BRIEF COMMUNICATIONS

Self-Similar Diffusion Flame

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In this communication, we develop a self-similar solution for a limiting case of a coaxial jet diffusion flame considered by Burke and Schumann [1]. Simple closed form expressions for both the flame height and shape are obtained and compared with the classical solution. The notation and terminology used closely follows Williams [2]. In the Burke-Schumann problem, fuel is fed through an inner tube while oxidizer flows through an outer coaxial tube. A steady flame is established at the mouth of the inner tube. With excess oxidizer, the flame ends on the axis and is called overventilated, while with excess fuel, it ends on the wall and is said to be underventilated. In this note, only the former case is considered and a self-similar solution is developed for the case in which the radius of the outer tube becomes infinite while that of the inner tube approaches zero. Like all self-similar solutions, the present one is useful in the far field, where initial/upstream conditions, no matter how complicated, become less important. Cylindrical (r, z) coordinates, are adopted, r being the radial coordinate and z the vertical or streamwise coordinate; the flame is assumed to remain axisymmetric.

As in Williams [2], radiation, diffusion due to pressure gradients, and the Soret and Dufour effects are all assumed negligible. The diffusivities of all species and temperature are assumed equal. Bulk viscosity is neglected. The Mach number is low and axial diffusion is neglected. Effects of buoyancy are neglected and thus the present results are valid in a momentum-dominated regime where flow instabilities are assumed to be absent. The thin flame approximation is made so that no penetration of fuel or oxidizer across the infinitesimally thin flame surface occurs. In general, reaction zone broadening occurs due to effects of equilibrium dissociation and finite rate chemistry. When all diffusivities are equal, the former situation can be handled relatively easily as discussed by Williams [2]. With simple finite rate kinetics, perturbation methods such as that due to Liñán may be used.

Single-step, irreversible, fast chemistry between fuel (F) and oxidizer (O) reacting to yield a product (P) is assumed:

$$\nu_F F + \nu_O O \rightarrow \nu_p P,$$

where the ν_i are the stoichiometric coefficients. Let $Y_{F,0}$ and $Y_{O,0}$ represent the fuel and oxidizer mass fractions at the upstream boundary. The following quantities are defined:

$$\alpha_F \equiv -\frac{Y_F}{W_F \nu_F}, \quad \alpha_O \equiv -\frac{Y_O}{W_O \nu_O},$$

$$\beta \equiv \alpha_O - \alpha_F, \tag{1}$$

where Y_F and Y_O are the fuel and oxidizer mass fractions, respectively, and W_F and W_O are their respective molecular weights. The quantity β is the coupling function in the Schvab-Zeldovich formulation. Under the stated assumptions, β is governed by the conserved scalar equation

$$\frac{1}{r}\frac{\partial}{\partial r} r\rho v_r \beta + \frac{\partial}{\partial z}\rho v_z \beta = \frac{1}{r}\frac{\partial}{\partial r} r\rho D \frac{\partial \beta}{\partial r}, \tag{2}$$

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where ρ is the density, v_r and v_z are the radial and streamwise velocity components, respectively, and D is the diffusion coefficient. Note that β ranges from $-Y_{O,0}/W_O\nu_O$ to $Y_{F,0}/W_F\nu_F$.

Consider the limiting case in which the radius of the outer tube becomes infinite while that of the inner tube approaches zero. This means that there is an infinitesimally thin jet of fuel, with finite mass flux, issuing into an infinite stream of oxidizer. For such a case one can expect a self-similar solution to Eq. 2. Since β decays asymptotically to $-Y_{O,0}/W_O\nu_O$, rather than to zero in the far field, it is convenient to define

$$\phi = \beta + \beta_0, \tag{3}$$

where

$$\beta_0 \equiv \frac{Y_{O,0}}{W_O\nu_O}, \tag{4}$$

so that ϕ goes to zero in the far field. In the thin flame approximation, the surface $\beta = 0$ or $\phi = \beta_0$ identifies the flame surface. Clearly ϕ also satisfies Eq. 2. Thus,

$$\frac{1}{r}\frac{\partial}{\partial r}r\rho v_r\phi + \frac{\partial}{\partial z}\rho v_z\phi = \frac{1}{r}\frac{\partial}{\partial r}r\rho D\frac{\partial\phi}{\partial r}. \tag{5}$$

