The decay of a viscous vortex pair

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The evolution of a viscous vortex pair is investigated through the use of a heuristic model. The model is based on the linear superposition of two Oseen vortices of opposite circulation spaced a distance $2b$ apart. The vortices are allowed to evolve through viscous diffusion and their mutual induction. The motion is unforced and as a consequence the total hydrodynamic impulse is exactly conserved for all time. In the model the total circulation in the upper half plane is assumed to remain initially constant. This constraint is applied up to a finite time when the model solution reaches its asymptotic form corresponding to a drifting Stokes dipole dominated by interdiffusion of vorticity across the plane of symmetry. The drift velocity of the vortex pair is determined by the condition that the integrated pressure force vanishes on the line of symmetry at all times. At large time this leads to an asymptotic value of the drift velocity which scales with the similarity properties of the Stokes solution. To provide a more rigorous foundation for the drift, the asymptotic behavior of the flow for large time is investigated through an expansion of the solution in inverse powers of the time. First the second-order pressure is determined as a solution of a Poisson equation with the source term generated by the first-order flow field. Surprisingly, the solution turns out to be independent of the drift. Nevertheless, an exact condition for the drift is found by considering the limiting form of the second-order pressure at infinity where the flow is irrotational and the pressure can be computed directly from the first-order velocity field using Bernoulli's equation. In this latter approach the far field pressure is determined up to an unknown function of time which upon comparison with the Poisson solution is identified as the drift. The exact drift obtained in this fashion differs by only 10% from the value obtained using the pressure field of the heuristic model. Finally, it is shown that the existence of the complete second-order asymptotic solution of the Navier–Stokes equations requires the inclusion of the same drift in the first-order solution that was found from the examination of the pressure. The second-order vorticity and streamfunction are determined; the latter contains a free constant to accommodate conditions at earlier times. Prospects for the existence of higher-order asymptotic terms are discussed.

I. INTRODUCTION; DEFINITION OF THE HEURISTIC MODEL

This study grew out of a desire to simulate the behavior of time-dependent high Reynolds number flow through the superposition of discrete viscous vortices. Each vortex element begins as a point of circulation surrounded by a sea of potential flow. For a finite time after the initiation of the motion and before significant overlap of adjacent rotational regions occurs, the flow can be regarded as closely approximating a solution of the Navier–Stokes equations. In this manner one can study some aspects of the effect of viscosity on the stability and behavior of point vortex arrays. To test this idea we chose to study the decay of a viscous vortex pair using the exact solution of Oseen[1] as our vortex element. This is the simplest nontrivial configuration we could think of and is representative of a flow which is commonly observed as an aspect of more complex flows.

During the course of our study we were deflected somewhat from this initial objective when we discovered that the simple superposition of Oseen vortices contained much useful information about the low Reynolds number, fully overlapped, flow field which prevails at large time. We were led to confront the problem of constructing an asymptotic expansion about the Stokes solution for a vortex pair in order to understand better the large time behavior predicted by the heuristic model. An attempt of this sort was carried out by Kambe and Oshima[2] for the case of a vortex ring. Expanding in inverse powers of the time about the Stokes solution of Phillips[3] they found a type of nonuniformity at large radii similar to the well-known Whitehead's paradox which arises when the Stokes solution for steady flow past a sphere is extended to second order. Following the procedure used for the sphere, they improved their solution by the method of matched asymptotic expansions. A similar problem arises in the expansion of the vortex pair. Guided by the results of the heuristic model we find that it is possible to construct a uniformly valid expansion to second order by adding a drift to the first-order Stokes solution.

The velocity components u, v and the vorticity ω of the Oseen vortex solution are given by

$$u = -(\Gamma y/2\pi r^2)(1 - e^{-r^2/4\nu t}),$$
$$v = (\Gamma x/2\pi r^2)(1 - e^{-r^2/4\nu t}), \quad (1)$$
$$\omega = (\Gamma/4\pi\nu t)e^{-r^2/4\nu t}. \quad (2)$$

The flow invariant is the circulation at infinity, Γ, and the Reynolds number is Γ/ν.

The subject of the present investigation is the problem of the decaying vortex pair flow in the plane. First a heuristic model is established. Use is made of the fact that the Oseen

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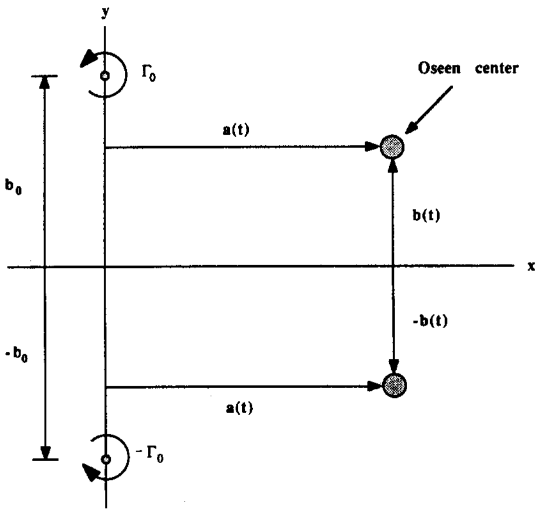

FIG. 1. Schematic drawing of the heuristic model showing moving Oseen centers with coordinates $a(t)$ and $\pm b(t)$. At $t = 0$, $a = 0$, and $b = b_0$.

vortex solution fulfills both the Navier–Stokes equations and the Stokes slow-flow equations. This is a consequence of the fact that along circular streamlines, the nonlinear vorticity convection terms have no effect. Upon superposition this property of the Oseen solution is lost. However, such a solution still fulfills the Stokes equations that are linear.

Figure 1 depicts the heuristic model in which we consider the superposition of two equal but counter-rotating Oseen vortices, Γ and $-\Gamma$, situated at $x = a$ and $y = \pm b$. The model parameters Γ, a, and b are allowed to be functions of time and the vortices are free to move under their mutual induction. The model flow field is given by

$$\begin{aligned} u = &(-(y-b)\Gamma/\{2\pi[(x-a)^2+(y-b)^2]\}) \\ &\times(1-e^{-\{[(x-a)^2+(y-b)^2]/4\nu t\}}) \\ &+((y+b)\Gamma/\{2\pi[(x-a)^2+(y+b)^2]\}) \\ &\times(1-e^{-\{[(x-a)^2+(y+b)^2]/4\nu t\}}), \end{aligned} \quad (3)$$

$$\begin{aligned} v = &((x-a)\Gamma/\{2\pi[(x-a)^2+(y-b)^2]\}) \\ &\times(1-e^{-\{[(x-a)^2+(y-b)^2]/4\nu t\}}) \\ &-((x-a)\Gamma/\{2\pi[(x-a)^2+(y+b)^2]\}) \\ &\times(1-e^{-\{[(x-a)^2+(y+b)^2]/4\nu t\}}), \end{aligned} \quad (4)$$

$$\begin{aligned} \omega = &(\Gamma/4\pi\nu t)(e^{-\{[(x-a)^2+(y-b)^2]/4\nu t\}} \\ &-e^{-\{[(x-a)^2+(y+b)^2]/4\nu t\}}). \end{aligned} \quad (5)$$

The above flow satisfies the Navier–Stokes equations in the limit $t \to 0$, where the vorticity field is confined to a pair of vanishingly small circles near $(x,y) = [a(0), \pm b(0)]$. The assumption that the system is force-free requires that the motion for $t < 0$ corresponds to a pair of point vortices translating at their inviscid speed of mutual induction. At $t = 0$ viscosity is "turned on" and the vortex pair decays.

For the slow flow emerging at $t \to \infty$, a similarity solution is expected. Such a solution is obtained from (3) to (5) by letting b go to zero while keeping the product Γb constant, i.e., by the classical procedure to obtain a dipole. In terms of the similarity variables

$$\xi = r/(4\nu t)^{1/2}, \quad \theta = \tan^{-1}[y/(x-a)], \quad (6)$$

the result of this procedure is

$$\begin{aligned} u = &[(I/\rho)/8\pi\nu t] \\ &\times[(1-2\sin^2\theta)(1-e^{-\xi^2})/\xi^2 + 2\sin^2\theta e^{-\xi^2}], \end{aligned} \quad (7)$$

$$\begin{aligned} v = &[(I/\rho)/8\pi\nu t] \\ &\times\{2\sin\theta\cos\theta[(1-e^{-\xi^2})/\xi^2 - e^{-\xi^2}]\}, \end{aligned} \quad (8)$$

$$\omega = [(I/\rho)/\pi][2/(4\nu t)^{3/2}]\sin\theta\xi e^{-\xi^2}, \quad (9)$$

where u and v are Cartesian velocity components and $r^2 = (x-a)^2 + y^2$.

