equation. Then the drift appears already as the necessary element for the solution in the first step.

The reformulation and generalization of the problem is considered now in terms of an initial value problem. Let the velocity field of an incompressible fluid, regular but otherwise unrestricted, be given at $t=0$; find the streamfunction for the rate of change of the velocity such that the flow evolves fulfilling the laws of motion as given by the Navier–Stokes equations. The determination of the rate streamfunction is already a difficult task. However, in the case in which the vortical flow region is surrounded by potential flow at infinity, the rate streamfunction and the rate potential have to describe the same flow at infinity. The rate potential, i.e., the time rate of change of the potential, is connected to the pressure. The pressure quadrupole is determined by the Poisson equation and will be interpreted as a dipole of fixed strength moving with the drift velocity.

This dynamic approach makes it evident that all formulas for the drift give basically an instantaneous value; only for steady or self-similar flows can a result be obtained that is valid for extended time intervals.

The connection between the dynamic view and the classical "kinematic" definition of the drift as the motion of the impulse centroid will be established.

There remains one unresolved question: Is the dynamic interpretation of the drift given here unique? To answer this question one needs to determine the far-field potential itself that corresponds to the vector potential at infinity. This turns out to be a difficult task beyond the dipole level; it is the subject of a separate paper.[2] Here we take advantage of the fact that the determination of the rate-potential at infinity does not require the explicit determination of the rate-streamfunction.

II. THE TIME-EVOLUTION PROBLEM FOR A FLOW VELOCITY FIELD

We consider the evolution of a given velocity field $\mathbf{u}$ in incompressible viscous flow, that is restricted at the time $t=0$ only by the condition

$$\operatorname{div} \mathbf{u}=0. \tag{1}$$

To assure that this condition is fulfilled, let $\mathbf{u}$ be derived from a streamfunction, or more generally, from a streamfunction vector $\mathbf{A}$:

$$\mathbf{u}=\operatorname{rot} \mathbf{A}. \tag{2}$$

It can *not* be said *a priori* that a solution of the Navier–Stokes equation will emerge for all $\mathbf{A}$, but evolution equations can be established and the construction of the solution will answer the question of existence.

The final analysis will be restricted to flows for which a scalar streamfunction can be defined, but the use of $\mathbf{A}$ is appropriate in Cartesian coordinates. Also, one can imagine that two components of $\mathbf{A}$ are chosen freely and the third is determined by the condition

$$\operatorname{div} \mathbf{A}=0 \tag{3}$$

which ensures that the components of the vorticity $\boldsymbol{\omega}$ and of the streamfunction $\mathbf{A}$ are connected by the Poisson equations

$$\boldsymbol{\omega}=\operatorname{rot} \mathbf{u}=-\nabla^2 \mathbf{A}. \tag{4}$$

The first solution to the problem posed starts with the determination of $\boldsymbol{\omega}$ and of the vector

$$\mathbf{q}=\mathbf{u}\times\boldsymbol{\omega} \tag{5}$$

from the initial data. Then the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t}-\nu \nabla^2 \boldsymbol{\omega}=\operatorname{rot} \mathbf{q} \tag{6}$$

is the time-evolution equation for $\boldsymbol{\omega}$, with the right-hand side (rhs) given at some fixed time t. This statement is obviously true for inviscid flow. It is proposed here to put the viscous term on the "unknown" left-hand side (lhs), as is done routinely for all step-by-step solutions of the full Navier–Stokes equations. Thus the determination of $\partial\boldsymbol{\omega}/\partial t$ involves the solution of the inhomogeneous linear diffusion Eq. (6).

As basic homogeneous solutions are known for Eq. (6), the particular inhomogeneous solution can be found analytically by the method of the "variation of constants." However, then the problem is not yet completely solved. The change in the streamfunction vector still has to be determined, by solving the Poisson equation (4) for the rate of change of $\mathbf{A}$, i.e.,

$$\nabla^2 \frac{\partial \mathbf{A}}{\partial t}=-\frac{\partial \boldsymbol{\omega}}{\partial t}. \tag{7}$$

This can be achieved by the same technique that has been proposed here for (6), but the solution of the Poisson equation by an integral which expresses source superposition is preferable. It leads to explicit results directly and in a shorter way. Then, $\partial\mathbf{A}/\partial t$ is used to update $\mathbf{u}$, $\boldsymbol{\omega}$, and $\mathbf{q}$.

