Archives of Mechanics • Archiwum Mechaniki Stosowanej • 31, 5, pp. 707–721, Warszawa 1979

Coherent turbulent structures as critical points in unsteady flow

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TURBULENT flows which are normally thought of as self-similar in space are re-examined in terms of a functional form which assumes similarity in time. It is suggested that such an assumption can provide a framework within which the coherent motions of turbulent shear flows can be analyzed in a unified way. Structural features, large eddies and so forth come out as critical points in a phase plane plot of particle trajectories. Some of the analytical properties of unsteady critical points are derived.

Przepływy turbulentne, traktowane zwykle jako samopodobne w przestrzeni fizycznej, rozważane są tutaj przy użyciu formy funkcjonalnej zakładającej podobieństwo w czasie. Proponuje się, aby to ostatnie założenie traktować jako podstawę teorii umożliwiającej jednolitą analizę spójnych ruchów dla turbulentnych przepływów ścinających. Cechy strukturalne, duże wiry, etc., ukazują się jako punkty krytyczne na wykresach trajektorii cząstek w płaszczyźnie fazowej. Wyprowadzono kilka cech analitycznych niestacjonarnych punktów krytycznych.

Турбулентные течения, трактованные обычно как автомодельные в физическом пространстве, рассматриваются здесь при использовании функциональной формы, предполагающей подобие во времени. Предлагается, чтобы это последнее предположение трактовать как основу теории дающей возможность однородного анализа связных движений для турбулентных течений сдвига. Структурные свойства, большие вихри и т. д. появляются как критические точки на диаграммах траекторий частиц в фазовой плоскости. Выведено несколько аналитических свойств нестационарных критических точек.

1. Introduction

FLUID motion can be studied from two distinct points of view. From the Lagrangian point of view the object of study is the moving fluid or, more precisely, its discrete particles which fill in a continuous manner some moving volume occupied by the fluid. In this view the primary variables of interest are particle trajectories and the various vector and scalar quantities characterizing the motion are examined as functions of time and of those data which distinguish one particle from another. As such data one may take, for example, the Cartesian coordinates of the fluid particles at some initial time.

From the Eulerian point of view, the object of study is not the fluid itself but a fixed space which is filled with a moving fluid. The primary variable of interest is the velocity field and the quantities characterizing the motion are considered as functions of time and space.

The two formulations are connected through the parametric equations for particle paths:

$$
\begin{aligned}
x(t) &= x_0 + \int_{t_0}^{t} u(x(s), y(s), z(s), s)\,ds, \\
y(t) &= y_0 + \int_{t_0}^{t} v(x(s), y(s), z(s), s)\,ds, \\
z(t) &= z_0 + \int_{t_0}^{t} w(x(s), y(s), z(s), s)\,ds.
\end{aligned} \tag{1.1}
$$

Information about particle paths carries with it complete information about the velocity field by simple differentiation of the above equations and, in principle, knowledge of the velocity field can be used to calculate particle paths.

If a frame of reference can be found in which the flow is steady, then the particle path equations reduce to an autonomous system with integral curves which coincide with the streamlines of the velocity field referred to the same frame. A number of authors have made use of this fact to explore the properties of solution trajectories in a variety of steady flow situations. OSWATITSCH (1958) and LIGHTHILL (1963) classified certain critical points which can occur in the neighborhood of a rigid boundary. PERRY and FAIRLIE (1974) reviewed critical point analysis in a general way and applied the technique to the problem of three-dimensional separation. They placed special emphasis on the fact that the method provides a wealth of topological language which is particularly well suited to the unambiguous description of fluid flow patterns. More recently HUNT, ABELL, PETERKA and WOO (1978) applied critical point theory to flow visualization studies of bluff obstacles.

If the flow is unsteady, then the integration of Eqs. (1.1) may be extremely complicated. Moreover, the physical interpretation of the integrated particle paths involves severe conceptual as well as analytical difficulties. If the integration is carried out over a volume of particles, then each point in space will be traversed by an infinite set of trajectories, each with a different slope corresponding to the passage of particles through the point at successive instants of time. The issue is further complicated by the fact that the pattern of particle paths, like the pattern of streamlines, depends on the frame of reference. Whereas our intuition tells us that physical phenomena should be describable in an invariant way, independent of the incidental motion of an observer.

Recently this problem has received new attention in connection with large scale coherent motions in turbulent shear flows. The main characteristics of these so-called large eddies are that they convect with the main flow and grow in size with length scales in the streamwise and cross-stream directions which, in most cases, are of the same order. They carry fluid with them and their induced motions entrain additional fluid.

The situation is this: the essential feature of free turbulence is growth and it now appears that an important element of this is local growth by moving and interacting large eddies. This suggests the need for a description of turbulence which focuses on the fluid composing the large eddy rather than on some fixed volume which is only momentarily occupied by it. As a first approximation it seems appropriate to choose a frame of reference which translates with some preferred point in the eddy. This approach can sometimes lead to a simplified qualitative description of the flow. However, the choice of convection speed can be an ambiguous and subjective process and, as noted above, conclusions regarding the physical processes taking place in the flow ought not to depend on this choice. What is needed is an approach which is Eulerian in the sense that governing equations are written in terms of familiar spatial variables but Lagrangian in the sense that trajectories of fluid particles can be described in a simple and invariant way.

