J. Fluid Mech. (2000), *vol.* 423, *pp.* 293–315. Printed in the United Kingdom 293

Self-similar, slightly compressible, free vortices

By KARL D. VON ELLENRIEDER AND BRIAN J. CANTWELL

Department of Aeronautics and Astronautics, Stanford University, Stanford, CA 94305, USA

(Received 7 July 1998 and in revised form 24 January 2000)

Exact and numerical similarity solutions for compressible perturbations to an incompressible, two-dimensional, axisymmetric vortex reference flow are presented. The reference flow consists of a set of two-dimensional, self-similar, incompressible vortices. Similarity variables, which give explicit expressions for the decay rates of the velocities and thermodynamic variables in the vortex flows, are used to reduce the governing partial differential equations to a set of ordinary differential equations. The ODEs are solved analytically and numerically for a Taylor vortex reference flow, and numerically for an Oseen vortex reference flow. The solutions are employed to study the dependences of the temperature, density, entropy, dissipation and radial velocity on the Prandtl number. Additionally, several integral relations, which allow one to trace the energy transfer in a slightly compressible vortex, are derived.

1. Introduction

Compressible vortices are a significant feature of many fluid flows of physical and technological interest: turbulent combustion in engines, strong wing-tip vortices, the generation of aeroacoustic noise by jet aircraft, shock diffraction around sharp corners, and astrophysical flows such as solar coronae. Compressible vortex models have become popular for testing non-reflecting boundary conditions and as a basic component of numerical simulations for studying shock/turbulence interaction. Yet, despite the frequent occurrence of compressible vortex flows in the literature, the number of analytical descriptions of this type of flow is quite small (Colonius, Lele & Moin 1991; Mandella 1987). In the light of this, we will examine analytical self-similar solutions for slightly compressible vortices. Since, in general, little is known about the physical nature of compressible vortices, the solutions are examined in detail with a particular emphasis placed on understanding the dissipation and energy transport occurring in the vortices and also the role of the Prandtl number.

Herein we concentrate on axisymmetric, compressible, viscous, free vortices with no axial flow and with constant far-field flow conditions. The fluid under consideration is taken to be an ideal gas which is thermally and calorically perfect. With these assumptions the motion of a fully compressible vortex is governed by the two-dimensional, cylindrically symmetrical, Navier–Stokes equations – which we will refer to as the *full equations*.

The dearth of analytical results for compressible free vortices is partly due to their complexity. In contrast, the two-dimensional motion of an isolated free vortex in an incompressible flow is well understood. Conservation of mass requires that there is no radial velocity in the vortex, and the Reynolds number Re is the only quantity which governs the evolution of the vortex. For very large Reynolds numbers, $Re \to \infty$, the flow is essentially inviscid, and the vortex motion is nearly steady. Otherwise, for $Re \sim O(1)$ or smaller, the viscous diffusion of angular momentum dominates the motion.

When the compressibility is important, additional parameters enter the problem. The flow has an analogous dependence on the Reynolds number, but one must also include the effects of heat conduction, viscous dissipation, compressibility, and radial convection on the motion and structure of the vortex (Bershader 1995). Compressible vortex flow requires that four other parameters, in addition to Re, be considered: the Mach number M, the Prandtl number Pr, as well as the temperature-dependent values of viscosity μ and thermal conductivity κ.

Since the full equations are nonlinear and there are a large number of parameters involved, the equations are difficult to solve analytically. In general, analytical solutions require substantial simplification of the boundary conditions, the initial conditions, and the form of the equations. For example, Chiocchia (1989) and Ardalan, Meiron & Pullin (1995) assumed steady, inviscid flow and applied hodograph transformations to solve ideal-fluid, compressible vortex problems. Earlier, Mack (1960) and Bellamy-Knights (1980) investigated steady, viscous, compressible vortex flows. In both of these works, the boundary conditions at the centre of the vortex were simplified by driving the flow with a rotating circular cylinder. Colonius *et al.* (1991) were able to find analytical solutions for compressible free vortices by examining the full equations in the limit of high Reynolds number and low Mach number. Two different non-self-similar initial conditions were used for the calculations. In the first case the vortex initially had a constant density; in the second case the vortex was initially homentropic.

Among techniques for finding analytical solutions to compressible vortex flow, the search for symmetry under a Lie group is a natural approach. But, since the full equations are not invariant under translations of the thermodynamic variables (von Ellenrieder 1998) they will not admit similarity solutions for compressible free vortex flow. This is one of the main reasons why analytical solutions for fully compressible, viscous, free vortices do not exist. However, as explained in § 2.3, the perturbation equations and the incompressible reference flows presented by Colonius *et al.* (1991) are invariant under a three-parameter Lie group. The group allows a rich set of similarity solutions to be found for a slightly compressible vortex.

The similarity equations and variables are developed in the following section. In § 3 solutions for an Oseen vortex (Oseen 1912) reference flow and a Taylor vortex (Taylor 1918) reference flow are given. Integrals which relate the dissipation and pressure work to the kinetic and internal energy in a slightly compressible vortex flow are derived. A summary of the key findings is given in § 4.

2. Governing equations

2.1. *Non-dimensional forms and perturbation expansions*

Our point of departure is the set of perturbation equations formulated by Colonius *et al.* (1991). These equations are derived with the assumptions that (i) there are no acoustic waves present in the flow, (ii) the convective and diffusive timescales are the same, (iii) the far-field flow conditions are constant, and (iv) μ and κ are constant.

The tangential and radial velocities (v and u, respectively) and the thermodynamic variables (pressure p, density ρ, and temperature T) are scaled as

$$\left.\begin{aligned} \tilde{r} = \frac{r}{l_i}, \quad \tilde{t} = \frac{v_m t}{l_i}, \quad \tilde{v} = \frac{v}{v_m}, \\ \tilde{u} = \frac{u}{v_m}, \quad \tilde{\rho} = \frac{\rho}{\rho_\infty}, \quad \tilde{p} = \frac{p - p_\infty}{\rho_\infty v_m^2}, \\ \tilde{T} = \frac{T}{T_\infty}, \quad \tilde{Re} = \frac{\rho_\infty v_m l_i}{\mu}, \quad Pr = \frac{\mu C_p}{\kappa}, \\ M = \frac{v_m}{a_\infty}, \quad \tau = \frac{\tilde{t}}{\tilde{Re}}. \end{aligned}\right\} \tag{2.1}$$

In (2.1) v_m is a reference velocity (the maximum tangential velocity in the vortex's initial velocity profile) and l_i is the radial location of v_m. Far-field flow quantities are denoted with the subscript ∞, r is the radial coordinate, and t is time. The term $\tilde{p}$ is similar to the standard definition of the dynamic pressure coefficient in aerodynamics and represents the normalized deviation of the local pressure p from the far-field pressure p_∞. The specific heat at constant pressure is denoted as C_p (assumed constant) and a is the speed of sound.

Each of the dependent flow variables is approximated as

$$\tilde{f} = \tilde{f}_0 + M^2 \tilde{f}_1 + O(M^4), \tag{2.2}$$

where $\tilde{f}$ represents any dependent flow variable. Terms of $O(1)$ are designated with a subscript 0 and belong to the incompressible reference flow. $O(M^2)$ terms are denoted by the subscript 1 and are the compressible perturbations to the reference flow.

2.1.1. *The simplified equations*

In the reference flow the radial velocity is necessarily zero, $\tilde{u}_0 = 0$, the density and temperature are uniform, $\tilde{\rho}_0 = \tilde{T}_0 = 1$, the pressure gradient balances the centrifugal forces created by the tangential velocity, and the tangential velocity is governed by a diffusion equation

$$\frac{\partial \tilde{p}_0}{\partial \tilde{r}} = \frac{\tilde{v}_0^2}{\tilde{r}}, \tag{2.3a}$$

$$\frac{\partial \tilde{v}_0}{\partial \tau} = \frac{\partial}{\partial \tilde{r}} \left[\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} (\tilde{r} \tilde{v}_0) \right]. \tag{2.3b}$$

The corresponding boundary conditions are

$$\tilde{r} = 0\colon \ \tilde{v}_0 = 0, \quad \frac{\partial \tilde{p}_0}{\partial \tilde{r}} = 0, \tag{2.4a}$$

and

$$\tilde{r} \to \infty\colon \ \tilde{v}_0 \to 0, \quad \tilde{p}_0 \to 0. \tag{2.4b}$$

The incompressible vorticity

$$\tilde{\omega}_0 = \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} (\tilde{r} \tilde{v}_0) \tag{2.5}$$

is governed by the equation

$$\frac{\partial \tilde{\omega}_0}{\partial \tau} = \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \frac{\partial \tilde{\omega}_0}{\partial \tilde{r}} \right), \tag{2.6}$$

where it is assumed that $\tilde{\omega}_0$ is bounded at $\tilde{r} = 0$, and $\tilde{\omega}_0 \to 0$ as $\tilde{r} \to \infty$.

To $O(M^2)$ the equations for conservation of mass, radial momentum, tangential momentum, energy, and the equation of state, respectively, are

$$\frac{\partial \tilde{\rho}_1}{\partial \tau} + \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{u}_1^*) = 0, \tag{2.7a}$$

$$\frac{\partial \tilde{p}_1}{\partial \tilde{r}} = \frac{\tilde{\rho}_1 \tilde{v}_0^2}{\tilde{r}} + \frac{2\tilde{v}_1\tilde{v}_0}{\tilde{r}}, \tag{2.7b}$$

$$\tilde{\rho}_1\frac{\partial \tilde{v}_0}{\partial \tau} + \frac{\partial \tilde{v}_1}{\partial \tau} + \tilde{u}_1^*\frac{\partial \tilde{v}_0}{\partial \tilde{r}} + \frac{\tilde{v}_0\tilde{u}_1^*}{\tilde{r}} = \frac{\partial}{\partial \tilde{r}}\left[\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{v}_1)\right], \tag{2.7c}$$

$$\frac{\partial \tilde{T}_1}{\partial \tau} + \frac{(\gamma-1)}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{u}_1^*) = \gamma(\gamma-1)\left(\frac{\partial \tilde{v}_0}{\partial \tilde{r}} - \frac{\tilde{v}_0}{\tilde{r}}\right)^2 + \frac{\gamma}{Pr}\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{T}_1}{\partial \tilde{r}}\right), \tag{2.7d}$$

$$\gamma\tilde{p}_0 = \tilde{\rho}_1 + \tilde{T}_1, \tag{2.8}$$

where $\tilde{u}_1^* = \tilde{u}_1\tilde{Re}$. Equation (2.8) implies that, for a slightly compressible vortex, the $O(M^2)$ density and temperature perturbations are of the same order of magnitude as the $O(1)$ pressure variation.