The evolution equations for $\mathbf{A}$ are now established; two inhomogeneous second-order partial differential equations have to be solved. Existence problems are reduced this way to the classical problem of the solution of inhomogeneous equations.

A simple and viable alternative to this scheme was noted by Cantwell and Rott.[1] It leads again to the solution of the same two equations, but the order in which they are solved is reversed. In this scheme, a new "intermediate" flow quantity is introduced instead of the vorticity $\boldsymbol{\omega}$. Let it be called the "rate-streamfunction" vector $\mathbf{B}$. It is defined by the equation

$$\mathbf{B}=\frac{\partial \mathbf{A}}{\partial t}-\nu \nabla^2 \mathbf{A} \tag{8}$$

so that

$$\operatorname{rot} \mathbf{B}=\frac{\partial \mathbf{u}}{\partial t}-\nu \nabla^2 \mathbf{u}. \tag{9}$$

Replacing the rhs by use of the Navier–Stokes equations gives

$$\text{rot}\,\mathbf{B}+\text{grad}\,H=\mathbf{u}\times\omega=\mathbf{q}, \tag{10}$$

where

$$H=p/\rho+(1/2)\mathbf{u}^2 \tag{11}$$

is the total pressure function. We obtain a Poisson equation to be solved for **B** by applying the operator rot to Eq. (10):

$$\nabla^2\mathbf{B}=-\text{rot}\,\mathbf{q} \tag{12}$$

with the restriction div $\mathbf{B}=0$, in analogy to (4). This is now the equation which has to be solved first, instead of (6). Once **B** is determined, **A** follows as a solution of the diffusion equation (8). Thus indeed in this new procedure the Poisson equation (12) is solved first and the diffusion equation (8) second.

Both schemes were discussed and tried by Cantwell and Rott,[1] and the examples showed clearly that the drift is needed as a free parameter; its proper choice assures the solubility of the problem at hand. However, results were obtained by actually calculating the solution of the two inhomogeneous linear second order equations, and it is disappointing to obtain results of limited applicability by this tedious process. Here, the aim is to find results of general validity.

III. FAR-FIELD DYNAMICS

The second scheme offers the possibility for a general expression for the drift in the case of a limited vortical domain moving in an unbounded fluid that is at rest at infinity. In this case, the vortical region is surrounded by irrotational flow and Eq. (8) for **A** reduces there to

$$\mathbf{B}_\infty=\frac{\partial\mathbf{A}_\infty}{\partial t}. \tag{13}$$

It would be much simpler to deal with the potential at infinity that exists both in inviscid and in viscous flow. The determination of this potential in general is a major problem, to be treated later in a separate paper.[2] However, there is a simple way to determine not the potential itself but the "rate potential" $\partial\phi/\partial t$ that corresponds to the rate streamfunction **B**. It follows from the equation that is obtained by applying the operation div to Eq. (10):

$$\nabla^2 H=\text{div}\,\mathbf{q}. \tag{14}$$

The far-field solution H_∞ is connected to the potential ϕ_∞ by Bernoulli's equation

$$-H_\infty=\frac{\partial\phi_\infty}{\partial t} \tag{15}$$

in the system in which the fluid is at rest at infinity. The generalization of this relation between H_∞ and ϕ_∞ such that the pressure comes out correctly in a moving system is simple, but to avoid confusion, all operations are now carried out in the system in which the fluid is at rest at infinity and where $p_\infty=0$.