The usefulness of the model hinges on the fact that, even though the connecting solution is only approximate, it can be assured that the limiting expressions for $t = 0$ and $t = \infty$ belong to the same solution, by noting that the hydrodynamic impulse I/ρ in the x direction remains the same for all times. Its value can be obtained by considering the flow at infinity associated with the creation of a vortex pair. The well-known result is

$$\frac{I}{\rho} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u\,dx\,dy = 2\Gamma(t)b(t) = 2\Gamma_0 b_0, \quad (10)$$

and implies an inverse relationship between the model parameters $\Gamma(t)$ and $b(t)$, which are yet to be detemined. Here, Γ_0 is the initial magnitude of the circulation for one vortex and $2b_0$ is the initial spacing of the vortices. That the impulse is conserved in the presence of viscosity can be easily shown using a control volume analysis.[4] The time-dependent Reynolds number, in the sense of Cantwell,[4] becomes

$$\mathrm{Re} = (I/\rho)^{2/3} t^{-1/3}/\nu. \quad (11)$$

Actually, the heuristic model represents a crude matching (or just a "patching") of the two limiting solutions: There is no finite region of overlap. However, the only one logical "matching point" can be found. To determine this point, the integrated vorticity or circulation in the upper half plane

$$C(t) = \int_{-\infty}^{+\infty}\int_0^{+\infty} \omega\,dy\,dx = \int_{-\infty}^{+\infty} u(x,0,t)dx \quad (12)$$

is calculated for the two limiting solutions. Note that the model parameter $\Gamma(t)$ and the circulation $C(t)$ are equal only at $t = 0$:

$$C_0 = \Gamma_0 = (I/\rho)/2b_0. \quad (13)$$

For small time, C is constant as all vorticity is concentrated in the core.

For large time, C becomes time dependent as a consequence of the dominant effect of vorticity diffusion across the plane of symmetry. Its asymptotic behavior is obtained by an elementary integration of (12) with ω introduced from (9):

$$C_{as}(t) = \int_{-\infty}^{\infty}\int_0^{\infty} \omega\,dy\,dx = \frac{I/\rho}{(4\pi\nu t)^{1/2}}. \quad (14)$$

The obvious matching point is the time t^*, where $C_0 = C_{as}$, that is,

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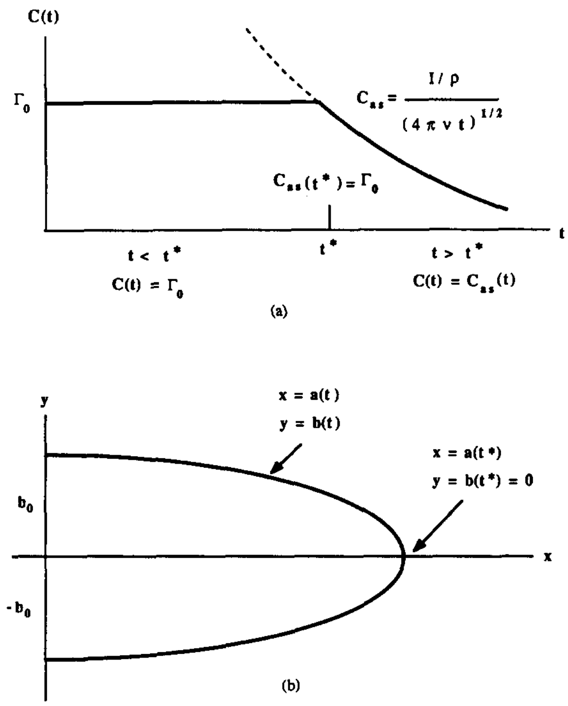

FIG. 2. Unsteady behavior of the heuristic model; (a) postulated time development of total circulation in the upper half plane, $C(t)$; (b) corresponding path of Oseen centers. See also Fig. 8.

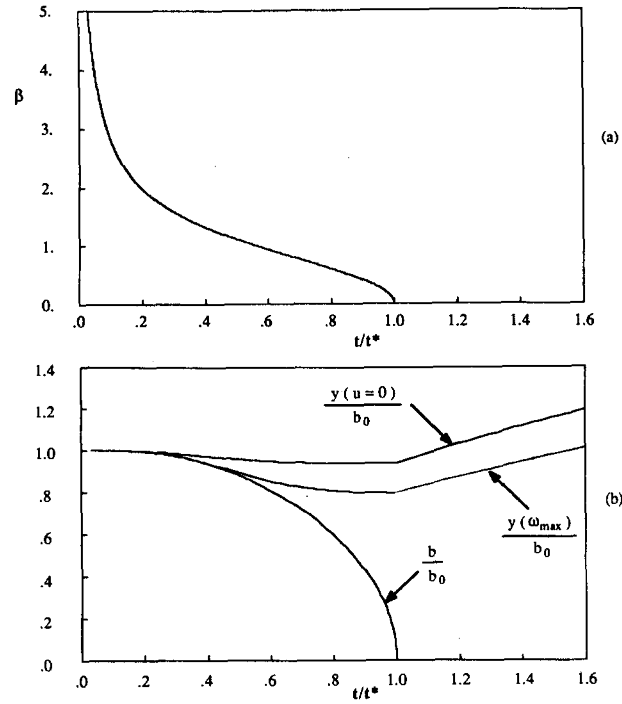

FIG. 3. Unsteady structure of the heuristic model; (a) β vs t/t^*, (b) the function $b(t)$ and the trajectories of the y coordinates of the point of maximum vorticity and point of flow reversal in a frame of reference moving with the vortex pair.

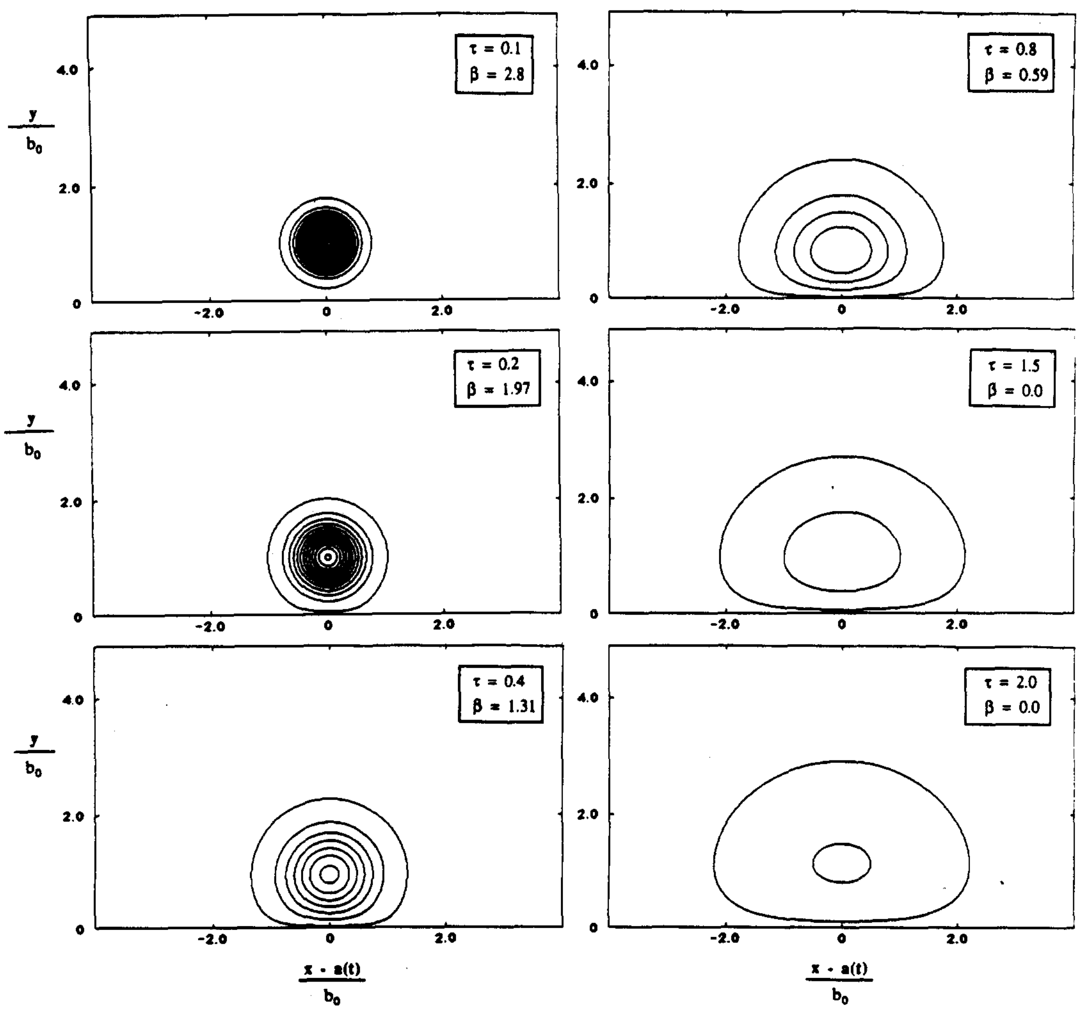

FIG. 4. Vorticity contours for the heuristic model. Here $\tau = t/t^*$.