Multiplying Eq. 5 by $r\,dr\,d\theta$ and integrating over $0 < r < \infty$ and $0 < \theta < 2\pi$ and noting that v_r vanishes at the boundaries gives the integral constraint

$$2\pi\int_0^\infty \rho v_z\phi r\,dr = \text{const} \equiv B, \tag{6}$$

which implies that the flux of ϕ is a constant, independent of z. The boundary conditions are

$$\frac{\partial\phi}{\partial r} < \infty \quad \text{as } r \to 0; \qquad r\frac{\partial\phi}{\partial r} \to 0 \quad \text{as } r \to \infty. \tag{7}$$

The following additional assumptions are made:

1. The flow is one dimensional in the sense that $\rho v = \text{const}$ everywhere. Here v is the velocity magnitude in the z direction ($v_r = 0$). This is a severe restriction and is justified only by the resulting simplification of the problem.
2. It is also assumed that $\rho D = \text{const}$.

Williams [2] presents an excellent discussion of the validity of these additional assumptions. Note that the integral constraint in Eq. 6 is valid regardless of these assumptions. It allows definition of a length scale, by which r and z may be made dimensionless:

$$L \equiv \sqrt{\frac{B}{\rho v\beta_0}}, \quad \bar{r} \equiv \frac{r}{L}, \quad \bar{z} \equiv \frac{z}{L}. \tag{8}$$

The scalar ϕ is normalized by β_0:

$$\bar{\phi} \equiv \frac{\phi}{\beta_0}. \tag{9}$$

A Peclet number is defined as follows:

$$\text{Pe}_f \equiv \frac{(\rho v)L}{(\rho D)}. \tag{10}$$

With the stated assumptions and dropping the overbars henceforth, Eq. 5 is written in dimensionless form as

$$\text{Pe}_f\frac{\partial\phi}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} = 0. \tag{11}$$

Note that $0 \le \phi \le 1 + Y_{F,0}/W_F\nu_F\beta_0$ and $\phi = 1$ at the flame. The integral constraint, Equation 6 becomes

$$2\pi\int_0^\infty \phi r\,dr = 1. \tag{12}$$

The following similarity variable is introduced:

$$\eta = \frac{\text{Pe}_f r^2}{z}, \tag{13}$$

and a solution to ϕ of the form

$$\phi(r, z) = Az^\kappa f(\eta) \tag{14}$$

is assumed. Substituting Eq. 14 into Eq. 11 yields the ordinary differential equation for f:

$$4\eta f'' + (4 + \eta)f' - \kappa f = 0. \tag{15}$$

Substitution of Eq. 14 into Eq. 12 allows determination of κ and A. Thus,

$$\frac{\pi A z^{\kappa+1}}{\text{Pe}_f}\int_0^\infty f(\eta)\,d\eta = 1. \tag{16}$$

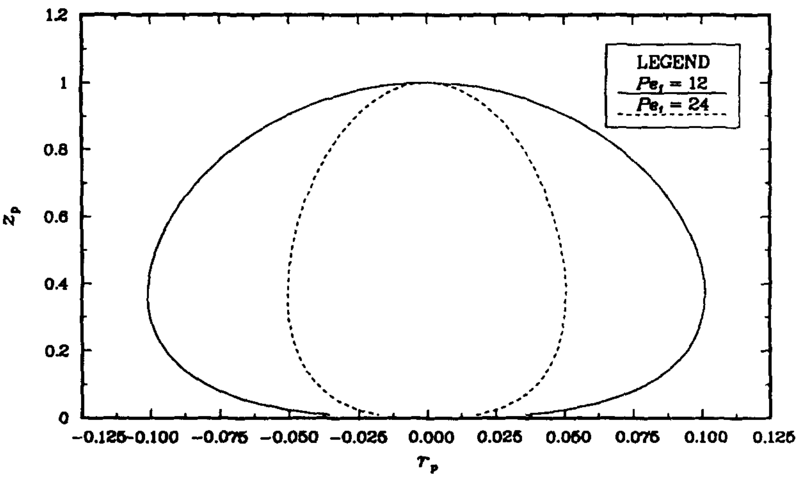

Fig. 1. Self similar diffusion flame shapes for $\mathrm{Pe}_f = 12, 24$. Length scale used is the flame height at each Peclet number.