The solution of (14) is

$$H=-\int(4\pi R)^{-1}\,\text{div}\,\mathbf{q}\,dV, \tag{16}$$

where

$$R=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2},\qquad dV=dx'\,dy'\,dz', \tag{17}$$

and div **q** is given in terms of $x'y'z'$. Now $(4\pi R)^{-1}$ is expanded about the origin of the xyz system. It is to be expressed with the help of the "harmonic source function"

$$\varphi=(4\pi\sqrt{(x^2+y^2+z^2)})^{-1} \tag{18}$$

so that

$$\begin{aligned}(4\pi R)^{-1}&=\varphi+\varphi_x x'+\varphi_y y'+\varphi_z z'\\&\quad+(1/2)[\varphi_{xx}x'^2+\varphi_{yy}y'^2+\varphi_{zz}z'^2]\\&\quad+\varphi_{xy}x'y'+\varphi_{xz}x'z'+\varphi_{yz}y'z',\end{aligned} \tag{19}$$

where the subscripts on φ stand for differentiations; thus φ and its derivatives are simply symbols for the normalized potentials of poles, dipoles, quadrupoles, etc. They occur in groups that are homogeneous in the ith powers of the expansion coordinates; to each such group belongs a set of asymptotic terms $H^{(i)}$. It is immediately seen that $H^{(0)}=0$ as $\int$ div $\mathbf{q}\,dV=0$, a trivial result. The next term is

$$-H^{(1)}=\varphi_x\int x'\,\text{div}\,\mathbf{q}\,dV+\varphi_y\int y'\,\text{div}\,\mathbf{q}\,dV+\varphi_z\int z'\,\text{div}\,\mathbf{q}\,dV=0. \tag{20}$$

The vanishing of $H^{(1)}$ follows as Lamb[3] has proven that

$$\int\mathbf{q}\,dV=0 \tag{21}$$

and partial integration of the terms in (20) leads to

$$\int x'\,\text{div}\,\mathbf{q}\,dV=\int\text{div}(x'\mathbf{q})dV-\int q_x\,dV=0,\ \text{etc.} \tag{22}$$

The result $H^{(1)}=0$ expresses the known fact that the pressure at infinity does not contain a dipole field (Cantwell,[4] 1986). Thus the leading term of H turns out to be $H^{(2)}$. Partial integrations applied to the coefficients of $H^{(2)}$ give

$$\int x'^2\,\text{div}\,\mathbf{q}\,dV=\int\text{div}(x'^2\mathbf{q})dV-2\int x'q_x\,dV,\ \text{etc.,} \tag{22a}$$

$$\int x'y'\,\text{div}\,\mathbf{q}\,dV=\int\text{div}(x'y'\mathbf{q})dV-\int(x'q_x+y'q_y)dV,\ \text{etc.} \tag{22b}$$

Now the linear moments of the components of **q** appear and these do not vanish. The expression for $H^{(2)}$ that follows from (16) and (19) is

$$H^{(2)}=\varphi_{xx}\int xq_x\,dV+\varphi_{yy}\int yq_y\,dV+\varphi_{zz}\int zq_z\,dV$$

$$+\varphi_{xy}\int (xq_y+yq_x)dV+\varphi_{xz}\int (xq_z+zq_x)dV$$

$$+\varphi_{yz}\int (yq_z+zq_y)dV. \qquad (23)$$

This result does not show the true symmetries of $H^{(2)}$ as the number of independent terms in (23) is only five: three quadrupole elements are connected by Laplace's equation

$$\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=0. \qquad (24)$$

Symmetry is obtained after choosing a privileged direction first (the x axis, say, to be identified later with the impulse direction). By use of (24) in the form

$$\varphi_{yy}=-(1/2)\varphi_{xx}+(1/2)(\varphi_{yy}-\varphi_{zz}), \qquad (24a)$$

$$\varphi_{zz}=-(1/2)\varphi_{xx}-(1/2)(\varphi_{yy}-\varphi_{zz}), \qquad (24b)$$

the number of terms in (23) is reduced to five. It is noted that the terms φ_{xy} and φ_{xz} represent the same flow except for a rotation around the x axis by $\pi/2$; the terms $\varphi_{yy}-\varphi_{zz}$ and $2\varphi_{yz}$ are related in the same way, by rotation by $\pi/4$. The final result is