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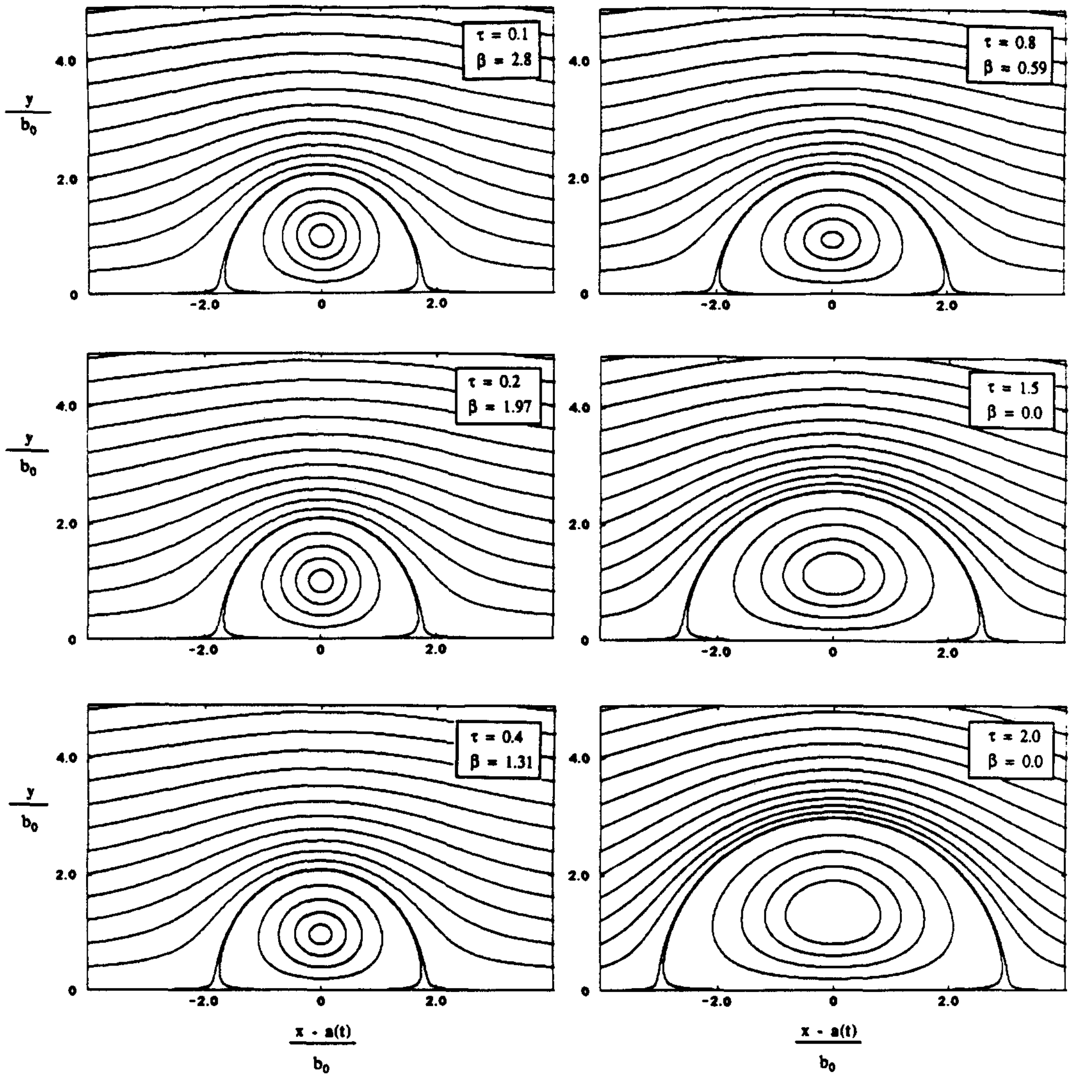

FIG. 5. Streamlines for the heuristic model in a frame of reference moving with the vortex pair. Here $\tau = t/t^*$.

$$t^* = b_0^2/\pi\nu, \tag{15}$$

and the value of C to be used is C_0 or $C_{as}(t)$, depending on whether t is smaller or bigger than t^* (see Fig. 2). The heuristic model is now defined.

Next the function $C(t)$ is evaluated from (12) using the general expression for $u(x,0,t)$ from (3):

$$u(x,0,t) = \{(I/\rho)/2\pi[(x-a)^2 + b^2]\} \times (1 - e^{-[(x-a)^2 + b^2]/4\nu t}). \tag{16}$$

With the variables

$$\beta = b/(4\nu t)^{1/2}, \quad \zeta = (x-a)/b, \tag{17}$$

the expression for C becomes

$$C(t) = [(I/\rho)/2\pi b]F_1(\beta), \tag{18}$$

where $F_1(\beta)$ is given by the integral

$$F_1(\beta) = \int_{-\infty}^{+\infty} (1 - e^{-\beta^2(1+\zeta^2)}) \frac{d\zeta}{1+\zeta^2}. \tag{19}$$

Differentiation of $F_1(\beta)$ with respect to β gives an elementary integral that can be evaluated and then integrated with respect to β; the result is

$$F_1(\beta) = 2\sqrt{\pi} \int_0^{\beta} e^{-\beta^2} d\beta = \pi \operatorname{erf}(\beta). \tag{20}$$

The value of C for $t > t^*$, C_{as}, is immediately recovered by approximating $F_1(\beta) = 2\beta(\pi)^{1/2}$ for $\beta \to 0$. Then (18) becomes identical with (14). For $t < t^*$, C in (18) is identified with C_0 in (13). This gives an implicit equation for b:

$$b/b_0 = \operatorname{erf}(\beta), \tag{21}$$

that is solved as a function of time by introducing t^* in the definition of β:

$$\beta = \frac{\sqrt{\pi}}{2}\left(\frac{t^*}{t}\right)^{1/2} \frac{b}{b_0}. \tag{22}$$

The resulting curve of β vs t/t^* is plotted in Fig. 3(a). The curve of b/b_0 vs t/t^* is plotted in Fig. 3(b). The slow change of b/b_0 for $t \ll t^*$ is reasonable, but the abrupt decrease to 0 at $t = t^*$, with a vertical tangent, is evidence of the inadequacy of the model around $t = t^*$, where the asymptotic value

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$b = 0$ is reached prematurely. Flow patterns drawn in Figs. 4 and 5 using these results show the transition to the final Stokes similarity pattern by the combined effect of the spreading and merging of the two vortex cores. These flow patterns can be compared with the trajectories of the vorticity maximum and point of zero streamfunction given in Fig. 3(b). The fore and aft symmetry of the vorticity and streamfunction patterns in these two figures is further evidence of the inadequacy of the model which does not permit any straining or distortion of the finite size vortex cores except for a flattening that occurs along the plane of symmetry as the vortices begin to overlap. Note that the horizontal coordinate is measured in terms of $[x - a(t)]/b_0$ and $a(t)$ is not yet determined.

II. CALCULATION OF THE DRIFT FROM THE HEURISTIC MODEL

The determination of $a(t)$ from the heuristic model that now follows has led to some unexpected results; the search for a more rigorous foundation of these results has in turn led to a systematic approach to the problem, to be treated in the subsequent sections. First, however, the heuristic approach is brought to a conclusion.

At time $t = 0$, each of the concentrated force-free vortices moves in the x direction with the velocity that is induced by the other vortex, namely, $\Gamma_0/4\pi b_0$. As the determination of the drift velocity by the induction law becomes very involved in viscous flow, it will be calculated here in a different way, namely, by use of the momentum balance in the y direction, applied to the upper half plane.