The $O(1)$ entropy variation of the reference flow is $\tilde{s}_0 = 0$. The $O(M^2)$ compressible entropy perturbation, local rate of entropy change, and vorticity are governed by

$$\tilde{s}_1 = \tilde{p}_0 - \tilde{\rho}_1, \tag{2.9a}$$

$$\frac{\partial \tilde{s}_1}{\partial \tau} = (\gamma-1)\left(\frac{\partial \tilde{v}_0}{\partial \tilde{r}} - \frac{\tilde{v}_0}{\tilde{r}}\right)^2 + \frac{1}{\tilde{r}Pr}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{T}_1}{\partial \tilde{r}}\right), \tag{2.9b}$$

and

$$\frac{\partial \tilde{\omega}_1}{\partial \tau} - \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{\omega}_1}{\partial \tilde{r}}\right) = \frac{\partial \tilde{p}_1}{\partial \tau}\tilde{\omega}_0 - \left(\frac{\partial \tilde{p}_1}{\partial \tilde{r}} + \tilde{u}_1^*\right)\frac{\partial \tilde{\omega}_0}{\partial \tilde{r}} - \tilde{\rho}_1\frac{\partial \tilde{\omega}_0}{\partial \tau}, \tag{2.9c}$$

respectively.

Using (2.7*a,d*), and (2.8) we can derive a partial differential equation for the evolution of $\tilde{T}_1$ in terms of the reference flow velocity $\tilde{v}_0$ and pressure $\tilde{p}_0$:

$$\frac{\partial \tilde{T}_1}{\partial \tau} - \frac{1}{\tilde{r}Pr}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{T}_1}{\partial \tilde{r}}\right) = (\gamma-1)\left[\left(\frac{\partial \tilde{v}_0}{\partial \tilde{r}} - \frac{\tilde{v}_0}{\tilde{r}}\right)^2 + \frac{\partial \tilde{p}_0}{\partial \tau}\right]. \tag{2.10}$$

Here, the equation set (2.7*a*–*c*), (2.8), and (2.10) will be used to describe the evolution of the $O(M^2)$ quantities; the related boundary conditions are

$$\tilde{r} = 0: \quad \tilde{u}_1 = 0, \quad \tilde{v}_1 = 0, \quad \frac{\partial \tilde{p}_1}{\partial \tilde{r}} = 0, \quad \frac{\partial \tilde{T}_1}{\partial \tilde{r}} = 0, \quad \frac{\partial \tilde{\rho}_1}{\partial \tilde{r}} = 0, \tag{2.11a}$$

$$\tilde{r} \to \infty: \quad \tilde{u}_1 \to 0, \quad \tilde{v}_1 \to 0, \quad \tilde{p}_1 \to 0, \quad \tilde{T}_1 \to 0, \quad \tilde{\rho}_1 \to 0. \tag{2.11b}$$

After the reference flow variables $\tilde{p}_0$ and $\tilde{v}_0$ are determined, one can solve for the $O(M^2)$ perturbation quantities.

2.2. *Reference flow solutions*

Following Colonius *et al.* (1991) the reference flow is given by de Neufville's (1957) incompressible vortex solutions. The reference flow vorticity is

$$\tilde{\omega}_0 = \frac{C_m e^{-\eta} L_m(\eta)}{(\tau + \tau_i)^{(m+1)}}. \tag{2.12}$$

Here, the parameter m is the eigenvalue of (2.6) when $\tilde{\omega}_0$ is separated in the variables τ and

$$\eta = \frac{\tilde{r}^2}{4(\tau + \tau_i)}. \tag{2.13}$$

The quantity $C_m/\tau_i^{(m+1)}$ is the amplitude (at $\tilde{r} = 0$) of the mth self-similar vorticity solution (when $\tau = 0$) and L_m is the mth-order Laguerre polynomial. The corresponding velocity and pressure are

$$\tilde{v}_0 = \begin{cases} \dfrac{2C_0}{\tilde{r}}[1 - \exp(-\eta)], & m = 0 \\ -\dfrac{2C_m}{\tilde{r}(\tau + \tau_i)^m} \exp(-\eta)[\mathrm{L}_m(\eta) - \mathrm{L}_{m-1}(\eta)], & m \geqslant 1, \end{cases} \tag{2.14}$$

and

$$\tilde{p}_0 = -\int_{\tilde{r}}^{\infty} \frac{\tilde{v}_0^2}{\tilde{r}} \mathrm{d}\tilde{r} = \frac{-C_m^2}{2(\tau + \tau_i)^{(2m+1)}} \int_{\eta}^{\infty} \frac{\exp(-2\eta)[\mathrm{L}_m(\eta) - \mathrm{L}_{m-1}(\eta)]^2}{\eta^2} \mathrm{d}\eta, \tag{2.15}$$

respectively.

The radial distributions of vorticity, tangential velocity, and pressure are plotted in figure 1(*a–c*) for the first three eigenvalues m, with $C_m = 1$, $\tau = 0$, and $\tau_i = 1$. Notice that $\tilde{\omega}_0 = 1$ at $\tilde{r} = 0$ for all three solutions shown. However, as m increases both the corresponding maximum tangential velocity and maximum pressure drop decrease. The reference flow solution for $m = 0$ is the classical Oseen vortex (Oseen 1912) and the solution for $m = 1$ is Taylor's vortex (Taylor 1918).

A general expression for the conserved quantity corresponding to each reference flow solution is given by

$$A_m = 2\pi \int_0^{\infty} \tilde{r}^{2m} \tilde{\omega}_0 \tilde{r} \mathrm{d}\tilde{r} \tag{2.16}$$

(von Ellenrieder 1998). Each A_m is the $2m$th area moment of the mth vorticity solution. When $m = 0$ the total circulation A_0 of the fluid is invariant. For $m = 1$ the total angular momentum of the flow $\mathscr{M}$ is constant and $A_1 = -2\mathscr{M}$. Evaluating (2.16) we find that C_m is related to the integral invariant A_m:

$$C_m = \frac{A_m}{(-1)^m 4^{(m+1)}(m!)\pi}. \tag{2.17}$$

2.3. *Similarity forms of perturbation terms and equations*

Both the incompressible reference flow equations and the compressible perturbation equations are invariant under a three-parameter Lie group. The group consists of stretching transformations in the radial coordinate and each dependent flow variable, as well as a translation in time. Similarity forms can be found for each dependent flow variable and explicitly yield the decay rate of each flow quantity; the mathematical details are given in von Ellenrieder(1998).

The similarity variables corresponding to the $O(M^2)$ perturbed quantities are

$$\sigma(\eta) = \tilde{T}_1(\tau + \tau_i)^{(2m+1)}, \quad \beta(\eta) = \tilde{\rho}_1(\tau + \tau_i)^{(2m+1)}, \tag{2.18a,b}$$

$$\alpha(\eta) = \tilde{u}_1^*(\tau + \tau_i)^{(2m+3/2)}, \quad \phi(\eta) = \tilde{v}_1(\tau + \tau_i)^{(3m+3/2)}, \tag{2.18c,d}$$

$$\psi(\eta) = \tilde{p}_1(\tau + \tau_i)^{(4m+2)}, \quad \zeta(\eta) = \tilde{s}_1(\tau + \tau_i)^{(2m+1)}, \tag{2.18e,f}$$

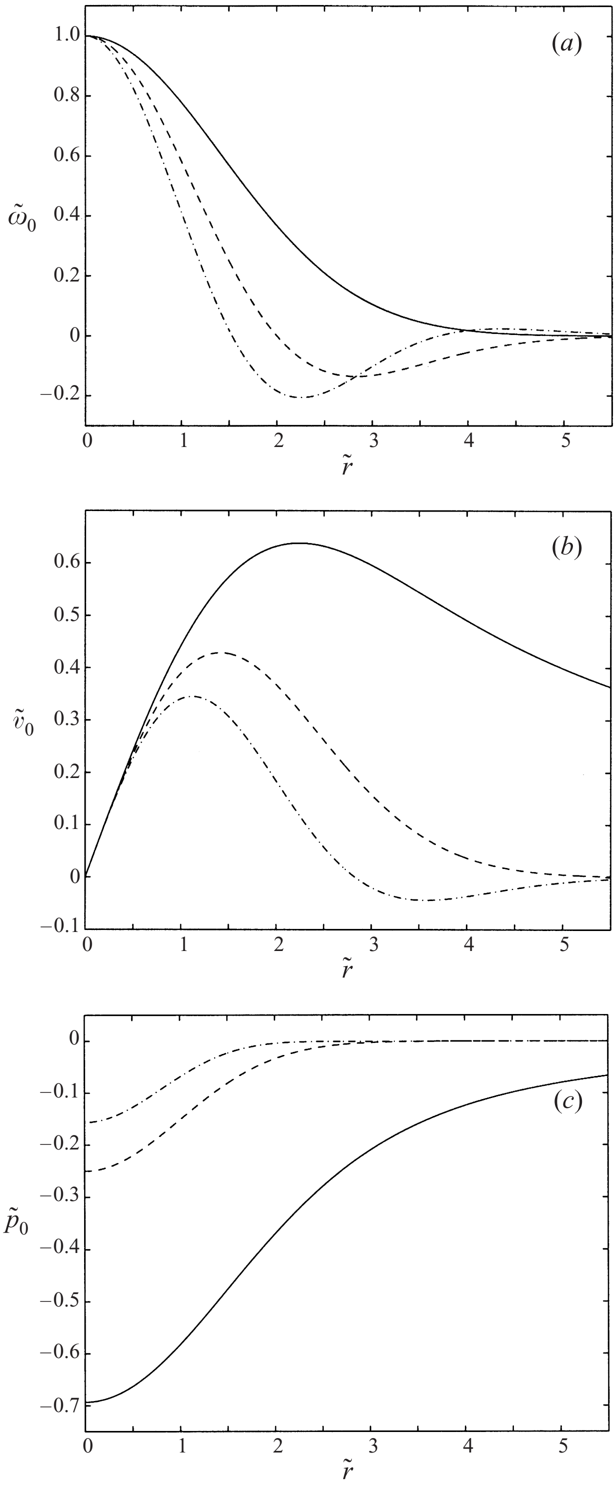

FIGURE 1. Radial distributions of (*a*) vorticity $\tilde{\omega}_0$, (*b*) tangential velocity $\tilde{v}_0$, and (*c*) pressure $\tilde{p}_0$, for $C_m = 1, \tau = 0$, and $\tau_i = 1$: ——, $m = 0$; – – – –, $m = 1$; —·—, $m = 2$.