$$H^{(2)}=\varphi_{xx}\int \left(xq_x-\frac{1}{2}(yq_y+zq_z)\right)dV$$

$$+\varphi_{xy}\int (xq_y+yq_x)dV+\varphi_{xz}\int (xq_z+zq_x)dV$$

$$+\frac{1}{2}(\varphi_{yy}-\varphi_{zz})\int (yq_y-zq_z)dV$$

$$+\varphi_{yz}\int (yq_z+zq_y)dV. \qquad (25)$$

It can be shown that among the integrals occurring in (25), the following relations hold

$$\int (yq_z-zq_y)dV=0, \quad \int (zq_x-xq_z)dV=0,$$
$$\int (xq_y-yq_x)dV=0. \qquad (26)$$

They express the conservation of angular momentum.

IV. USE OF POLAR COORDINATES. RESULTS FOR AXISYMMETRIC FLOW

Results are presented now by using polar coordinates r, ϑ in the yz plane:

$$y=r\cos\vartheta, \quad z=r\sin\vartheta. \qquad (27)$$

The components of $\mathbf{q}$ are now expressed in this system:

$$q_x=u_r\omega_\vartheta-u_\vartheta\omega_r, \quad q_r=u_\vartheta\omega_x-u_x\omega_\vartheta,$$
$$q_\vartheta=u_x\omega_r-u_r\omega_x. \qquad (28)$$

To be introduced in (25) are q_x and

$$q_y=q_r\cos\vartheta-q_\vartheta\sin\vartheta, \quad q_z=q_r\sin\vartheta+q_\vartheta\cos\vartheta. \qquad (29)$$

We evaluate the coefficient of φ_{xx} in (25) and obtain

$$\int \left(xq_x-\frac{1}{2}(yq_y+zq_z)\right)dV$$

$$=\int \left(xu_r+\frac{1}{2}ru_x\right)\omega_\vartheta\,dV$$

$$-\int \left(x\omega_r+\frac{1}{2}r\omega_x\right)u_\vartheta\,dV=S-S^*, \qquad (30)$$

where the first integral S is the only part left in axisymmetric flow $u_\vartheta=0$. We also ignore all nonaxisymmetric terms in (25). Then the leading term in (15) is

$$\frac{\partial\phi_\infty}{\partial t}=-H^{(2)}=-S\varphi_{xx}. \qquad (31)$$

This form of the evolution equation for the axisymmetric far-field potential is valid in the system in which the fluid at infinity is at rest.

Certain results specific to the axisymmetric case are now derived. The vorticity equation (6) becomes (with the subscript ϑ for ω omitted)

$$\frac{\partial\omega}{\partial t}+\frac{\partial(u_x\omega)}{\partial x}+\frac{\partial(u_r\omega)}{\partial r}=\nu\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial r^2}+\frac{\partial(\omega/r)}{\partial r}\right). \qquad (32)$$

Integration of (32) over a meridional xr plane gives

$$\frac{\partial}{\partial t}\left(\iint \omega\,dx\,dr\right)=-\nu\int_{-\infty}^{+\infty}\left(\frac{\partial\omega}{\partial r}+\frac{\omega}{r}\right)_{r=0}dx \qquad (33)$$

showing that the vorticity integral over the half-infinite xr plane is invariant only in inviscid flow.

The formula $\int q_x\,dV=0$ [see (21)], with q_x given by (28) and $dV=2\pi r\,dx\,dr$, states that

$$\iint u_r\omega r\,dx\,dr=0. \qquad (34)$$

Multiplying (32) by r^2 and integrating again over the xr plane gives, by use of (34),

$$\iint \omega r^2\,dx\,dr=\text{const}, \qquad (35)$$

as the integral over r^2 times the viscous terms of (32) vanishes. [They can be brought into the form of the rhs of (32) by replacing ω with ωr^2, and changing the coefficient of the last term to -3.] The invariant (35) will be related presently to the impulse.