In plane incompressible flow in which (i) there is a symmetry line (the x axis, say); (ii) the flow is at rest at infinity; (iii) there are no external forces or submerged bodies that can support a force; and (iv) the velocity is finite everywhere, it is found that the resultant force on the symmetry line in the y direction, F_y, is zero. Upon accounting for the conditions (i)–(iii), the pressure-momentum balance in the y direction gives for this quantity,

$$F_y = \frac{dJ}{dt}, \tag{23}$$

where J is in the y component of the hydrodynamic impulse in the upper half plane

$$\frac{J}{\rho} = \int_0^\infty \int_{-\infty}^{+\infty} v\,dx\,dy = \int_0^\infty V\,dy, \tag{24}$$

and V is given by

$$V(y,t) = \int_{-\infty}^{+\infty} v(x,y,t)dx, \quad V(0,t) = 0. \tag{25}$$

Conditions (ii) and (iv) permit the evaluation of $\partial V/\partial y$ as

$$\frac{\partial V}{\partial y} = -\int_{-\infty}^{+\infty} \frac{\partial u}{\partial x}\,dx = u(-\infty) - u(+\infty) = 0, \tag{26}$$

and in consequence, $J = 0$ and $F_y = 0$, that is,

$$\int_{-\infty}^{+\infty} \left(-p(x,0,t) + \mu \frac{\partial v}{\partial y}\right) dx = -\int_{-\infty}^{+\infty} p(x,0,t)dx + \mu[u(+\infty) - u(-\infty)] = -\int_{-\infty}^{+\infty} p(x,0,t)dx = 0. \tag{27}$$

To apply this result to the flow defined in Sec. I, the pressure is calculated from the momentum equation on the symmetry line.

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\nabla^2 u, \tag{28}$$

with u inserted from (3). The expression for $\partial u/\partial t$ contains three terms:

$$\frac{\partial u}{\partial t} = \left.\frac{\partial u}{\partial t}\right|_{a,b} + \dot a\frac{\partial u}{\partial a} + \dot b\frac{\partial u}{\partial b}. \tag{29}$$

It is a property of the heuristic model that

$$\left.\frac{\partial u}{\partial t}\right|_{a,b} = \nu\nabla^2 u, \tag{30}$$

because keeping a and b constant defines a system fixed with the Oseen vortex elements. In such a system Eq. (30), which is the linear slow-flow equation, is fulfilled at every point. The second term represents the contribution of the field translation, relative to the fluid at rest at infinity, with the velocity $da/dt \equiv U$:

$$\dot a\frac{\partial u}{\partial a} = -U\frac{\partial u}{\partial x}. \tag{31}$$

The third term, found from (3) after a brief calculation, is

$$\dot b\frac{\partial u}{\partial b} = \dot b\frac{u}{b} + \frac{b\dot b}{x-a}\frac{\partial u}{\partial x}. \tag{32}$$

This expression remains finite as $\partial u/\partial x = 0$ for $x = a$, and is symmetric with respect to $x - a$, in contrast to (31), which is antisymmetric. Thus (32) produces [in contrast to (31)] an antisymmetric pressure distribution, which does not have a net contribution to the integral (27). The moment about $x = a$ associated with this term is infinite, a reminder that inevitable flaws of the heuristic model do exist. On the positive side, however, is the fact that the whole distribution (32) vanishes for $t = 0$ and $t = -\infty$; thus its omission does not affect the limits that are securely connected by the use of the flow invariant I/ρ.

The upshot of this discussion is an approximation to (28) in which all terms are eliminated except those that occur also in the inviscid case, so that p fulfills Bernoulli's equation

$$p(x,0,t) = \rho(Uu - \tfrac{1}{2}u^2). \tag{33}$$

Introducing this result in (27), and with u given by (16), the following formula for U emerges:

$$U = [(I/\rho)/4\pi b^2]F_2(\beta)/F_1(\beta), \tag{34}$$

where

$$F_2(\beta) = \int_{-\infty}^{+\infty} (1 - e^{-\beta^2(1+\zeta^2)})^2 \frac{d\zeta}{(1+\zeta^2)^2}. \tag{35}$$

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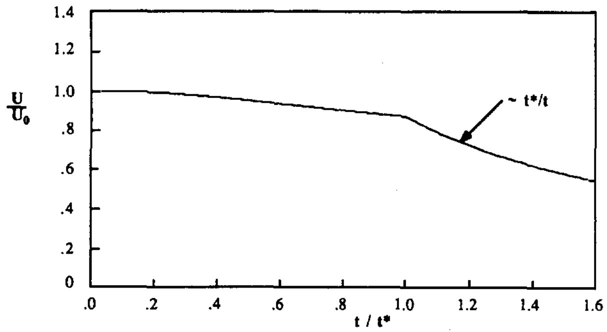

FIG. 6. Drift velocity of the vortex pair, $U = da/dt$; U_0 is the inviscid velocity.

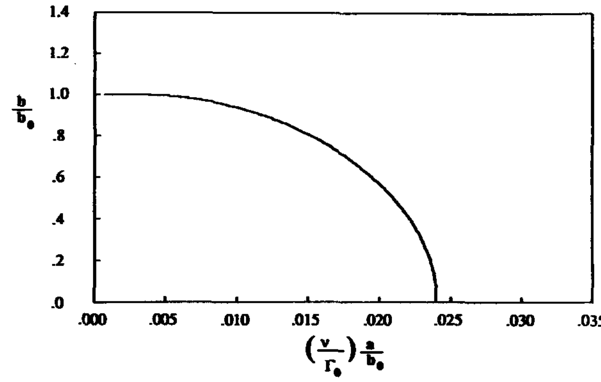

FIG. 8. Trajectory in space of the Oseen centers.

Differentiating $F_2(\beta)$ with respect to β gives

$$\frac{dF_2(\beta)}{d\beta} = 4\beta\,[F_1(\sqrt{2}\beta) - F_1(\beta)] \tag{36}$$

[see also (20)]. Integrating (36) gives

$$F_2(\beta) = \tfrac{1}{2}F_3(\sqrt{2}\beta) - F_3(\beta), \tag{37}$$

$$F_3(\beta) = (2\beta^2 - 1)\frac{2}{\sqrt{\pi}}\int_0^{\beta} e^{-\beta^2}\,d\beta + \frac{2}{\sqrt{\pi}}\beta e^{-\beta^2}. \tag{38}$$

This is used together with (20) and (21) for the numerical evaluation of (34).

The limiting value U_0 for $t = 0$ is obtained for $\beta = \infty$, $F_1 = 1$, and $F_2 = \frac{1}{2}$, and is in agreement with the inviscid convection speed:

$$U_0 = (I/\rho)/8\pi b_0^2\,. \tag{39}$$

The limiting function U_{as} for $t \to \infty$, $\beta \to 0$ is obtained from (34) using the approximations

$$F_1(\beta) \cong (2/\sqrt{\pi})\beta,$$
$$F_2(\beta) \cong (2/\sqrt{\pi})(4/3)(\sqrt{2} - 1)\beta^3. \tag{40}$$

With the definition (17) of β, it follows that

$$U_{as} = [(I/\rho)/12\pi\nu t](\sqrt{2} - 1). \tag{41}$$

The existence of this "asymptotic drift," which is dimensionally and physically compatible with the asymptotic slow-flow solution, is an unexpected flow property of the heuristic model. Its examination based on a more rigorous analysis appears desirable.

It is interesting to note that U_{as} has been expressed in (41) without the explicit use of t^*, the matching point of the heuristic model. In Fig. 6, however, U/U_0 is plotted against t/t^*. From (34), with (39) and (15), it follows that

$$\frac{U}{U_0} = \frac{2b_0^2}{b^2}\frac{F_2(\beta)}{F_1(\beta)} = \frac{2F_2(\beta)}{[F_1(\beta)]^2} \tag{42}$$

and the asymptotic value is

$$\frac{U_{as}}{U_0} = \frac{2\pi(\sqrt{2} - 1)}{3}\frac{t^*}{t} = 0.87\frac{t^*}{t}. \tag{43}$$

At the matching point $t = t^*$, U is continuous but has a jump in slope.

The total displacement of the vortex pair since the onset of the motion is found by integrating U with respect to t to give $(a/b_0)(\nu/\Gamma_0)$ shown in Fig. 7. The resulting trajectory of $b(t)/b_0$ vs $a(t)/b_0$ is shown in Fig. 8. For $t > t^*$ the total displacement is given by

$$\frac{a(t)}{b_0} = \left[0.024 + 0.022\ln\left(\frac{t}{t^*}\right)\right]\frac{\Gamma_0}{\nu}, \tag{44}$$

which expresses two important properties of the heuristic model: first, that the total displacement of the vortex pair at a given time is linearly proportional to the initial vortex Reynolds number; and, second, that the displacement increases logarithmically without limit.

The positioning of the origin of coordinates under the vortex center in Figs. 4 and 5 implies that the flow variables plotted in these figures are represented in a moving frame of reference whose origin is located at $x = a(t)$. In this frame there is an opposing wind $-da/dt$ that causes the vortex pair to be enclosed in a circular streamline of finite radius. The data in Fig. 6 have been used to provide the drift velocity at various times required for the streamfunction plots in Fig. 5. When plotted in this fashion, Figs. 4 and 5 are "universal" in the sense that they are independent of the physical constants of the flow. The instantaneous streamlines plotted in the absolute coordinates of the heuristic model represent a Stokes dipole without a closed streamline.