$$\delta(\eta) = \frac{\partial \tilde{s}_1}{\partial \tau}(\tau+\tau_i)^{2(m+1)}, \quad \varpi(\eta) = \tilde{\omega}_1(\tau+\tau_i)^{(3m+2)}. \tag{2.18g, h}$$

Note that the linearity of (2.7*a*–*c*), (2.8), and (2.10) allows one to superpose $O(M^2)$ self-similar solutions, which are found using (2.18), to construct any $O(M^2)$ initial condition. But, since the decay rate of each flow variable is dependent upon the eigenvalue *m*, the motion of a decaying vortex which is not initially self-similar will asymptotically approach the self-similar solution that corresponds to the lowest eigenvalue.

2.3.1. *Perturbation equations*

Substituting the similarity forms of the perturbation variables (2.18*a*–*e*) into equations (2.7*a*–*c*), (2.8), and (2.10), the partial differential equations describing the $O(M^2)$ perturbations are reduced to the following set of ordinary differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}(\alpha\eta^{1/2}) = \eta\frac{\mathrm{d}\beta}{\mathrm{d}\eta} + (2m+1)\beta, \tag{2.19a}$$

$$2\eta\frac{\mathrm{d}\psi}{\mathrm{d}\eta} = \beta V^2 + 2\phi V, \tag{2.19b}$$

$$\eta\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2} + (\eta+1)\frac{\mathrm{d}\phi}{\mathrm{d}\eta} + \left[3(m+\tfrac{1}{2}) - \frac{1}{4\eta}\right]\phi = (\alpha - \beta\eta^{1/2})\Omega - m\beta V, \tag{2.19c}$$

$$\beta = \gamma\mathscr{P} - \sigma, \tag{2.19d}$$

$$\frac{\eta}{Pr}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\eta^2} + \frac{(\eta Pr+1)}{Pr}\frac{\mathrm{d}\sigma}{\mathrm{d}\eta} + (2m+1)\sigma = (\gamma-1)\left[(2m+1)\mathscr{P} + \frac{V^2}{2} - \left(\Omega - \frac{V}{\eta^{1/2}}\right)^2\right]. \tag{2.19e}$$

Here V and $\mathscr{P}$ are the similarity forms of the incompressible velocity and pressure:

$$V(\eta) = \tilde{v}_0(\tau+\tau_i)^{(m+1/2)}, \quad \mathscr{P}(\eta) = \tilde{p}_0(\tau+\tau_i)^{(2m+1)}. \tag{2.20a, b}$$

Transforming (2.11*a*) and (2.11*b*), gives the boundary conditions for (2.19*a*–*e*):

$$\eta = 0: \quad \alpha = 0, \quad \phi = 0, \quad \eta^{1/2}\frac{\mathrm{d}\psi}{\mathrm{d}\eta} = 0, \quad \eta^{1/2}\frac{\mathrm{d}\sigma}{\mathrm{d}\eta} = 0, \quad \eta^{1/2}\frac{\mathrm{d}\beta}{\mathrm{d}\eta} = 0, \tag{2.21a}$$

and

$$\eta \to \infty: \quad \alpha \to 0, \quad \phi \to 0, \quad \psi \to 0, \quad \sigma \to 0, \quad \beta \to 0. \tag{2.21b}$$

The self-similar forms of the equations for the $O(M^2)$ entropy, dissipation, and vorticity, (2.9*a*–*c*) respectively, are

$$\zeta = -(\gamma-1)\mathscr{P} + \sigma, \tag{2.22a}$$

$$\delta = (\gamma-1)\left(\Omega - \frac{V}{\eta^{1/2}}\right)^2 + \frac{1}{Pr}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}\right), \tag{2.22b}$$

and

$$\eta\frac{\mathrm{d}^2\varpi}{\mathrm{d}\eta^2} + (\eta+1)\frac{\mathrm{d}\varpi}{\mathrm{d}\eta} + (3m+2)\varpi = m\beta\Omega + \eta\frac{\mathrm{d}\beta}{\mathrm{d}\eta}\left(\Omega + \frac{\mathrm{d}\Omega}{\mathrm{d}\eta}\right) + \eta^{1/2}(\alpha - \eta^{1/2}\beta)\frac{\mathrm{d}\Omega}{\mathrm{d}\eta}. \tag{2.22c}$$

The solutions for the homogeneous parts of (2.19*e*) and (2.22*c*) are

$$\sigma_h = B_m \exp(-\eta_1) \mathrm{L}_{2m}(\eta_1) \tag{2.23}$$

and

$$\varpi_h = W_m \exp(-\eta) \mathrm{L}_{3m+1}(\eta), \tag{2.24}$$

where $\eta_1 = \eta Pr$, B_m and W_m are arbitrary constants, and the subscript h refers to the homogeneous part of each solution. The homogeneous part of the equation for the $O(M^2)$ tangential velocity (2.7*c*) and the $O(1)$ tangential velocity equation (2.3*b*) have the same form. Using (2.3*b*), (2.20), and (2.19*c*) one can show that the forms of ϕ_h and V will be somewhat alike:

$$\phi_h = -\frac{(3m+1)Q_m \exp(-\eta)}{\eta^{1/2}} [\mathrm{L}_{3m+1}(\eta) - \mathrm{L}_{3m}(\eta)], \tag{2.25}$$

where Q_m is an arbitrary constant.

In the next section (2.19)–(2.25) are used to find self-similar solutions for the $O(M^2)$ compressible perturbations to both an incompressible Oseen vortex and an incompressible Taylor vortex.

3. Solutions

3.1. *Analytical solutions for the slightly compressible Taylor vortex*

For $m = 1$, (2.12), (2.14), and (2.15), can be used to reduce (2.19*e*) to the following relation:

$$\eta_1 \frac{\mathrm{d}^2\sigma}{\mathrm{d}\eta_1^2} + (\eta_1 + 1)\frac{\mathrm{d}\sigma}{\mathrm{d}\eta_1} + 3\sigma = -C_1^2(\gamma - 1)\exp(-2\eta)\left[\eta^2 - \frac{\eta}{2} + \frac{3}{4}\right], \tag{3.1}$$

where, again, we take $\eta_1 = \eta Pr$. Analytical solutions to (3.1) for a general value of Pr can be found using the Method of Frobenius. The particular solution is

$$\sigma_p = -C_1^2(\gamma - 1)\eta_1 \exp(-2\eta_1/Pr)\sum_{n=0}^{\infty} D_n(2\eta_1)^n, \tag{3.2}$$

where

$$\left.\begin{aligned}
D_0 &= \frac{3}{4}, \\
D_1 &= \left(\frac{1}{2Pr} - \frac{3}{8}\right), \\
D_2 &= \left(\frac{15Pr^2 - 44Pr + 32}{144Pr^2}\right), \\
&\vdots \\
D_n &= -\frac{(4-2Pr)D_{n-2} + [(2n+6)Pr^2 - 4Pr(2n+1)]D_{n-1}}{4(n+1)^2Pr^2}.
\end{aligned}\right\} \tag{3.3}$$

Using (2.23) we find that the homogeneous part of the solution for $m = 1$ is

$$\sigma_h = \frac{B_1}{2}\exp(-\eta_1)(\eta_1^2 - 4\eta_1 + 2), \tag{3.4}$$

and the complete solution is $\sigma = \sigma_h + \sigma_p$. Using (2.15), (2.19*d*), (2.22*a*), and (3.2)–(3.4) β and ζ are easily found. However, solving equations (2.19*a–c*) for α, ψ, and ϕ is difficult because (3.2) contains a slowly converging series.

Luckily, when $Pr = 1$, it is relatively easy to find simple closed-form solutions for most of the $O(M^2)$ terms. In this case $\eta = \eta_1$, and the particular solution to (3.1) is

$$\sigma_p = -\frac{C_1^2(\gamma-1)}{4}\exp(-2\eta)(2\eta+1). \tag{3.5}$$

The total solution for σ is given by

$$\sigma = -\frac{C_1^2(\gamma-1)}{4}\exp(-2\eta)(2\eta+1) + \frac{B_1}{2}\exp(-\eta)(\eta^2-4\eta+2). \tag{3.6}$$

Using this result and (2.15) in (2.19*d*) and (2.22*a*) gives the solutions for β and ζ:

$$\beta = \frac{C_1^2}{4}\exp(-2\eta)[2\eta(\gamma-1)-1] - \frac{B_1}{2}\exp(-\eta)(\eta^2-4\eta+2), \tag{3.7}$$

and

$$\zeta = -\frac{C_1^2(\gamma-1)}{2}\eta\exp(-2\eta) + \frac{B_1}{2}\exp(-\eta)(\eta^2-4\eta+2). \tag{3.8}$$

Integrating (2.19*a*), we find the similarity parameter corresponding to the $O(M^2)$ radial velocity

$$\alpha = \frac{C_1^2\exp(-2\eta)}{4\eta^{1/2}}\left[(2-\gamma)+\eta(1-2\gamma)+2\eta^2(\gamma-1)\right] - \frac{B_1\eta^{1/2}\exp(-\eta)}{2}[\eta^2-6\eta+6] + \frac{G_1}{\eta^{1/2}}. \tag{3.9}$$

The homogeneous solution for ϕ, which can be found using (2.25), is

$$\phi_h = -\frac{Q_1\eta^{1/2}\exp(-\eta)}{6}(\eta^3-12\eta^2+36\eta-24), \tag{3.10}$$

where Q_1 is a constant.