V. THE DRIFT: DYNAMIC INTERPRETATION

It is impossible to find ϕ_∞ from $\partial\phi_\infty/\partial t$ without the use of further information. However, only two potential functions that have a regular zero at infinity can give (31), namely, a dipole φ_x and a quadrupole φ_{xx}. Each term can contribute to $\partial\phi_\infty/\partial t$ in two ways, namely, (i) by drifting; then the order of the pole in the time derivative is raised by one so that this effect is only to be considered for the dipole; and (ii) by the change of the coefficient of these terms. As the dipole strength is known to be a constant (=impulse), this second kind of contribution can come only from the quadrupole φ_{xx}.

It will be shown that the interpretation of $\partial\phi_\infty/\partial t$ being generated to leading order by the drifting dipole potential alone agrees with the classical definition of the drift as the velocity of the impulse centroid. The question of uniqueness will be considered in Paper II.[2]

The determination of the dipole potential itself follows here the theory of Lamb, who states that[5] "...the velocity potential at any point, due to a single reentrant vortex, is equal to the product of $\kappa/4\pi$ into the solid angle which a surface bounded by the vortex subtends at that point." For a circular ring we have $\kappa=\omega\, dr\, dx$ and the solid angle in the far field is equal to the surface πr^2 projected normal to a ray with the components x, r, and divided by $x^2+r^2(=R^2)$. The potential of the ring is thus

$$(4\pi)^{-1}\omega\, dr\, dx\, \pi r^2(x/R)(1/R^2)=-(1/2)\omega r\, dV\, \varphi_x, \tag{36}$$

where $dV=2\pi r\, dr\, dx$ and the dipole potential is normalized in accordance with the definition $\varphi=(4\pi R)^{-1}$, introduced in (18). Thus the farfield potential of an axisymmetric vortex configuration is

$$\phi_\infty=-I\varphi_x, \tag{37}$$

where

$$I=\frac{1}{2}\int \omega r\, dV \tag{38}$$

is recognized as the invariant impulse [proportional to (35)]. The potential is given in a coordinate system fixed with the dipole configuration, i.e., x in (37) is $x_{(\text{fluid fixed})}-x_0(t)$. Thus from (37) we have

$$\frac{\partial\phi_\infty}{\partial t}=-I\varphi_{xx}\frac{dx_0}{dt}. \tag{39}$$

Comparison with (31) gives the final result for $U=dx_0/dt$:

$$U=\frac{S}{I}=\frac{\int\int(r^2u_x+2xru_r)\omega\, dr\, dx}{\int\int r^2\omega\, dr\, dx}, \tag{40}$$

where S and I have been introduced from (30) and (38), with $dV=2\pi r\, dr\, dx$. The integration is to be extended over a meridional plane.

The formula (40) has the proper independence of the choice of the origin of x, which remains arbitrary at any time. According to Eq. (34), it is permissible to change the value of x [which occurs in the numerator of (40)] to $x-x_0=x'$, without affecting the result. This also holds for u_x; replacing $u_x(x)dx$ by $u_x(x')dx'$ has no effect. However, the operation has to be performed in a system in which the fluid at infinity is at rest, so that $u_x(\infty)=0$ is an essential and necessary condition for the validity of our results. Replacing u_x by $u'_x=u_x+u_0$ gives $U'=U+u_0$, i.e., the formula fails.

It seems then for a moment that the result given by (40) is useless, as knowing the velocity u_x that vanishes at infinity apparently means to know everything. However, this is not the case. The correct interpretation of U is that it gives the *instantaneous* velocity with which the whole vortex *configuration* moves. The instantaneous character of the drift is evident as only time-rate equations were needed for the derivation. The velocity is associated with a particular flow element of the configuration, namely, the asymptotic dipole. It remains to be shown that this is also the velocity of the centroid of the integral that represents the invariant impulse I of the flow.

VI. THE DRIFT: CLASSICAL DEFINITION

To derive the classical formula for the drift in inviscid flow, first the simple argument used by Lamb[3] will be given, which has its roots in the approach initiated by Helmholtz[6] in his basic paper of 1858. Then, the verification of this result for viscous flow will follow, which was first presented by Saffman[7] in 1970.