The first step toward a more rigorous treatment of the drift is taken in the next section, by the accurate calculation of the second-order pressure.

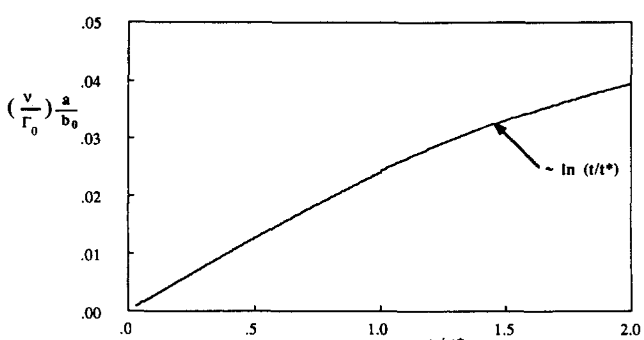

FIG. 7. The total displacement of the vortex pair. Note that the total displacement is proportional to the initial vortex Reynolds number.

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III. CALCULATION OF THE SECOND-ORDER PRESSURE FIELD

The viscous incompressible time-dependent flow equations are written for plane flow in terms of the streamfunction:

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}. \tag{45}$$

The vorticity is

$$\omega = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\nabla^2\psi. \tag{46}$$

With the help of the quantity

$$P = p/\rho + \tfrac{1}{2}(u^2 + v^2), \tag{47}$$

and with ω defined in (46), the Navier–Stokes equations are written in the form

$$-\frac{\partial P}{\partial x} = \frac{\partial u}{\partial t} + \nu\nabla^2 u - v\omega, \tag{48}$$

$$-\frac{\partial P}{\partial y} = \frac{\partial v}{\partial t} + \nu\nabla^2 v + u\omega. \tag{49}$$

It follows by using (45) and (46) that P fulfills the Poisson equation

$$\nabla^2 P = \omega^2 - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial y}. \tag{50}$$

By use of the identity

$$\nabla^2(\psi\omega) = \psi\nabla^2\omega + \omega\nabla^2\psi + 2\left(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial y}\right), \tag{51}$$

a more convenient form of Eq. (50) is obtained:

$$\nabla^2 P^* = \tfrac{1}{2}(\omega^2 + \psi\nabla^2\omega), \tag{52}$$

where

$$P^* = p/\rho + \tfrac{1}{2}(u^2 + v^2 + \omega\psi). \tag{53}$$

The flow is determined by the convection–diffusion equation for the vorticity obtained from (48) and (49) by eliminating P:

$$\nu\nabla^2\omega - \frac{\partial\omega}{\partial t} = u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y}. \tag{54}$$

For Stokes flow, the vorticity fulfills the linear part of Eq. (54), i.e., a homogeneous diffusion equation:

$$\nu\nabla^2\omega_1 - \frac{\partial\omega_1}{\partial t} = 0, \tag{55}$$

and the streamfunction follows from (53).

Linearization reduces Eqs. (52) and (53) for the pressure to

$$\nabla^2 p_1 = 0. \tag{56}$$

In the case where no singularities are admitted and there are no bodies submerged in the flow, (56) has only the trivial solution $p_1 = \text{const}$, or a linear function of space to be discussed presently. The viscous vortex pair problem for $t > 0$ is in this category, and the nontrivial leading-order pressure is the quadratic approximation, p_2. All terms in (52) and (53) are quadratic except the pressure; thus the quantity p_2 follows from these equations upon introduction of the linear solution (subscript 1) in all other terms. Let p_2 be split into two parts. The first part corresponds to the homogeneous solution of (52), which is $P_2^* = \text{const}\ (=0)$. Thus the corresponding part of p_2 is

$$(1/\rho)p_{2_h} = -\tfrac{1}{2}(u_1^2 + v_1^2 + \omega_1\psi_1). \tag{57}$$

The second part of p_2 is obtained from the particular solution of the inhomogeneous equation (52) for P_2^*. Thus this part of p_2 is governed by the equation

$$(1/\rho)\nabla^2 p_{2_p} = \tfrac{1}{2}(\omega_1^2 + \psi_1\nabla^2\omega_1). \tag{58}$$

For the problem at hand, namely, the asymptotic similarity Stokes flow solution with the vorticity (9), the streamfunction ψ_1 is found by differentiation with respect to y of the Oseen vortex streamfunction. Thus ψ_1 is proportional to the x velocity component of the Oseen flow, given in (1). With an additional term added to accommodate the drift anticipated in Sec. II, ψ_1 is written in terms of the similarity variables (6) as follows:

$$\psi_1 = [(I/\rho)/2\pi\sqrt{4\nu t}\,][(1/\xi)(1 - e^{-\xi^2}) + c_0\xi]\sin\theta. \tag{59}$$

The drift is proportional to the constant c_0, which is not yet determined. The velocity vector has the radial and circumferential components

$$u_{r_1} = \frac{1}{r}\frac{\partial\psi_1}{\partial\theta} = \frac{I/\rho}{2\pi(4\nu t)}\left(\frac{1}{\xi^2}(1 - e^{-\xi^2}) + c_0\right)\cos\theta, \tag{60}$$

$$u_{\theta_1} = -\frac{\partial\psi_1}{\partial r} = \frac{I/\rho}{2\pi(4\nu t)}\left(\frac{1}{\xi^2}(1 - e^{-\xi^2}) - 2e^{-\xi^2} - c_0\right)\sin\theta, \tag{61}$$

and the vorticity (9) renamed ω_1 is given here again for completeness:

$$\omega_1 = [(2I/\rho)/\pi(4\nu t)^{3/2}]f_1\sin\theta, \quad f_1 = \xi e^{-\xi^2}. \tag{62}$$

The meaning of the constant c_0 requires some clarification. The system used in (59)–(62) has its origin in the center of the Stokes dipole, and the drift appears as a relative wind, uniform at infinity. The velocity and vorticity fields of the heuristic model (3)–(5) are written as functions of the coordinates $(x - a(t), y)$, where x is measured from an origin fixed in an inertial frame with the fluid at rest at infinity. The vortex pair moves in the frame (x,y) with the velocity da/dt. Thus in effect we have made in (59) a change to a system in which there is a relative wind, $-da/dt$, a system that has been already defined at the end of Sec. II by coordinates used in Figs. (4) and (5). As the relative wind, $-da/dt$, decays with time, the system is noninertial, and in consequence the deceleration of the uniform wind must be balanced by a decaying pressure field

$$(1/\rho)p_1 = [(I/\rho)/2\pi(4\nu)^{1/2}t^{3/2}]c_0\xi\cos\theta. \tag{63}$$

This is a linear function of x that is a solution of the homogeneous equation (56). Upon transformation back to the inertial frame, the first-order pressure is restored to zero.

It remains to introduce (59)–(62) in the pressure equations (57) and (58). For the first part, straightforward alge-

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bra gives

$$-\frac{1}{\rho}p_{2_h}=\frac{(I/\rho)^2}{(4\pi\nu t)^2}\Big[\frac{1}{8\xi^4}(1-e^{-\xi^2})^2$$
$$+\frac{c_0}{4\xi^2}(1-e^{-\xi^2})\cos 2\theta$$
$$+\Big(\frac{1}{4}e^{-\xi^2}-\frac{1}{4\xi^2}e^{-\xi^2}(1-e^{-\xi^2})$$
$$+\frac{c_0}{4}(1+\xi^2)e^{-\xi^2}\Big)(1-\cos 2\theta)\Big]. \quad (64)$$

The first two lines of this equation contain the terms considered in Sec. II; the other terms vanish on the symmetry line $y=0$. Introduction of this part of the pressure in the condition (27) for vanishing force on the symmetry line leads to a value of c_0 corresponding to the drift given in Sec. II by (41), namely,

$$c_0=\tfrac{2}{3}(\sqrt{2}-1)=-0.276. \quad (65)$$

The determination of the second part of the pressure involves a solution of the Poisson equation. This operation has no counterpart in the pressure calculations of Sec. II. The discussion of this significant point is deferred: first, the calculation of p_2 is completed.