3.1.1. *Initial/boundary conditions*

The constants C_1 and τ_i are determined so that $\tilde{v}_0 = 1$, at $\tilde{r} = 1$, when $\tau = 0$:

$$\tau_i = \tfrac{1}{2}, \qquad C_1 = \mathrm{e}^{1/2}/2.$$

The radial velocity in the centre of the vortex must be zero from (2.21*a*), so

$$G_1 = -\frac{C_1^2(2-\gamma)}{4}.$$

For this value of G_1, (3.9) shows that far from the vortex core, as $\eta \to \infty$, the radial velocity is

$$\tilde{u}_1^* \sim \frac{G_1}{\eta^{1/2}(\tau+\tau_i)^{7/2}} = -\frac{C_1^2(2-\gamma)}{2\tilde{r}(\tau+\tau_i)^3}. \tag{3.11}$$

This result matches the prediction for the far-field radial velocity in the Taylor vortex made by Colonius *et al.* (1991).

The solutions for σ, β, and ζ (3.6)–(3.8) contain polynomials in η. All powers of η in these polynomials are greater than zero. For this reason the boundary conditions (2.21*a*) are trivially satisfied for any value of B_1. Here we will determine B_1 so that

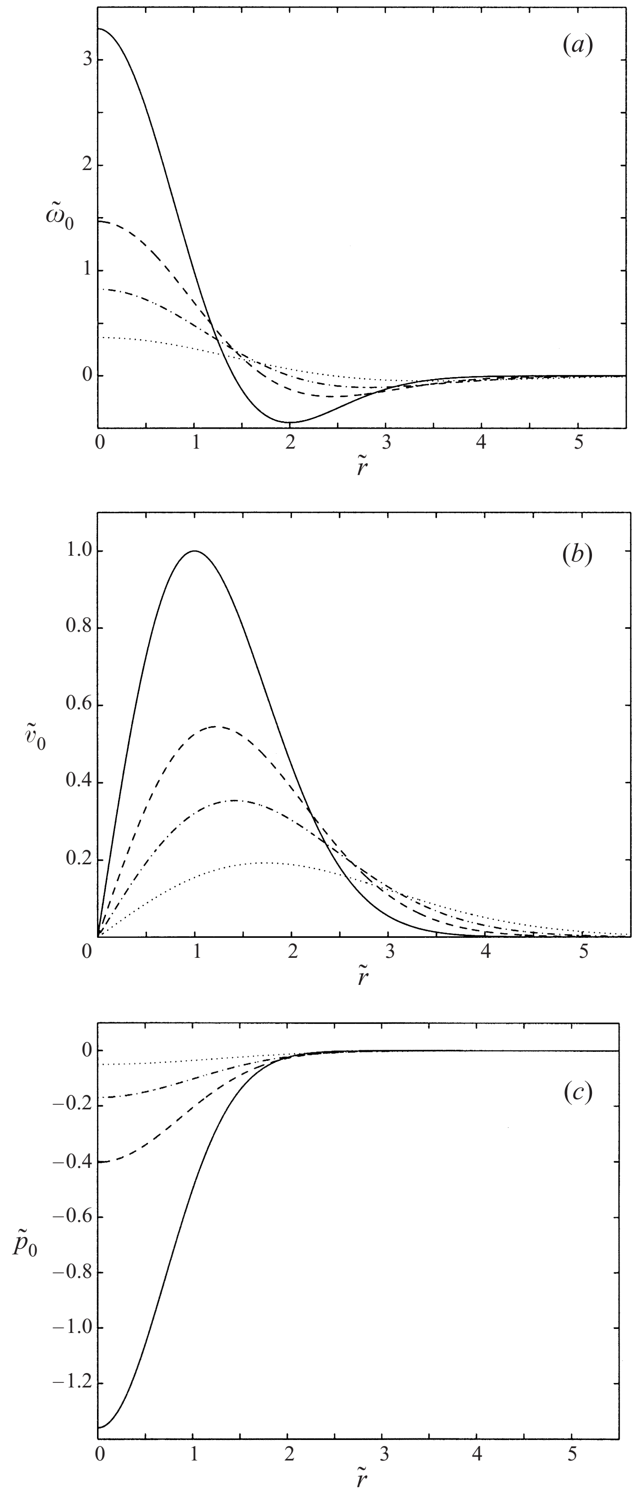

FIGURE 2 (*a*–*c*). For caption see page 304.

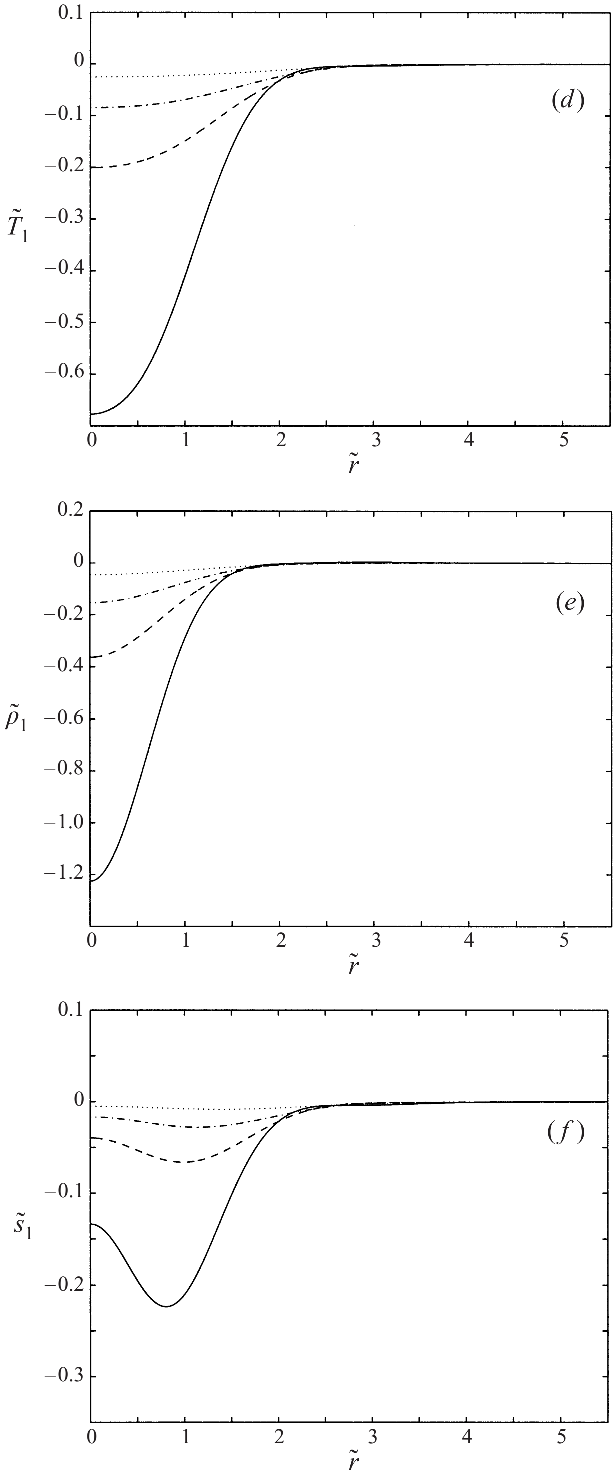

FIGURE 2 (*d–f*). For caption see page 304.

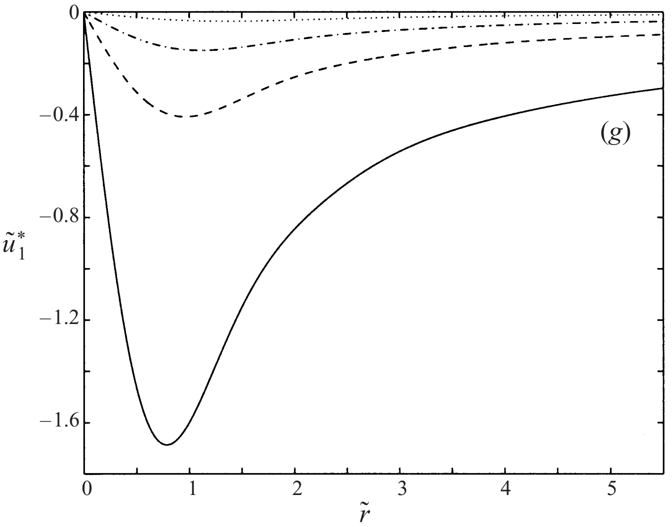

FIGURE 2. Exact solutions for a slightly compressible Taylor vortex with $\tau_i = \frac{1}{2}$, $Pr = 1$, $\gamma = 1.4$, $C_1 = e^{1/2}/2$, $B_{1e} = -1.67 \times 10^{-2}$. Results for four different times are shown: ——, $\tau = 0$; – – – –, $\tau = \frac{1}{4}$; —•—, $\tau = \frac{1}{2}$, and ······, $\tau = 1$. Reference flow quantities are plotted in (*a*) vorticity, (*b*) tangential velocity, and (*c*) pressure. The compressible perturbations are shown in (*d*) temperature, (*e*) density, (*f*) entropy, and (*g*) radial velocity.