Lamb[3] introduces a second invariant which exists only in inviscid flow, obtained from (33) when its rhs vanishes for $\nu=0$ so that the meridional vorticity integral becomes a flow constant. Let it be connected to (35) by a new constant r_0, defined by the relation

$$\int\int \omega\, dx\, dr=r_0^{-2}\int\int \omega r^2\, dx\, dr=\text{const.} \tag{41}$$

Now let x_0 be defined as the centroid of the impulse integral, given by the equation

$$\int\int \omega r^2(x-x_0)dx\, dr=\int\int (r^2x-r_0^2x_0)\omega\, dx\, dr=0. \tag{42}$$

The drift $U=dx_0/dt$ is obtained by differentiating (42) with respect to time. The first possibility is to introduce $\partial\omega/\partial t$ in the integrand in (42). However, it is also possible to keep $\omega\, dx\, dr$ constant and to differentiate the bracket in the second form in (42) with respect to time. This is a change from the Eulerian to the Lagrangian point of view, which Lamb supports (following Helmholtz[6]) by discretizing the integral in (42) and introducing individual vortices of the strength $\omega\, dA$. Differentiation of the second integral (42) gives in this sense

$$\int\int \omega(2xru_r+r^2u_x-r_0^2U)dx\, dr$$
$$=\int\int [2xu_r+r(u_x-U)]\omega r\, dx\, dr=0 \tag{43}$$

so that (40) reappears in the form

$$U\int\int \omega r^2\, dx\, dr=\int\int u_x\omega r^2\, dx\, dr$$
$$+2\int\int u_r\omega xr\, dx\, dr. \tag{44}$$

Both Helmholtz and Lamb consider this result as incomplete, as further analysis is needed for the solution of the classical vortex ring problem (to be discussed later). It is noted that (41) and the introduction of r_0 is not really needed for the derivation of (44), which can also be based solely on the first form in (42). This removes one paradoxical aspect of this short derivation, which gives a result valid in viscous flow by using arguments valid only in inviscid flow.

However, the second step in the derivation of (44), namely, the Lagrangian differentiation of (42), is also valid only in inviscid flow. Thinking of the discretized definition of the integrals, the Lagrangian differentiation is admissible only if the thin elementary rings are always formed by the same particles. The Helmholtz theorem which states that this is a permissible assumption is valid only in inviscid flow.

The proof of (44) for viscous flow has to revert to the first idea mentioned above, namely, to the evaluation of the integral $\int\int (\partial\omega/\partial t) r^2 x \, dx \, dr$ based on the first integral in (42) [without using (41)], and inserting $\partial\omega/\partial t$ from (32), which follows from the Navier–Stokes equations. Then it is found that no viscous effect is manifest in the result as the operation $\int\int \cdots r^2 x^m \, dx \, dr$, when applied to the viscous term in (32), gives zero, both for $m=0$ [the case considered for the derivation of the invariance of the integral (35)] and for $m=1$, needed in (42). The operations with the inviscid terms are simple and confirm (44). This verification is the essence of Saffman's proof.

Thus it has been shown that the drift (44) of the impulse centroid is the same as the drift of the dipole potential that represents the flow asymptote at infinity, both in inviscid and in viscous flow.

The classical approach to the ring problem requires further analysis, as already noted, because the system in which $u_x(\infty)=0$ is not *a priori* known. Conversely, however, if such an information does exist, then (40) has to suffice for the determination of U. Such is the case for the determination of the asymptotic drift, given by Cantwell and Rott[1] for the terminal stages of plane flow generated by a vortex pair; the derivation of the analogous result in three dimensions now follows.