In calculating the inhomogeneous term in (58), it is advantageous to replace $\nabla^2\omega_1$ by $\partial\omega_1/\partial t$, using (55). The resultant equation is

$$(1/\rho)\nabla^2_{\xi,\theta}(p_{2_p})$$
$$=[(I/\rho)^2/(4\pi\nu t)^2]V_p(\xi)(1-\cos 2\theta), \quad (66)$$

where

$$V_p(\xi)=2e^{-2\xi^2}-2e^{-\xi^2}$$
$$+(1-2c_0)\xi^2e^{-\xi^2}+c_0\xi^4e^{-\xi^2}. \quad (67)$$

The Laplacian in (66) has been transformed to the similarity coordinates. The solution of (66) is anticipated in the form

$$(1/\rho)p_{2_p}=[(I/\rho)^2/(4\pi\nu t)^2][h_0(\xi)-h_2(\xi)\cos 2\theta], \quad (68)$$

where h_0 and h_2 fulfill the equations

$$\frac{d^2h_0}{d\xi^2}+\frac{1}{\xi}\frac{dh_0}{d\xi}=V_p(\xi), \quad (69)$$

$$\frac{d^2h_2}{d\xi^2}+\frac{1}{\xi}\frac{dh_2}{d\xi}-\frac{4}{\xi^2}h_2=V_p(\xi). \quad (70)$$

The function h_0 is determined by the two integrations

$$H_0=\int\xi V_p\,d\xi,\quad h_0=\int\xi^{-1}H_0\,d\xi, \quad (71)$$

and with the help of the homogeneous solution of (70), ξ^{-2}, the inhomogeneous solution is obtained, using standard methods, by the integrals

$$H_2=\int\xi^{-1}V_p\,d\xi,\quad h_2=\xi^{-2}\int\xi^3H_2\,d\xi. \quad (72)$$

Constants of integration are determined by the requirement that no singularities occur for $\xi=0$. The result of the lengthy calculation is

$$h_0=-\frac{1}{2}\int_\xi^\infty(e^{-\xi^2}-e^{-2\xi^2})\frac{d\xi}{\xi}$$
$$+\frac{1}{4}e^{-\xi^2}+\frac{c_0}{4}(1+\xi^2)e^{-\xi^2} \quad (73)$$

and

$$h_2=\frac{1}{2}\xi^2\int_\xi^\infty(e^{-\xi^2}-e^{-2\xi^2})\frac{d\xi}{\xi}+\frac{1}{8}e^{-2\xi^2}$$
$$-\frac{1}{16\xi^2}(1-e^{-2\xi^2})$$
$$+\frac{c_0}{4}(1+\xi^2)e^{-\xi^2}-\frac{c_0}{4\xi^2}(1-e^{-\xi^2}). \quad (74)$$

The sum of the two parts determined thus far gives the final formula for p_2:

$$-\frac{1}{\rho}p_2=\frac{(I/\rho)^2}{(4\pi\nu t)^2}\Big[\frac{1}{8\xi^4}(1-e^{-\xi^2})^2-\frac{1}{4\xi^2}(e^{-\xi^2}-e^{-2\xi^2})+\frac{1}{2}\int_\xi^\infty(e^{-\xi^2}-e^{-2\xi^2})\frac{d\xi}{\xi}+\Big(\frac{1}{4\xi^2}(e^{-\xi^2}-e^{-2\xi^2})$$
$$-\frac{1}{16\xi^2}(1-e^{-2\xi^2})-\frac{1}{4}e^{-\xi^2}+\frac{1}{8}e^{-2\xi^2}+\frac{1}{2}\xi^2\int_\xi^\infty(e^{-\xi^2}-e^{-2\xi^2})\frac{d\xi}{\xi}\Big)\cos 2\theta\Big]. \quad (75)$$

This result contains a genuine surprise, namely, that all terms resulting from the drift, i.e., those multiplied by c_0, have canceled! This means that p_2 is the same for all values of the drift.

The first step in the discussion of (75) is the verification of two immediate consequences. First, the pressure integral along the whole symmetry line (27) has to vanish for p_2 as given by (75). This is easily verified, particularly as the less common integrals that occur have already been evaluated in previous sections. Second, the formulas for the pressure, (47) and (50), must be invariant under translations of the frame of reference, including the case of nonuniform motion. For the full Navier–Stokes equations this is known to be true.[5,6] The verification for the present case is straightforward, using (47) and (50).

IV. THE EXACT VALUE OF THE ASYMPTOTIC DRIFT VELOCITY

It seems, at first, that the plan to find an improved value for the drift by the determination of p_2 has failed. Moreover, the results cast serious doubt on the usefulness of the heuristic model. The way out for both difficulties is found by considering the second-order pressure p_2 at infinity.

It is noted that at infinity the flow is not vortical, in spite of the presence of a steadily growing vortical region. Outside

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this region contaminated by vorticity, potential flow is found; the potential flow pattern drifts relative to the fluid at rest, but is otherwise not disturbed by the unsteadiness caused by the growing vortical region of fixed impulse. The first-order potential $\phi_{1\infty}$ that corresponds to the streamfunction ψ_1 given by (59) evaluated at infinity is now written in the inertial frame as

$$\phi_{1\infty}(x,y,t) = -\frac{I/\rho}{2\pi}\frac{x-a(t)}{[x-a(t)]^2+y^2} = \frac{I/\rho}{2\pi}\frac{\cos\theta}{r}. \tag{76}$$

The pressure is given by the classical Bernoulli equation, that is,

$$\frac{p}{\rho} = -\frac{\partial\phi}{\partial t} - \frac{1}{2}(u^2+v^2) = U_{\rm as}u - \frac{1}{2}(u^2+v^2), \tag{77}$$

where $U_{\rm as} = da/dt$ has been introduced. The second-order pressure at infinity is now obtained from the first-order velocity components. With the velocity derived from the dipole potential (76), Eq. (77) gives the second-order pressure at infinity as

$$\frac{p_{2\infty}}{\rho} = U_{\rm as}\frac{I/\rho}{2\pi}\frac{\cos 2\theta}{r^2} - \frac{(I/\rho)^2}{8\pi^2 r^4}. \tag{78}$$

The quadrupole component of the pressure contains the unknown drift.

Now the pressure (78) is compared with p_2 at infinity found from (75), an expression that is valid over the whole vortical field. At infinity, (75) gives

$$\frac{p_{2\infty}}{\rho} = \frac{(I/\rho)^2}{64\pi^2\nu t}\frac{\cos 2\theta}{r^2} - \frac{(I/\rho)^2}{8\pi^2 r^4}. \tag{79}$$

In this expression the coefficient of the quadrupole term is determined. Its value is part of the pressure solution obtained from the Poisson equation (52). Comparison of (78) and (79) gives the result

$$U_{\rm as} = (I/\rho)/32\pi\nu t. \tag{80}$$

This exact result has to be compared with (41). In terms of the coefficient c_0, which is a nondimensional measure of the drift, the exact value is $c_0 = -\frac{1}{4}$, which is only 10% less than the value (65) obtained from the heuristic model.

Actually, the fact that the heuristic model allows for the drift is the explanation, in retrospect, for its usefulness. The

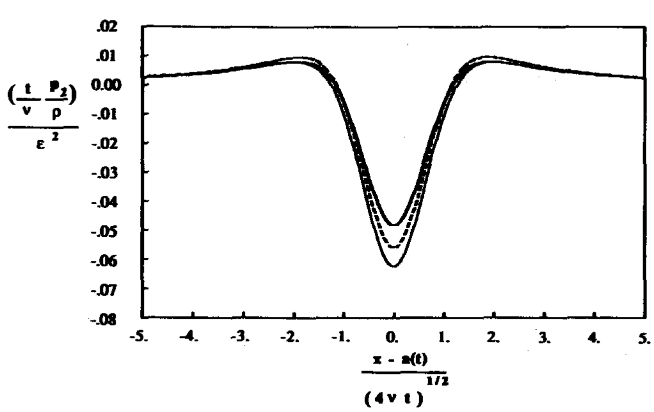

FIG. 9. Pressure distribution along the plane of symmetry ($y=0$). The pressure derived from the heuristic model [Eq. (64)] is shown by the dotted line for $c_0 = -0.25$ and by the dashed line for $c_0 = -0.276$. Solid line is the exact second-order pressure [Eq. (75)].

specific value of the drift is obtained by applying Bernoulli's equation (77) on the symmetry line, where it is expected to give a rough approximation only. Nevertheless, this pressure, together with the condition that the resultant force on the symmetry line vanishes, leads to an error of only 10% in the drift. Therefore the distribution of p_2 on the symmetry line according to the exact result (75) and the pressure derived from Bernoulli's equation, that is, (64) for $\theta=0$, should show reasonable agreement. This is confirmed in Fig. 9.

The result (80) can be confirmed using the drift velocity derived from the integral representation of the vortex centroid in the manner of Kambe and Oshima[2] or of Stanaway, Cantwell, and Spalart.[8] However, from the point of view of the present analysis of the whole asymptotic field, the logical next step is to solve the second-order vorticity transport equation and to show that the existence of this solution requires the same drift as was found from the far field pressure.