$\tilde{\rho}_1(\tilde{r} = 0, \tau = 0)$ is close to the value found in the experiments of Mandella (1987). The depth of the vortex density well is defined as

$$\frac{\rho(\tilde{r} = 0, \tau) - \rho_\infty}{\rho_\infty} = M^2 \tilde{\rho}_1(\tilde{r} = 0, \tau). \tag{3.12}$$

If, at $\tau = 0$, the depth of the density well is 0.55, and the Mach number of the vortex is $M = 0.67$, then $\tilde{\rho}_1(\tilde{r} = 0, \tau = 0) = -1.23$. The forms of (3.2) and (3.5) differ, so in order for the series and exact solutions for the $O(M^2)$ density, temperature, and entropy perturbations to match, (2.19*d*), (3.2), and (3.5) require that B_1 for the series solution (denoted B_{1s}), and B_1 for the exact solution (denoted B_{1e}) are related in the following way:

$$B_{1s} = B_{1e} - \frac{C_1^2(\gamma - 1)}{4}. \tag{3.13}$$

Using an initial density-well depth of 0.55, $\gamma = 1.4$, and $B_{1s} = -8.47 \times 10^{-2}$, gives $B_{1e} = -1.67 \times 10^{-2}$. The exact solutions for both the reference flow and the compressible perturbations are plotted in figure 2. Also, the series solutions with $Pr = 0.5, 0.72$, and 1.0 are given in figure 3.

The specification of the initial density-well depth is a departure from the boundary conditions in (2.21*a*). Rather than only requiring that the gradient of the density is zero at $\tilde{r} = 0$, we are also specifying an initial value for the density at the origin. The boundary conditions in (2.21*a*) are invariant under the similarity transformation group, but the initial density-well depth condition for $\tilde{\rho}_1$ is not. The resulting shape of the density distribution is dependent on the initial value chosen for $\Delta\rho^*$. Since the coefficient B_1 is determined by this initial $\Delta\rho^*$ the distributions of the self-similar solutions are dependent on the value of B_1 and can differ substantially as B_1 is varied. When B_1 is increased the temperature $\tilde{T}_1$ at $\tilde{r} = 0$ increases and can become positive;

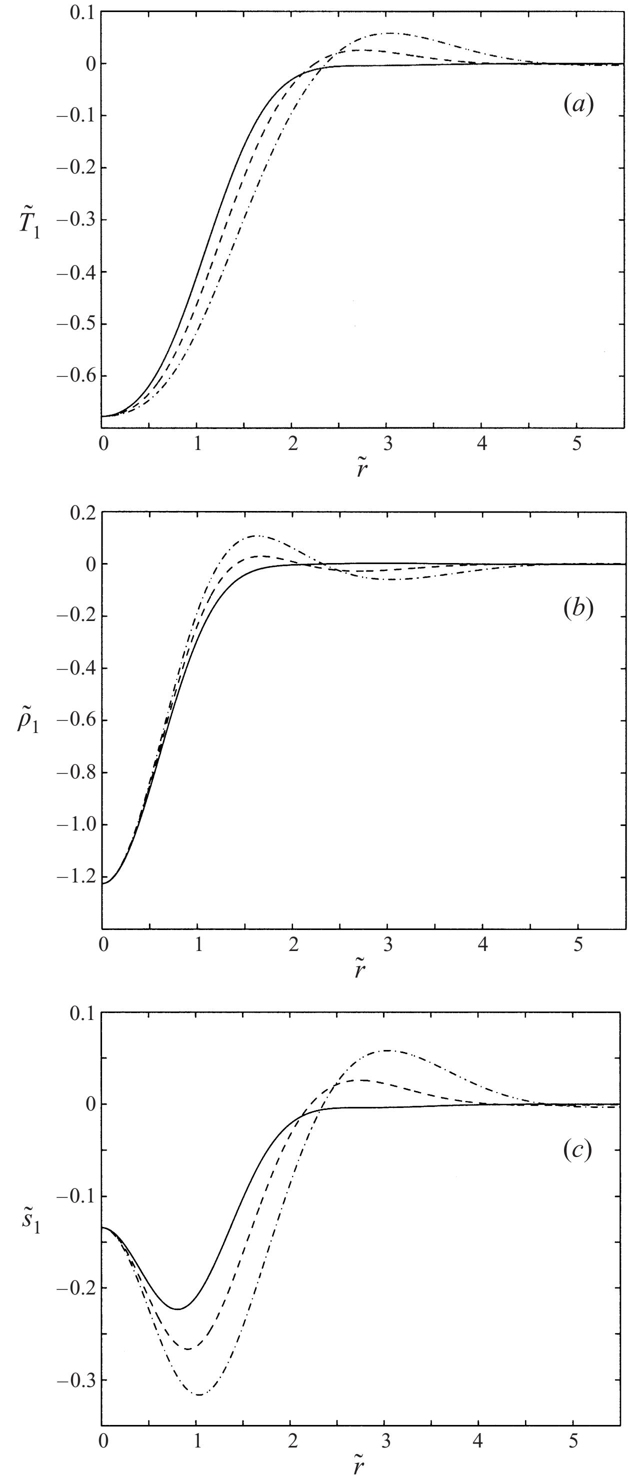

FIGURE 3. Pr dependence of the series solution with $\tau_i = \frac{1}{2}$, $\tau = 0$, $\gamma = 1.4$, $B_{1s} = -8.47 \times 10^{-2}$, $C_1 = e^{1/2}/2$. Radial distributions of $O(M^2)$ (*a*) temperature, (*b*) density, and (*c*) entropy, and are shown for three different values of Pr: ——, $Pr = 1.00$; – – – –, $Pr = 0.72$; —·—, and $Pr = 0.50$. The solution is approximated by the first 100 terms of the infinite series (3.2).

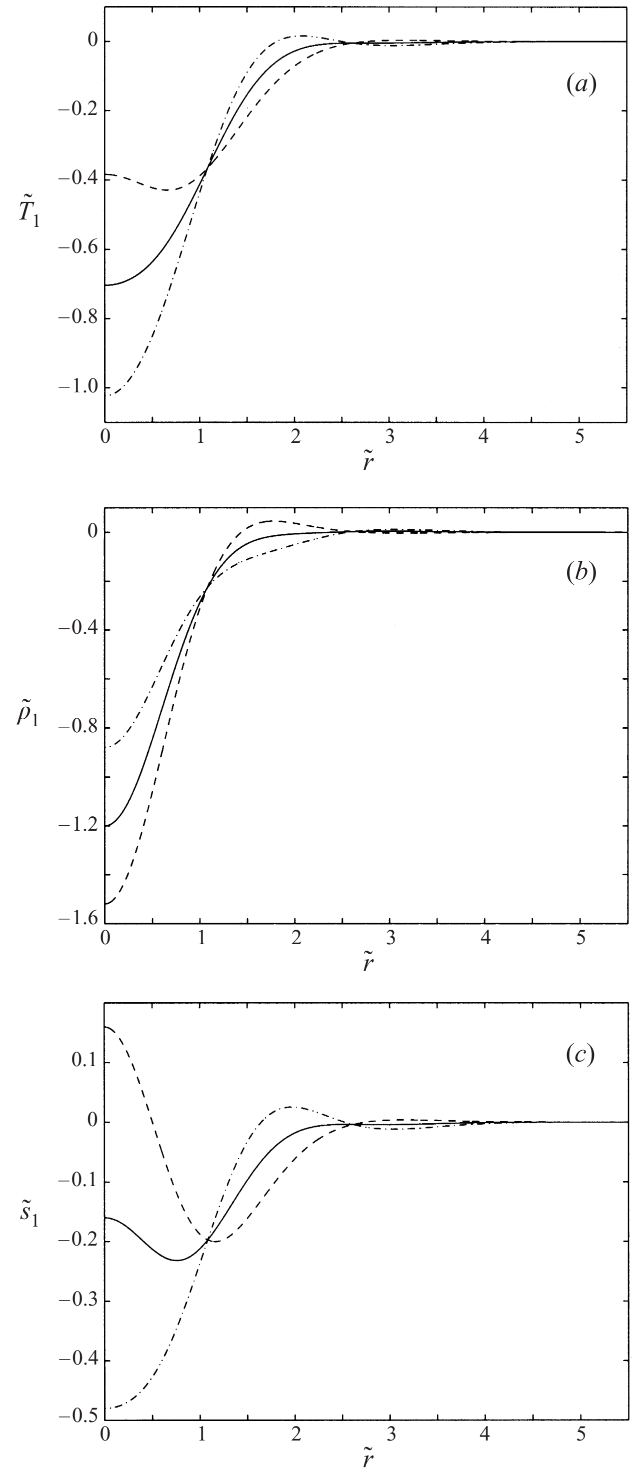

FIGURE 4 (*a*–*c*). For caption see facing page.

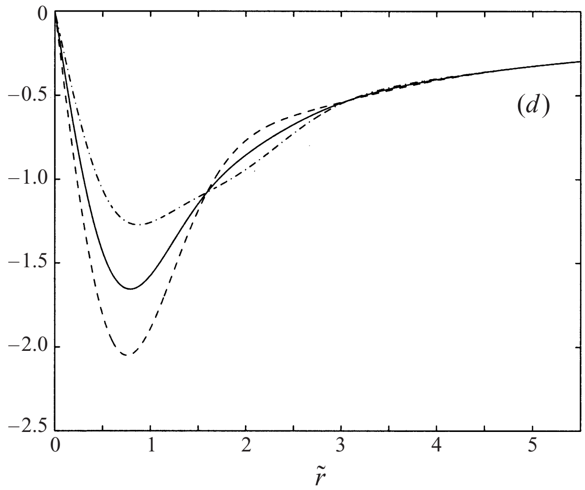

FIGURE 4. Variation of the exact solution with $\tau_i = \frac{1}{2}$, $\tau = 0$, $Pr = 1$, $\gamma = 1.4$, $C_1 = e^{1/2}/2$. Radial distributions of $O(M^2)$ (*a*) temperature, (*b*) density, (*c*) entropy, and (*d*) radial velocity, are shown for three different values of B_{1e}: ----, $B_{1e} = 2 \times 10^{-2}$; ——, $B_{1e} = -2 \times 10^{-2}$; —•—, and $B_{1e} = -6 \times 10^{-2}$.

m	0	1	2	3	4
C_m	0.699	$\frac{1}{2}e^{1/2}$	1.6402	4.5879	15.507
τ_i	0.199	$\frac{1}{2}$	0.7964	1.0922	1.387

TABLE 1. Values of C_m and τ_i that give $\tilde{v}_0(\tilde{r} = 1, \tau = 0) = 1$ for the first five eigenvalues.

the corresponding value of density $\tilde{\rho}_1$ decreases. The trends in $\tilde{\rho}_1$ and $\tilde{T}_1$ are reversed when B_1 is reduced (see figure 4). Since (2.21*a*) is still satisfied after applying the initial density-well depth condition the solutions are self-similar in time.