VII. THE ASYMPTOTIC DRIFT

The general result is now applied to the flow that represents the final stage of decay of a vortical flow region of limited extent in a viscous fluid. In the case that there is an initial impulse, the flow field that prevails for $t \to \infty$ is given by the dipole solution of the time-dependent linear Stokes-flow equation. Its streamfunction is (Phillips[8])

$$\psi = \psi_\infty G(\xi), \tag{45}$$

where

$$\psi_\infty = I r^2 / 4\pi (r^2 + x^2)^{3/2} \tag{46}$$

is the streamfunction in the far-field potential flow region, corresponding to the potential ϕ_∞ given by (37), and I is the impulse (38). The function G depends on the variable

$$\xi = r(4\nu t)^{-1/2} \tag{47}$$

which is the similarity variable of the diffusion equation. The function G is

$$G(\xi) = \frac{2}{\sqrt{\pi}} \left(\int_0^\xi e^{-\xi^2} d\xi - \xi e^{-\xi^2} \right). \tag{48}$$

It is noted that $G(\infty)=1$, and that $G=O(\xi^3)$ for small ξ.

According to the new point of view that has been the starting point of this paper, we do not "iterate" or "improve" on the asymptotic solution given by Eqs. (45)–(48). Instead, we determine U from Eq. (40) which assures that the flow given by Eqs. (45)–(48) at an initial time t will evolve in accordance with the laws of motion that are given by the Navier–Stokes equations. In general, U would be found as an instantaneous value and the proper evolution is assured only for a small time step dt. However, we are dealing with a similarity solution in which time is absorbed in the spatial similarity variable ξ given by (47). Thus the result will give the time dependence of U in the range of validity of the similarity solution.

With the streamfunction given, velocity and vorticity follow; they have to be inserted in the formula (40). The evaluation of U requires lengthy operations with elementary integrals. The result

$$U = (7/15) I (8\pi\nu t)^{-3/2} \tag{49}$$

has already been published (without proof) by Cantwell and Rott[1] and was tested by numerical experiments of Stanaway, Cantwell, and Spalart,[9] who have found a four-digit agreement with (49). Their step-by-step method in time is a numerical experiment for a theory using time evolution.

Cantwell and Rott[1] have derived the asymptotic drift velocity in plane flow in several different ways; the result is

$$U = \frac{I}{32\pi\nu t}, \quad I = \int\int y\omega_z dx \, dy \tag{50}$$

for the impulse aligned with the x axis. The integral for I has to be extended over the whole xy plane.

VIII. APPLICATION OF LAMB'S IDENTITY

The theory as presented thus far is adequate in the course of a step-by-step calculation of a vortical velocity field at any time, and was used in this sense by Stanaway *et al.*[9] For the classical initial value problem in the sense of Helmholtz, however, only the initial vorticity is postulated at the time $t=0$, and the direct connection to the result given by (40) can be established only after the velocity field is also known. This process can be simplified by a relation in which the integrals that determine U are connected directly to the streamfunction. Such a relation was already found by Helmholtz[10] for axisymmetric flow. The general formula of Lamb[3] is valid for any geometry in which a vortical region of finite extent moves in an unbounded three-dimensional incompressible (and possibly viscous) fluid which is at rest at infinity. Lamb gives the following "useful expression"[11] for the total instantaneous kinetic energy:

$$E \equiv \frac{1}{2} \int (u_x^2 + u_y^2 + u_z^2) dV = \int [\mathbf{u} \cdot (\mathbf{r} \times \boldsymbol{\omega})] dV. \tag{51}$$

The usefulness of this relation becomes evident when combined with the expression for E by the vector streamfunction $\mathbf{A}$:

$$E=\frac{1}{2}\int (\mathbf{A}\cdot\omega)dV. \tag{52}$$

In axisymmetric flow, $\mathbf{A}$ can be expressed by the Stokes streamfunction ψ:

$$A_x=0, \quad A_y=-z\psi/r^2, \quad A_z=y\psi/r^2 \tag{53}$$

so that

$$ru_x=\frac{\partial\psi}{\partial r}, \quad ru_r=-\frac{\partial\psi}{\partial x}. \tag{54}$$

Then equating the expressions (51) and (52) leads to the relation

$$\iint (xru_r-r^2u_x)\omega\, dx\, dr=\frac{1}{2}\iint \psi\omega\, dx\, dr. \tag{55}$$

This is used to eliminate u_x in the integrands, in the relations (40) and (44). The new result that emerges is

$$U\iint \omega r^2\, dx\, dr=\frac{1}{2}\iint \psi\omega\, dx\, dr+3\iint xru_r\omega\, dx\, dr. \tag{56}$$

Saffman[7] derived this result directly from the inviscid equations of motion and confirmed later its validity for viscous flow.