V. THE SECOND TERM OF THE ASYMPTOTIC SOLUTION

The first approximation to the vorticity, ω_1, is a solution of the homogeneous diffusion equation (55). The second-order vorticity ω_2 has to balance the convection terms in (54) associated with the first-order solution. Thus ω_2 fulfills the inhomogeneous equation

$$\nu\nabla^2\omega_2 - \frac{\partial\omega_2}{\partial t} = u_1\frac{\partial\omega_1}{\partial x} + v_1\frac{\partial\omega_1}{\partial y}. \tag{81}$$

The second-order streamfunction ψ_2 follows then from the Poisson equation

$$\nabla^2\psi_2 = -\omega_2. \tag{82}$$

The solution has to fulfill the condition that the vorticity vanishes exponentially at infinity.

This scheme involves the solution of two linear equations: the inhomogeneous diffusion equation first, and the Poisson equation second. It turns out that a different method in which the order of these operations is reversed offers considerable advantages. The method is first given here in full generality, starting with the Navier–Stokes equations in the form (48) and (49). Let a flow quantity B be defined as follows:

$$B = \frac{\partial\psi}{\partial t} + \nu\omega = \frac{\partial\psi}{\partial t} - \nu\nabla^2\psi. \tag{83}$$

With the help of this quantity, (48) and (49) are rewritten as follows:

$$-\frac{\partial P}{\partial x} = \frac{\partial B}{\partial y} - v\omega, \tag{84}$$

$$\frac{\partial P}{\partial y} = \frac{\partial B}{\partial x} - u\omega. \tag{85}$$

Elimination of P leads to a Poisson equation for B:

$$\nabla^2 B = u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y}. \tag{86}$$

It is noted that if P is known as the solution of the Poisson equation that results from the elimination of B, i.e., as a solution of Eq. (50), then inserting those values of P in (84)

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and (85) gives equations for B that fulfill the integrability condition. Thus the perfect differential dB is known, and it is possible to obtain B by only one integration from either (84) or (85), without the need of solving (86). Conversely, when B is known from (86), then P can be found with one integration only.

It is not intended to take advantage of this connection between P and B here for the analysis that follows, but is pointed out that any condition that was found to be necessary for the existence of P will have its counterpart in a condition for B. This assures that the subsequent analysis using B will lead to the same value for the drift as was found in Sec. III.

The equations (86) and (83) for the second-order quantities B_2 and ψ_2 are

$$\nabla^2 B_2 = u_1 \frac{\partial \omega_1}{\partial x} + v_1 \frac{\partial \omega_1}{\partial y}, \tag{87}$$

$$\nu \nabla^2 \psi_2 - \frac{\partial \psi_2}{\partial t} = -B_2, \tag{88}$$

and it is seen that now the operation that ultimately leads to the determination of the second-order streamfunction involves first the solution of a Poisson equation and then the solution of the inhomogeneous diffusion equation.

The inhomogeneous term in the first equation (of both schemes) is

$$u_1 \frac{\partial \omega_1}{\partial x} + v_1 \frac{\partial \omega_1}{\partial y} = \frac{(\Gamma/p)^2}{(4\pi\nu t)^3} V(\xi) \sin 2\theta, \tag{89}$$

where

$$V(\xi) = (1/\xi^2)(1 - e^{-\xi^2}) e^{-\xi^2} - e^{-\xi^2} - c_0 \xi^2 e^{-\xi^2}. \tag{90}$$

This suggests a solution of (87) of the form

$$B_2 = (\Gamma/\rho/4\pi\nu t)^2 q_2(\xi) \sin 2\theta. \tag{91}$$

The resulting differential equation for q_2 has the same homogeneous part as Eq. (70) for h_2; the inhomogeneous term in the equation for q_2 is V defined by (89) and (90), which replaces V_p in (70) defined by (66) and (67). The solution is obtained by use of the scheme (72), and the result is

$$\begin{aligned} q_2 = & -\frac{1}{4} e^{-\xi^2} + \frac{1}{8}\left(1 - \frac{1}{2\xi^2}\right) e^{-2\xi^2} \\ & + \frac{1}{2}\xi^2 \int_\xi^\infty (e^{-\xi^2} - e^{-2\xi^2}) \frac{d\xi}{\xi} \\ & - \frac{1}{4} c_0 \left(1 + \frac{1}{\xi^2}\right) e^{-\xi^2} + c_1 \xi^2 + c_2 \xi^{-2}. \end{aligned} \tag{92}$$

The last two terms are the homogeneous solutions. The choice of the fixed limit of the integral in (92) at infinity assures $B_2(\infty) = 0$ for $c_1 = 0$. Regularity at the origin requires that

$$c_2 = \tfrac{1}{16} + c_0/4. \tag{93}$$

For nonvanishing c_2, the leading term of B_2 at infinity decays algebraically:

$$B_{2\infty} = c_2[(\Gamma/\rho)^2/4\pi^2 \nu t](\sin 2\theta)/r^2. \tag{94}$$

One more condition to be fulfilled by the solution is given by the relationship between B and ψ at infinity, which according to (83) holds for all orders when the vorticity at infinity vanishes exponentially:

$$B_{2\infty} = \frac{\partial \psi_{2\infty}}{\partial t}. \tag{95}$$

Looking at (94) and (95) makes it clear that a nonvanishing constant c_2 means logarithmic variation of ψ_2 with time, which is impossible. Thus $c_2 = 0$, a condition that can be fulfilled thanks to the drift term in (93) by setting $c_0 = -\frac{1}{4}$. This agrees with the result of Sec. III.

The possibility that (95) implies a time variation of ψ_2, which is caused by moving coordinates, is excluded, as such a term would be varying with 3θ and thus affecting the third harmonics. For the same reason, the drift implies a correction of the first-order streamfunction ψ_1 given by (59), and this affects the solution of the second-order equation (88) for ψ_2. From the point of view of a systematic asymptotic expansion, the drift is a type of second-order effect which resembles what has been called elsewhere[7] a "switchback," a first-order effect determined by a second-order condition.

The expression (91) for B_2 suggests the following form for the solution ψ_2 of (88):

$$\psi_2 = [(\Gamma/\rho)^2 t/(4\pi\nu t)^2] g_2(\xi) \sin 2\theta = \nu \epsilon^2 g_2(\xi) \sin 2\theta. \tag{96}$$

The quantity ϵ defined by the second expression for ψ_2 is

$$\epsilon = (\Gamma/\rho)/2\pi\nu\sqrt{4\nu t} = (1/4\pi)(\text{Re})^{3/2}. \tag{97}$$

The second expression for ϵ in (97) uses Re defined in (11). The first-order streamfunction (59) has, in terms of ϵ, the form

$$\psi_1 = \nu \epsilon g_1(\xi) \sin \theta, \tag{98}$$

so that ϵ is the small parameter of the asymptotic expansion.

Introduction of (96) and (91) in (88) leads to the following differential equation for g_2:

$$g_2'' + (1/\xi + 2\xi) g_2' + 4(1 - 1/\xi^2) g_2 = -4 q_2. \tag{99}$$

Two homogeneous solutions are ξ^{-2} and a solution that is regular everywhere:

$$g_{2_h} = (1/\xi^2)(1 - e^{-\xi^2}) - e^{-\xi^2}. \tag{100}$$

The solution (100) is recognized as a Stokes quadrupole, which is obtained from the dipole solution by differentiation with respect to x. It is a valid part of the second-order streamfunction, and has an unknown coefficient. Its value is affected by a shift of the origin of the first-order streamfunction in the x direction, as can be seen by the following expansion for a function ψ_1 that is shifted by x_0:

$$\psi_1(x - x_0, y, t) = \psi_1(x, y, t) - x_0 \frac{\partial \psi_1(x, y, t)}{\partial x} \cdots. \tag{101}$$

The second term on the right-hand side is a Stokes quadrupole.