Lastly, note that by reducing the number of independent variables used to describe the problem ($\tilde{r}, \tau \to \eta$), the need to arbitrarily assign initial conditions for each flow variable at every point in the flow is eliminated. One does not need to invent an artificial, non-physical, initial condition (such as a compressible vortex with an initially constant density at all radial locations) – as is required for non-self-similar solutions.

3.2. *Numerical solutions*

The Oseen vortex ($m = 0$) and the Taylor vortex ($m = 1$) are both commonly used to model two-dimensional incompressible vortices. Therefore, it is interesting to compare the self-similar solutions of the compressible perturbations to these reference flows.

The $O(M^2)$ perturbation equations (2.19*a, d, e*) and (2.22*a*) are easy to numerically solve. As in § 3.1.1 above, the reference flow constants C_m and τ_i are fixed such that $\tilde{v}_0 = 1$ at $\tilde{r} = 1$ when $\tau = 0$. For reference, table 1 lists the values of C_m and τ_i which satisfy this condition for the first five eigenvalues. Equation (2.19*e*) is first numerically integrated to find the similarity parameter σ, which corresponds to the $O(M^2)$ temperature variation. The second-order Runge–Kutta scheme is used to perform the integration for $0 \leqslant \eta \leqslant 10$ on a 1000 point grid. The integration is started at $\eta = 0$ and is initialized so that there is no heat flux at the origin ($\partial \tilde{T}_1/\partial \tilde{r} = 0$), and the temperature variation at $\tilde{r} = 0$ corresponds to a density variation with a density-

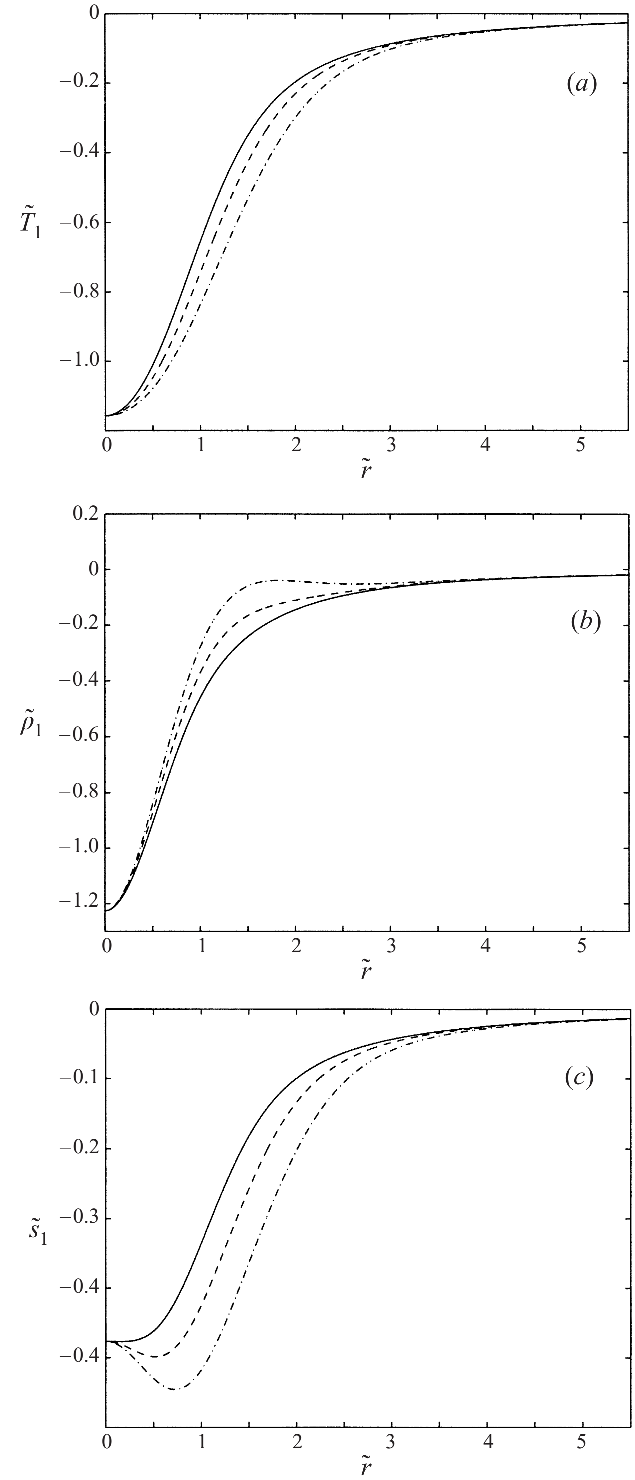

FIGURE 5 (*a*–*c*). For caption see facing page.

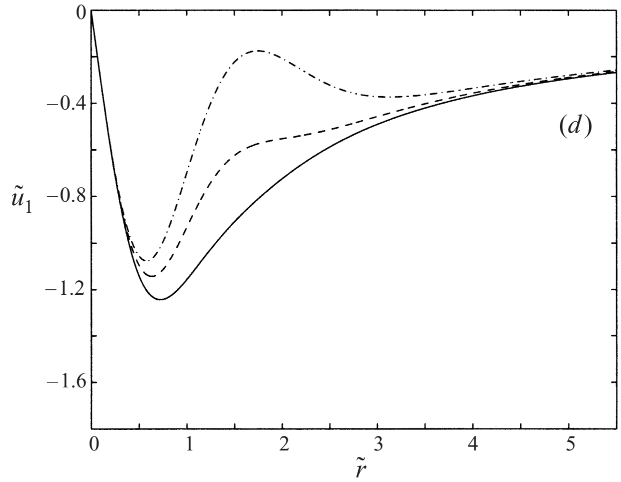

FIGURE 5. Pr dependence of the numerical solution for an Oseen reference flow. $\tau_i = 0.199$, $\tau = 0$, $\gamma = 1.4$, $C_0 = 0.699$. Radial distributions of $O(M^2)$ (*a*) temperature, (*b*) density, (*c*) entropy, and (*d*) radial velocity are shown for three different values of Pr: ——, $Pr = 1.00$; – – – –, $Pr = 0.72$; —•—, and $Pr = 0.50$.

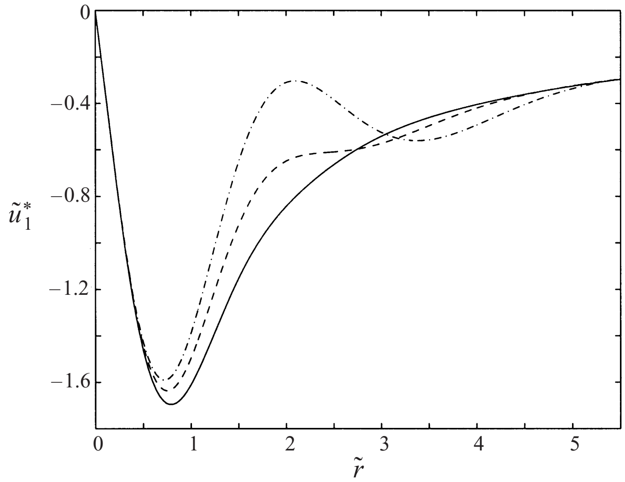

FIGURE 6. Pr dependence of the solution for a Taylor reference flow. $\tau_i = 1/2$, $\tau = 0$, $\gamma = 1.4$, $C_1 = e^{1/2}/2$. Radial distribution of the $O(M^2)$ radial velocity is shown for three different values of Pr: ——, $Pr = 1.00$; – – – –, $Pr = 0.72$; —•—, and $Pr = 0.50$.

well depth of 0.55. The forcing terms in the equations are determined using the $O(1)$ analytical solutions for the velocity (2.14) and pressure (2.15). Equations (2.18*a*–*c*, *f*) together with (2.19*a*, *d*) and (2.22*a*) are then used to calculate $\tilde{u}_1^*$, $\tilde{\rho}_1$, and $\tilde{s}_1$, respectively.

The numerical solutions for an Oseen vortex reference flow with $Pr = 0.5, 0.72$, and 1.0 are given in figure 5. The numerical solutions of $\tilde{T}_1$, $\tilde{\rho}_1$, and $\tilde{s}_1$ for the Taylor vortex and these values of Pr duplicate the series solutions already shown in figure 3. Therefore, only the solution for $\tilde{u}_1^*$ is shown in figure 6.

The radial extent of both the Oseen and the Taylor vortex reference flows is larger for smaller values of Pr. However, as shown in figure 3 the temperature, density, and entropy distributions for the Taylor vortex exhibit peaks near the core radius at the smaller Prandtl numbers. The entropy distributions of both reference flows contain local maxima at the origin when $Pr = 0.5$ and 0.72, but entropy peaks at the core radius of the vortex do not appear when the reference flow is an Oseen vortex.