Since the above integral is independent of z,

$$\kappa = -1, \quad A = \frac{\mathrm{Pe}_f}{\pi \displaystyle\int_0^{\infty} f(\eta)\, d\eta}. \tag{17}$$

The solution of Eq. 15 subject to the boundary conditions is

$$f(\eta) = \exp\left(-\frac{\eta}{4}\right). \tag{18}$$

Thus,

$$\phi(r, z) = \frac{\mathrm{Pe}_f}{4\pi z} \exp\left(-\frac{\mathrm{Pe}_f\, r^2}{4z}\right). \tag{19}$$

In the flame sheet approximation, $\phi = 1$ corresponds to the flame surface. Eq. 19 then gives a simple expression for the locus of the flame (r^*, z^*):

$$r^* = 2\sqrt{\frac{z^*}{\mathrm{Pe}_f} \ln\left(\frac{\mathrm{Pe}_f}{4\pi z^*}\right)}. \tag{20}$$

From Eq. 20, the flame height z_H is obtained; it is the location at which the stoichiometric surface intersects the axis:

$$z_H = \frac{\mathrm{Pe}_f}{4\pi}. \tag{21}$$

The linear dependence of the nondimensional flame height on the Peclet number is explicitly demonstrated by Eq. 21. As expected, the dimensional flame height decreases with increasing mass fraction of the oxidizer in the far field. From Eq. 20, the coordinates (r_W, z_W), corresponding to the maximum half-width of the flame are

$$r_W = \frac{1}{\sqrt{\pi e}}, \quad z_w = \frac{z_H}{e}, \tag{22}$$

where e is the base of the natural logarithm.

In order to plot flame shapes, it is convenient to normalize on the flame height:

$$r_p \equiv \frac{r^*}{z_H}, \quad z_p \equiv \frac{z^*}{z_H}. \tag{23}$$

In terms of (r_p, z_p), Eq. 20 is written as

$$r_p = \frac{4}{\mathrm{Pe}_f}\sqrt{\pi \ln {z_p}^{-z_p}} \tag{24}$$

In Fig. 1, flame shapes are plotted for two different values of Pe_f. As Pe_f increases, the flame becomes thinner in these coordinates.

In Fig. 2, the self-similar flame shape is compared with the classical Burke–Schumann solution [2] for two different Peclet numbers. Using the definition of the flux B (the integration being from $r = 0$ to the outer tube radius) and the length scale L, an effective Pe_f for the classical solution is ob-

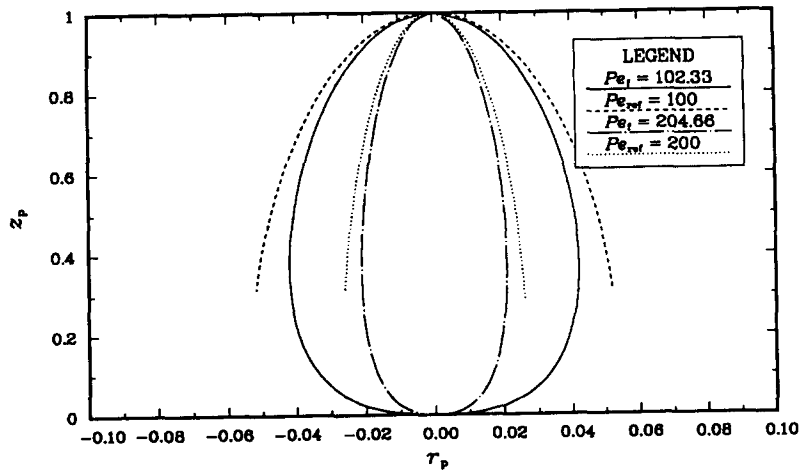

Fig. 2. Burke-Schumann [2] and self similar flame. Only the downstream portion of the Burke-Schumann flame is drawn for emphasis. For specified Pe_{ref} for the classical solution, equivalent Pe_f is given.

tained:

$$\mathrm{Pe}_f = \frac{v}{D} r_i \sqrt{\pi \left[1 + \frac{Y_{F,0}}{W_F \nu_F \beta_0}\right]}, \tag{25}$$

where r_i is the inner tube radius. The following values were assumed for the classical solution: $Y_{F,0}/W_F \nu_F \beta_0 = 2$, the inner-to-outer tube radius ratio $r_i/r_o = \frac{1}{3}$ and the Peclet number based on the outer tube radius, $\mathrm{Pe}_{\mathrm{ref}} \equiv v r_o / D = 100$ and 200. As expected, the agreement improves further downstream. Both the flame shapes compare very well for higher Peclet numbers, for which the flame becomes thinner. For small Peclet numbers, the agreement improves as r_i/r_o is decreased. Both flames exhibit linear dependence of flame height on Peclet number.

The expression for the flame shape developed in this note is simpler than the series solution obtained by Burke and Schumann [1] as well as the solution for $r_o \to \infty$ given by Williams [2] in terms of integrals of Bessel functions. It is a good approximation to the classical solution in the far field. The solution given here has fewer parameters and gives closed form analytical values for all the key flame shape quantities.

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