The inhomogeneous solution of (99) is obtained by the following scheme:

$$G_2 = e^{-\xi^2} \int \xi^{-1} e^{\xi^2} q_2 \, d\xi, \quad g_2 = -4\xi^{-2} \int \xi^3 G_2 \, d\xi, \tag{102}$$

where q_2 has to be introduced from (92) with $c_1 = c_2 = 0$,

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$c_0 = -\frac{1}{4}$. As the homogeneous solution exhausts the permissible algebraic behavior of the second-order streamfunction, it is no loss of generality to require that the particular inhomogeneous solution of (99) given by the operations (102) should have exponential decay at infinity. The corresponding function g_2, obtained after lengthy calculation, is

$$\begin{aligned} g_2 = {} & \frac{1}{32\xi^2}(e^{-\xi^2} - e^{-2\xi^2}) + \frac{1}{32}(e^{-\xi^2} - 2e^{-2\xi^2}) \\ & + \frac{1}{4}\xi^2\left(\int_0^\xi (e^{-\xi^2} - e^{-2\xi^2})\frac{d\xi}{\xi} - \frac{1}{2}\log 2\right) \\ & - \frac{1}{8\xi^2}\int_0^\xi (e^{-\xi^2} - e^{-2\xi^2})\frac{d\xi}{\xi} \\ & + \frac{1}{8}\left(1 + \frac{1}{\xi^2}\right)e^{-\xi^2}\int_0^\xi (1 - e^{-\xi^2})\frac{d\xi}{\xi} \\ & + \frac{1}{16}\log 2\left(\frac{1}{\xi^2}(1 - e^{-\xi^2}) - e^{-\xi^2}\right). \end{aligned} \tag{103}$$

This analysis completely bypasses the use of the vorticity, yet the expression for the second-order vorticity might be of some interest. The simplest way to obtain it is by the introduction of the known solution ψ_2 into Eq. (82). The appropriate form for ω_2 expressed with the help of ϵ, Eq. (97), is

$$\omega_2 = (1/t)\epsilon^2 f_2(\xi)\sin 2\theta. \tag{104}$$

This expression and (96) are introduced in (82), and give

$$g_2'' + (1/\xi)g_2' - (4/\xi^2)g_2 = -4f_2. \tag{105}$$

The combination of (105) and (99) gives

$$2\xi g_2' + 4g_2 = 4(f_2 - q_2), \tag{106}$$

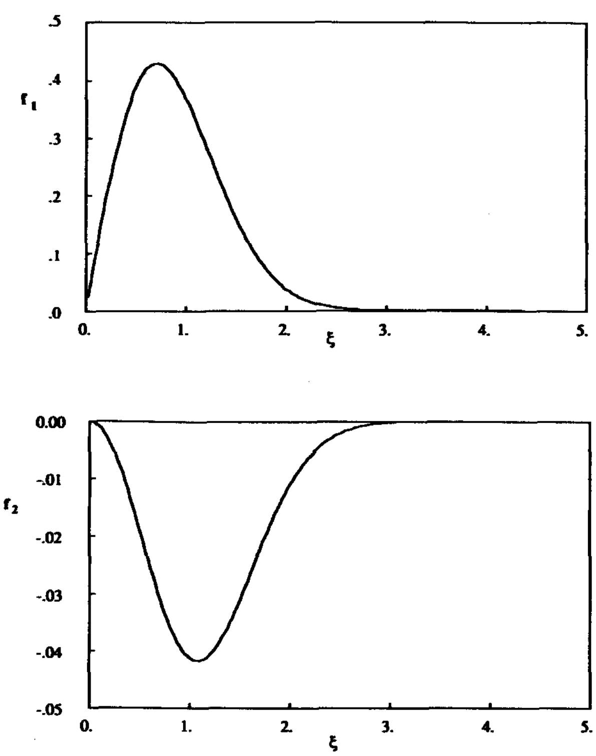

FIG. 10. First- and second-order vorticity functions $f_1(\xi)$ and $f_2(\xi)$.

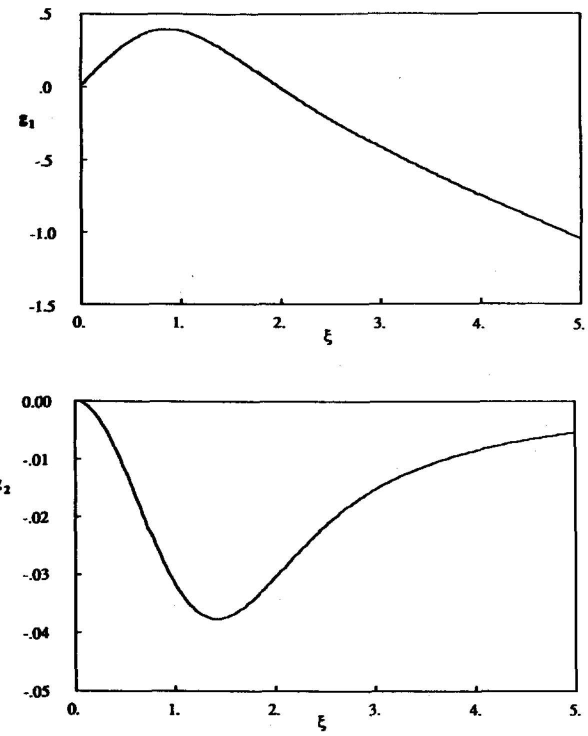

FIG. 11. First- and second-order streamfunctions $g_1(\xi)$ and $g_2(\xi)$.

which is solved for f_2:

$$f_2 = q_2 + \frac{1}{2\xi}\frac{d}{d\xi}(\xi^2 g_2). \tag{107}$$

The result is

$$\begin{aligned} f_2 = {} & -\frac{1}{8}\left(e^{-\xi^2} - \frac{1}{2}e^{-2\xi^2}\right) \\ & + \frac{1}{16\xi^2}(e^{-\xi^2} - e^{-2\xi^2}) \\ & - \frac{1}{8}\xi^2 e^{-\xi^2}\int_0^\xi (1 - e^{-\xi^2})\frac{d\xi}{\xi} \\ & + \left(\frac{3}{32} + \frac{1}{16}\log 2\right)\xi^2 e^{-\xi^2}. \end{aligned} \tag{108}$$

The vorticity functions F_1 and F_2 and the streamfunctions G_1 and G_2 are plotted in Figs. 10 and 11.

VI. OUTLOOK

The next question that arises is whether the same scheme that was followed for the determination of the second-order term permits the continuation of the asymptotic expansion to higher orders. The need of a "switchback" at second order clearly shows that the continuation is not a matter of routine. A sufficient number of degrees of freedom, i.e., undetermined constants, is needed at every level to make the continuation possible. However, not every free constant can be used when the need for a switchback arises. The free constant obtained at second order was interpreted as the indeterminacy of the first-order solution. It can be shown that this constant cannot be used to assure the existence of the third-order solution, and that it remains undetermined even

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in the hypothetical case that the third-order solution exists.

That the need for switchback continues can be seen from the discussion of the Poisson equation for B, Eq. (86). The inhomogeneous term in this equation vanishes exponentially at infinity, and it can be easily verified that its integral over the whole flow plane is always zero. Thus a solution of the Poisson equation exists; to find it, a harmonic analysis of the inhomogeneous term with respect to the angular coordinate θ has to be performed first. To each harmonic component belongs a term in B that decays algebraically at infinity. According to Eq. (95), a corresponding term appears in the streamfunction; it is found by integration with respect to time and thus decays slower with time than the corresponding term in B. The critical question is whether this time behavior is admissible. This was the difficulty encountered at second order, and a more detailed discussion shows that the same problem will arise at all orders. Actually it turns out that for the odd and even orders of B all odd and even harmonics occur in the inhomogeneous term up to the order of B. Thus, at third order, a new type of switchback is found: a solution appears that is proportional to the first harmonic, i.e., to the dipole, unless its coefficient happens to be zero or is canceled by a free constant that is disposable. Certainly no time-dependent dipole component is admissible, as the dipole moment is proportional to the impulse.

In an asymptotic analysis, it is not sufficient to show the existence of higher-order solutions; the possibility is needed to accommodate more and more information from earlier times. The only information used in the asymptotic analysis presented here is the fact that the impulse is an invariant. Thus the flow might have been caused not only by a vortex pair, but also by any superposition of vortex pairs that were produced before a certain time $t = 0$. An attempt to reach this multiplicity of possible solutions going backward in time is certainly hopeless.

What has been shown in this paper is that for all flows that are force-free for $t > 0$, the leading-order asymptotic solution involves the drift, and that a second-order solution exists, which already contains a certain indeterminacy connected with the positioning of the origin of the asymptotic first-order solution. This information is particularly useful as a final check on a numerical analysis that was performed progressing forward in time.

Work applying the same kind of analysis to the vortex ring is in progress. Both Kovasznay and Lee[9] and Kambe and Oshima[2] anticipated the asymptotic drift of vortex rings. Recent numerical calculations by Stanaway, Cantwell, and Spalart[8] show that the large time solution approaches the drifting Stokes dipole. The computed drift velocity is unique for a variety of initial conditions and is equal to 0.003 7038 $\times (I/\rho)/(\nu t)^{3/2}$. Using the expansion procedure outlined in this paper, the asymptotic drift of a vortex ring was determined to be $(7/240\pi^{3/2}2^{1/2})(I/\rho)/(\nu t)^{3/2}$, which agrees to five significant figures with the computed value. This excellent agreement strengthens our faith in both the computation and the analysis.

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