3.3. *General observations*

3.3.1. *Prandtl number dependence*

While the reference flow terms are independent of the Prandtl number, Pr exerts its influence on all of the $O(M^2)$ quantities through the forcing terms of (2.19*a*–*e*). Consider the Prandtl number dependence of the $O(M^2)$ solutions (figure 3). Pr represents the ratio of viscous to thermal diffusion. When $Pr = 1$, heat and viscosity diffuse at the same rate; when $Pr < 1$, heat diffuses faster. Therefore, when the temperature profiles for different values of Pr are plotted on the same graph, and each profile is set to the same temperature at $\tilde{r} = 0$ and $\tau = 0$, the temperature distribution for the curve with the smaller value of Pr will extend further from the origin (figure 3*a*). Also, as Pr is lowered, the entropy distribution changes because the heat conduction term in the dissipation equation (2.9*b*) varies inversely with Pr. As Pr decreases, heat conduction has a greater effect on the entropy variation $\tilde{s}_1$.

3.3.2. *Radial velocity*

The Prandtl number dependence of the solutions also reveals a connection between the local amount of compression in a vortex and the local radial velocity. By symmetry, the radial velocity is necessarily zero at the origin. In the far field the vortex behaves like a two-dimensional, incompressible sink flow with $\tilde{u}_1^* \sim -1/\tilde{r}$; hence, the vortex is always compressed (Colonius *et al.* 1991). Between these limiting values of $\tilde{r}$, Pr has a strong effect on the local values of $\tilde{u}_1^*$ and $\tilde{\rho}_1$. From figures 5(*d*) and 6 one can see that, starting from just inside the vortex core, the magnitude of the radial velocity can decrease non-monotonically for $Pr \neq 1$ (for both the Oseen and the Taylor vortex). The corresponding plots of $\tilde{\rho}_1$ (figure 5*b* and 3*b*, respectively) reveal that, from curve to curve, at those radial locations where the fluid is more compressed, the magnitude of the radial velocity is smaller. Hence, the amount of compression in the flow affects the radial velocity on both a large scale (the far-field flow behaviour) and on a small scale, local sense.

3.3.3. *The entropy*

Can the compressible self-similar vortex be homentropic? In the homentropic case, $\tilde{s}_1 = \zeta = 0$. If this is true, then (2.22*a*) requires

$$(\gamma - 1)\mathscr{P} = \sigma. \tag{3.14}$$

Using this relation to replace σ with $\mathscr{P}$ in (2.19*e*), we see that $\mathscr{P}$ must depend on Pr in order for the vortex to be homentropic. However, since the base flow is incompressible the base flow pressure must be independent of Pr. Therefore, a slightly compressible, self-similar vortex cannot be homentropic. Note that if a slightly compressible vortex is not self-similar, one can specify an initial condition where the vortex is homentropic (Colonius *et al.* 1991). However, this condition is only valid at the initiation of the flow; the vortex may then evolve to a self-similar state.

In accordance with the Second Law of Thermodynamics, the total amount of entropy in the flow must increase. Far from the core of the vortex the entropy is constant, so for $\tilde{r} \to \infty$ the entropy flux is zero. Therefore, the following inequality must hold for the total entropy in the vortex flow:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\infty \tilde{s}_1 \tilde{r}\,\mathrm{d}\tilde{r} \geqslant 0. \tag{3.15}$$

As a partial check of the $O(M^2)$ perturbation solutions for the Taylor vortex, we substitute (2.13), (2.18*f*) and (3.8) into integral (3.15) to verify that the Second Law is not violated. The result is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\infty \tilde{s}_1 \tilde{r}\,\mathrm{d}\tilde{r} = \frac{C_1^2(\gamma-1)}{2(\tau+\tau_i)^3}, \tag{3.16}$$

which is always $\geqslant 0$ and so satisfies (3.15). Because the total far-field heat flux into or out of the vortex is zero (see § 3.3.4), the net entropy produced in the flow is due to viscous dissipation. The magnitude of this dissipation depends on C_1.

3.3.4. *Entropy production*

The first term on the right-hand side of (2.9*b*) is proportional to the rate of entropy production by viscous dissipation, and the second term is proportional to the rate of change of entropy due to heat conduction. This equation has an interesting consequence for the total heat flux in a slightly compressible vortex. Using the fact that $\tilde{s}_1 \to 0$ in the far field, together with the $O(1)$ solutions for $\tilde{v}_0$ (2.14), one can show that the divergence of the heat flux into or out of the vortex in the far field is zero,

$$\lim_{\tilde{r}\to\infty} \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{T}_1}{\partial \tilde{r}}\right) = 0. \tag{3.17}$$

Integrating (3.17) with respect to $\tilde{r}$ reveals that the far-field heat flux must also be zero,

$$\lim_{\tilde{r}\to\infty} \tilde{r}\frac{\partial \tilde{T}_1}{\partial \tilde{r}} = 0$$

in order to satisfy the requirement that $\tilde{T}_1 \to 0$ for $\tilde{r} \to \infty$. Because of this, the net heat flux within the vortex is also zero

$$\int_0^\infty \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{T}_1}{\partial \tilde{r}}\right)\tilde{r}\,\mathrm{d}\tilde{r} = 0.$$

Therefore, regardless of whether or not a weakly compressible vortex is self-similar, if the overall temperature in the core of the vortex increases, the temperature rise is not caused by the conduction of heat from the far field into the vortex. It can only be caused by viscous dissipation and pressure work: the total viscous work in a flow can be split into a part which is responsible for deforming fluid particles (the dissipation) and a second part which accelerates fluid particles (Thompson 1988) – the acceleration part of the viscous work affects the kinetic energy in the flow, but does not increase the temperature of the fluid; see § 3.3.5. To state this in another way, the form of the perturbation equations limits the solutions to those cases for which one asymptotically obtains a uniform temperature (as the vortex decays) without the ‘feed’ of heat from infinity.

Most of the local entropy changes occurring in the core of a self-similar, slightly compressible Taylor vortex are caused by heat conduction; since there is no net heat flux into the vortex the total entropy change produced by heat conduction is zero. To see this, examine (2.9*b*) again. The rate of change of entropy is plotted in figure 7 for $Pr = 1$ at several different times. The maxima of $\partial\tilde{s}_1/\partial\tau$ correspond to the minima

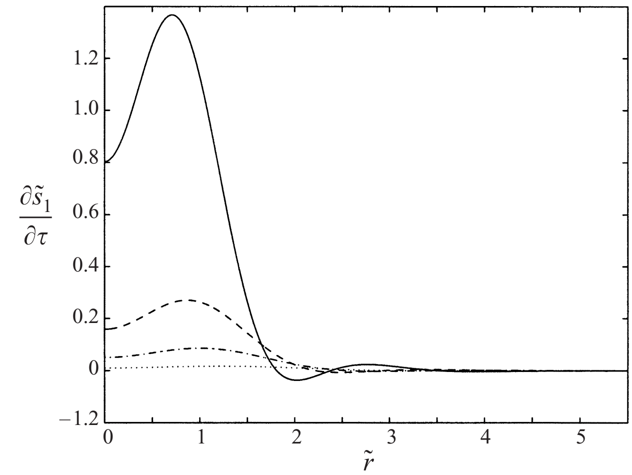

FIGURE 7. Entropy production in the Taylor vortex when $Pr = 1$, $C_1 = e^{1/2}/2$, $\tau_i = \frac{1}{2}$. ——, $\tau = 0$; – – – –, $\tau = \frac{1}{4}$, —•—, $\tau = \frac{1}{2}$, $\cdots\cdots$, $\tau = 1$.

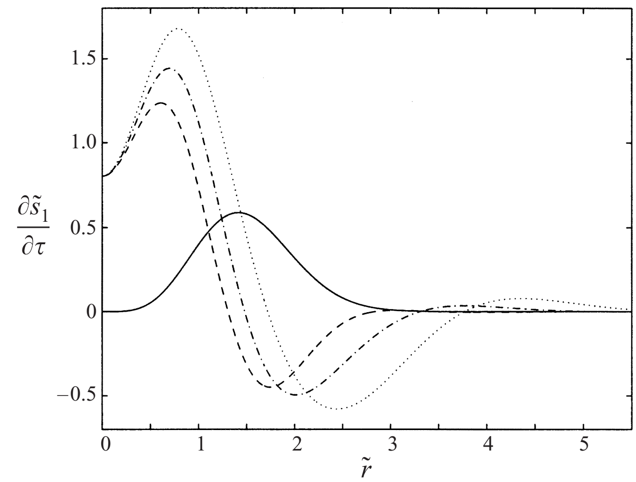

FIGURE 8. Comparison of entropy production by heat conduction and viscous dissipation in the Taylor vortex when $\tau = 0$, $C_1 = e^{1/2}/2$, $\tau_i = \frac{1}{2}$: ——, viscous dissipation terms; and heat conduction terms; for – – – –, $Pr = 1$, —•—, $Pr = 0.72$, $\cdots\cdots$, $Pr = 0.5$.

of $\tilde{s}_1$ (figure 2*f*). In figure 8 each of the terms in (2.9*b*) is plotted separately. This allows us to compare the relative magnitudes of the rate of entropy changes due to viscous dissipation and due to heat conduction. In the core of the vortex $\tilde{r} \leqslant 1$, the magnitude of the heat conduction term is larger than the viscous dissipation term. Note that the part of $\partial \tilde{s}_1/\partial \tau$ due to heat conduction is larger than zero, because heat is flowing into the cold core of the vortex. For $\tilde{r} \approx 1$ and longer, the rate of entropy change from heat conduction is less than zero because heat is flowing away from this region and into the vortex core. The rate of dissipation from viscous diffusion is, of course, always greater than zero. As shown in figure 8, the series solution for σ_p, (3.2), can be used with (2.18*f*) and (2.22*b*) to reveal that the rate of entropy change from heat conduction increases in magnitude as the Prandtl number decreases; when μ is constant, heat diffuses faster, but the rate of viscous diffusion remains the same.