The applicability of (56) hinges on information available on the behavior of the streamfunction ψ at infinity, just as (44) needed information on $u_x(\infty)$. Thus the condition

$$\psi(\infty)=0 \tag{57}$$

is an integral part of the results of this analysis. This condition is fulfilled by the asymptotic streamfunction (45), and the drift can be obtained either from (44) or from (56). For the vortex ring, the derivation of ψ starts with the determination of the flux through the ring in a system in which the fluid at infinity is at rest, and (56) permits the direct use of ψ, instead of the velocity component u_x.

In an inviscid fluid, a flow with a streamfunction ψ' (say) can exist which has a steady contour $\psi'=0$ enclosing the vortical region, in the system fixed with the vortex centroid. Then the streamfunction ψ to be introduced in (56) is

$$\psi=\psi'+\frac{1}{2}Ur^2 \tag{58}$$

giving

$$U\iint \omega r^2\, dx\, dy=\frac{2}{3}\iint \psi'\omega\, dx\, dr+4\iint u_r\omega xr\, dx\, dr. \tag{59}$$

For the Hill vortex of radius a and with $\omega=\Omega r$, the streamfunction is $\psi'=(\Omega/10)r^2(a^2-x^2-r^2)$, and one finds $U=(2/15)\Omega a^2$. Lamb[3] derives this result by expressing the continuity of the velocity (and thus of the pressure) on the separating streamline $\psi'=0$ between the vortical and the potential flow regions.

In summary, it is seen that Lamb's identity is not needed whenever the drift is determined in a flow field where at the time considered and in the system used the velocity is known to vanish at infinity. This result holds for all times for a step-by-step calculation which is properly started at $t=0$.

IX. THE PROBLEM OF IMPROVED ASYMPTOTICS

We are now in the position to investigate the role of the drift for expansions in which solutions of the Navier–Stokes equations are sought with the Stokes solution as the leading term. Such a procedure was applied by Kambe and Oshima[12] to the asymptotic flow describing the last stages of a viscous vortex ring. Cantwell and Rott[1] treated the analogous case of the vortex pair. They found that a regular second order solution does not exist unless the proper drift U is introduced first; this served as a means for calculating U. This issue was not encountered by Kambe and Oshima,[12] who carried out their analysis in a vortex-fixed system. The explanation offered here for the difference in the behavior of the two expansion processes is based on the different orders of magnitude of U in the two cases. In plane flow U varies as t^{-1} and the asymptotic displacement grows as $\log t$. In axisymmetric flow, U decreases with $t^{-3/2}$ and the displacement varies with $t^{-1/2}$. Now the success of determining a function Ψ (say) in the "wrong" coordinate system depends on the convergence properties of the expansion of $\Psi(x+a)$ in powers of a, i.e.,

$$\Psi(x+a)=\Psi(x)+\left(\frac{\partial\Psi}{\partial x}\right)a+\frac{\partial^2\Psi}{\partial x^2}\frac{a^2}{2}+\cdots.$$

Identifying the displacement with a, it is clear why the choice of the proper coordinate system is much more important for plane flow than in the axisymmetric case. This is one more case in which there is a marked difference in the behavior of Stokes flow in two and three dimensions.

There is a second reason that Kambe and Oshima[12] were not led to an analytic expression for the drift. They did not complete a full cycle leading to a new streamfunction; they only determined a new vorticity. It is therefore not possible to extract the final drift formula from their work, as was maintained originally by Cantwell and Rott[1] and later by Shariff and Leonard.[13]

This paper provides an advance knowledge of the drift which should greatly facilitate the formulation of a method that intends to give improved asymptotics. The reconsideration of Kambe and Oshima's problem using this point of view is a straightforward generalization, but the explicit determination of the higher-order flow fields leads to problems that are forbidding by the shear bulk of the analysis involved.

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