3.3.5. *Vortex energetics*

The total amount of pressure work performed on a fluid volume is given by the integral of the product of the pressure and fluid velocity at the volume's surface:

$$\int_{\partial V} p\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} = \int_{V} \nabla \cdot (p\boldsymbol{u})\mathrm{dV} = \int_{V} (\boldsymbol{u} \cdot \nabla p + p\nabla \cdot \boldsymbol{u})\mathrm{dV}. \tag{3.18}$$

Here we have used Stokes' Theorem to convert the surface integral into a volume integral and split the total pressure work into two terms. The first term $\boldsymbol{u}\cdot\nabla p$ represents the work performed by the pressure gradients in moving fluid inside the volume and the second term $p\nabla \cdot \boldsymbol{u}$ is the work done by pressure forces in locally compressing the fluid at each point within the volume. In the far field, $u \sim -1/r$, the pressure is constant, and the surface area of the two-dimensional flow field is proportional to r. Therefore, from the surface integral in (3.18) we see that the external pressure field surrounding a slightly compressible free vortex performs a net amount of work on the flow as the vortex decays. As viscous shear stresses decrease the tangential velocity, the centrifugal force of the flow's rotation relaxes and the fluid in the vortex core is compressed by the external pressure field. This pressure work, in addition to heat released by the dissipation of the flow's kinetic energy, increases both the temperature and density of the vortex. To see this in more detail, we will consider the two parts of the total pressure work and their separate effects on the kinetic and internal energy of the flow.

The scalar product of the velocity and the momentum equations gives an expression for the balance of kinetic energy in a flow. Note that the kinetic energy associated with the radial component of the velocity does not enter our analysis because it is $O(M^4)$. Integrating the product of $\tilde{v}_0$ and the tangential momentum equation (2.3*b*) gives an expression for the total $O(1)$ kinetic energy in the vortex:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\infty \frac{\tilde{v}_0^2}{2}\tilde{r}\mathrm{d}\tilde{r} = -\int_0^\infty \left(\frac{\partial \tilde{v}_0}{\partial \tilde{r}} + \frac{\tilde{v}_0}{\tilde{r}}\right)^2 \tilde{r}\mathrm{d}\tilde{r}. \tag{3.19}$$

We see that since the reference flow is incompressible, only the work involved in viscous dissipation decreases the total $O(1)$ kinetic energy of the flow (Landau & Lifshitz 1987).

The product of $\tilde{v}_0$ and (2.7*c*), as well as the use of (2.3*a*), gives an equation for the $O(M^2)$ kinetic energy:

$$\tilde{\rho}_1\frac{\partial}{\partial \tau}\left(\frac{\tilde{v}_0^2}{2}\right) + \tilde{u}_1^*\frac{\partial}{\partial \tilde{r}}\left(\frac{\tilde{v}_0^2}{2}\right) + \frac{\partial}{\partial \tau}(\tilde{v}_0\tilde{v}_1)$$
$$= -\tilde{u}_1^*\frac{\partial \tilde{p}_0}{\partial \tilde{r}} + \tilde{v}_1\frac{\partial}{\partial \tilde{r}}\left[\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{v}_0)\right] + \tilde{v}_0\frac{\partial}{\partial \tilde{r}}\left[\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{v}_1)\right]. \tag{3.20}$$

Integrating this expression over the entire flow field we find the total $O(M^2)$ kinetic energy in the flow

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\infty \left(\frac{\tilde{\rho}_1\tilde{v}_0^2}{2} + \tilde{v}_1\tilde{v}_0\right)\tilde{r}\mathrm{d}\tilde{r}$$
$$= -\int_0^\infty \tilde{u}_1^*\frac{\partial \tilde{p}_0}{\partial \tilde{r}}\tilde{r}\mathrm{d}\tilde{r} - 2\int_0^\infty \left(\frac{\partial \tilde{v}_0}{\partial \tilde{r}} + \frac{\tilde{v}_0}{\tilde{r}}\right)\left(\frac{\partial \tilde{v}_1}{\partial \tilde{r}} + \frac{\tilde{v}_1}{\tilde{r}}\right)\tilde{r}\mathrm{d}\tilde{r}. \tag{3.21}$$

From this equation, (2.13), (2.20), and (2.18*b*,*d*), we see that the time derivative of the total $O(M^2)$ kinetic energy varies as $1/\tau^{4m+2}$. Since the total $O(M^2)$ kinetic energy is a perturbation to the total $O(1)$ kinetic energy, it may be either positive or negative in sign. Since this term may be negative, the second term on the right-hand side of (3.21), which represents the contribution of viscous forces, is not necessarily positive-definite. However, in the case of both the Oseen and the Taylor vortex, the first term on the right-hand side of (3.21), which gives the $\boldsymbol{u} \cdot \nabla p$ part of the pressure work in (3.18), always increases the $O(M^2)$ kinetic energy of the vortex. This happens because the radial velocity in these two vortices is negative for $\tilde{r} > 0$.

Turning now to the internal energy of the flow e, let $\tilde{e} = C_v T_\infty e$. With this choice of scaling, $\tilde{e} = \tilde{T}$ and from (2.7*d*) we find an equation for the balance of internal energy in the flow:

$$\frac{\partial \tilde{e}_1}{\partial \tau} = \gamma(\gamma-1)\left(\frac{\partial \tilde{v}_0}{\partial \tilde{r}} - \frac{\tilde{v}_0}{\tilde{r}}\right)^2 + \frac{\gamma}{Pr}\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{T}_1}{\partial \tilde{r}}\right) - \frac{(\gamma-1)}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{u}_1^*). \tag{3.22}$$

Integrating this expression over the entire flow field gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\infty \tilde{e}_1 \tilde{r}\mathrm{d}\tilde{r} = \gamma(\gamma-1)\int_0^\infty \left(\frac{\partial \tilde{v}_0}{\partial \tilde{r}} - \frac{\tilde{v}_0}{\tilde{r}}\right)^2 \tilde{r}\mathrm{d}\tilde{r} - (\gamma-1)\int_0^\infty \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{u}_1^*)\tilde{r}\mathrm{d}\tilde{r}; \tag{3.23}$$

note that the integral of the heat flux is zero (see § 3.3.4).

Because the first term on the right-hand side of this equation is positive definite, and since fluid is moving into the vortex, the total internal energy of the vortex always increases in time. The second term on the right-hand side of (3.23) is proportional to the $O(M^2)$ divergence of the flow and represents the total amount of work performed by the pressure in compressing the fluid within the vortex. To see this note that, to $O(M^2)$, the term $-p\nabla \cdot \boldsymbol{u}$ in (3.18) is given by

$$-p\nabla \cdot \boldsymbol{u} \to -\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{u}_1^*), \tag{3.24}$$

where (2.1) has been used to make the terms non-dimensional, and the Rayleigh–Janzen expansion (2.2) was applied.

4. Concluding remarks

The equations which describe a two-dimensional, slightly compressible, vortex flow are invariant under a three-parameter Lie group. The similarity solutions found with this group give explicit expressions for the decay rates of each flow variable, and reveal that non-self-similar, slightly compressible vortices will decay to a self-similar state. Further, the solutions provide a rational means for specifying the initial conditions in a slightly compressible vortex and to study numerical and experimental flows in the light of their evolution to self-similar conditions.

The solutions for the radial velocity and density perturbations in weakly compressible Oseen and Taylor vortices show that the amount of local compression in a vortex strongly influences the local magnitude of the radial velocity – the magnitude of the radial velocity is smaller in the more compressed regions of the flow.

Generally, self-similar, slightly compressible vortices are not isentropic, and there is no net heat flux between the vortex centre and the far field. The total pressure work on the vortex can be split into two parts, each of which separately affects the internal and kinetic energy in the flow. The work performed by the pressure gradients in moving fluid increases the $O(M^2)$ kinetic energy of the flow, and the pressure work of compressing the fluid increases the vortex's internal energy. The effects of viscous dissipation form the remainder of the terms in both the $O(M^2)$ kinetic and internal energy balance in the vortex.

This work was supported, in part, by the NASA Graduate Student Researchers Program and the NASA Ames-Stanford Joint Institute for Aeronautics and Acoustics NCC 2-55.

REFERENCES

Ardalan K., Meiron, D. I. & Pullin, D. I. 1995 Steady compressible vortex flows: the hollow-core vortex array. *J. Fluid Mech.* **301**, 1–17.

Bellamy-Knights, P. G. 1980 Viscous compressible heat conducting spiralling flow. *Q. J. Mech. Appl. Maths* **33**, 321–336.

Bershader, D. B. 1995 Compressible vortices. In *Fluid Vortices* (ed. S. I. Green). Kluwer.

Chiocchia, G. 1989 A hodograph approach to the rotational compressible flow of an ideal fluid. *Q. App. Maths* **47**, 513–528.

Colonius, T., Lele, S. K. & Moin, P. 1991 The free compressible viscous vortex. *J. Fluid Mech.* **230**, 45–73.

Ellenrieder, K. von 1998 Self-similar compressible free vortices. PhD thesis, Dept. of Aeronautics & Astronautics, Stanford University, 1998. (Also available as NASA JIAA manuscript TR 121.)

Landau, L. D. & Lifshitz, E. M. 1987 *Fluid Mechanics*, 2nd Edn. Pergamon.

Mack, L. M. 1960 The compressible viscous heat-conducting vortex. *J. Fluid Mech.* **8**, 284–292.

Mandella, M. J. 1987 Experimental and analytical studies of compressible vortices. PhD thesis, Dept. of Applied Physics, Stanford University.

Neufville, A. de 1957 The dying vortex. In *Proc. Fifth Midwestern Conf. on Fluid Mechanics*, pp. 365–375. University of Michigan.

Oseen, C. W. 1912 Über Wirbelbewegung in einer reibenden Flüssigkeit. *Ark. f. Mat. Astron. Fys.* **7**, 14.

Taylor, G. I. 1918 On the dissipation of eddies. Reprinted in *The Scientific Papers of Sir Geoffrey Ingram Taylor*, vol. II (ed. G. K. Batchelor). Cambridge University Press, 1971.

Thompson, P. A. 1988 *Compressible-Fluid Dynamics*. RPI.