Now it is axiomatic in the study of turbulence that some method be used which simplifies the complex, fluctuating motions which compose all such flows. The usual procedure is to take a simple long time average thereby removing any explicit time dependence. Here we shall replace this with an assumption that part of the unsteady motion in turbulent shear flows is truly *coherent*. Flows which are normally considered to be self-similar in space will be re-examined in terms of a functional form for the dependent variables which assumes similarity in time. The *limited* objective of this paper is to explore some of the consequences of this assumption and to see whether it can provide a framework within which the coherent motions of turbulent flows can be usefully described.

The first and most important consequence of the assumption is that the particle path Eq. (1.1) can be reduced to an autonomous system (CANTWELL 1978) as in steady flow. Moreover, the resulting phase plane pattern of particle trajectories is invariant under certain transformations of a moving observer (CANTWELL, COLES and DIMOTAKIS 1978).

A second consequence is that, as in Reynolds averaging, some of the physics of turbulence is lost. The large eddies that are observed in mixing layers (BROWN and ROSHKO 1974), wakes (BEVILAQUA and LYKOUDIS 1971) and jets (CROW and CHAMPAGNE 1971) interact with each other. In mixing layers and jets the interactions are strong and occur with great rapidily whereas in wakes the interactions occur infrequently so that the eddies retain their identity over many characteristic lengths of the flow (TANEDA 1959). In either case a vortex observed *now* will, with the passage of sufficient time, be incorporated into a vortex of larger scale. Under the assumption of this paper the coherent part of the unsteady motion is to be thought of as the ultimate result of many such interactions. In this sense the coherent motion is unseen in an instantaneous picture of the flow, just as the long time averaged flow is unseen. It is an average over many interactions of an unsteady motion which is at a scale larger than the eddies which are actually observed.

However, it is not an average referred in the usual sense to a laboratory observer; for this would simply retrieve the ordinary Reynolds averaged velocity field. Rather, it is an average referred to an observer in a contracting space with distances which are scaled by some power of time as determined by the global parameters which govern the flow. In two dimensions we can think of this as an average referred to an observer who is receding out of the plane of the flow at a rate which fixes (in his view) the position and size of the various structural features of the flow. See CANTWELL, COLES and DIMOTAKIS (1978) for a specific example of this averaging process. See also TURNER (1964) where a similarity form is used to plot particle trajectories in a rising and expanding Hill's spherical vortex.

2. Similarity assumption

We consider incompressible, constant property flows governed by

$$\frac{\partial u_j}{\partial x_j} = 0,$$

(2.1)

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\varrho}\frac{\partial p}{\partial x_i} + \frac{1}{\varrho}\frac{\partial \tau_{ij}}{\partial x_j},$$

where the stress terms have been purposely left undefined. For turbulent flows the stresses would involve products of fluctuations averaged over a suitably defined ensemble.

Similarity and group-theoretical methods have often been used to reduce the complexity of the equations of fluid motion and an interesting early discussion of this subject may be found in BIRKHOFF (1950). Here we use invariance of the system (2.1) under the two parameter group of stretchings and translations to construct similarity variables of the following form:

$$\xi_i = \frac{x_i - V_i t}{M^\alpha t^k},$$

$$U_i(\xi) = \frac{u_i - V_i}{M^\alpha t^{k-1}},$$

(2.2)

$$P(\xi) = \frac{p}{\varrho M^{2\alpha} t^{2k-2}},$$

$$T_{ij}(\xi) = \frac{\tau_{ij}}{\varrho M^{2\alpha} t^{2k-2}},$$

where M is a parameter of the motion for a given flow and α and k are chosen so that $M^\alpha t^k$ has the dimensions of a length.

Upon application of the transformation (2.2) the system (2.1) becomes

$$\frac{\partial U_j}{\partial \xi_j} = 0,$$

(2.3)

$$(k-1)U_i + (U_j - k\xi_j)\frac{\partial U_i}{\partial \xi_j} = -\frac{\partial P}{\partial \xi_i} + \frac{\partial T_{ij}}{\partial \xi_j}$$

and the particle path Eq. (1.1) become

(2.4) $$\frac{d\xi_i}{d\tau} = U_i(\xi) - k\xi_i,$$

where $\tau \equiv \ln t$.

Critical points of the system (2.4) occur where

(2.5) $$U_i(\xi) = k\xi_i.$$

If a non-steady similarity assumption is to have any applicability at all to turbulent flows, then it should satisfy the minimum requirement of giving spatial growth and decay laws of characteristic lengths and velocities which correspond with those with which we are already familiar. In this context we will consider flows where there is a uniform velocity to the right in the x-direction so that $V = (u_\infty, 0, 0)$ and flows where there is no externally imposed velocity $V = (0, 0, 0)$. Consider the trajectory of a critical point shown schematically as a stable focus in Fig. 1. If we take ξ_1 in the direction of the external

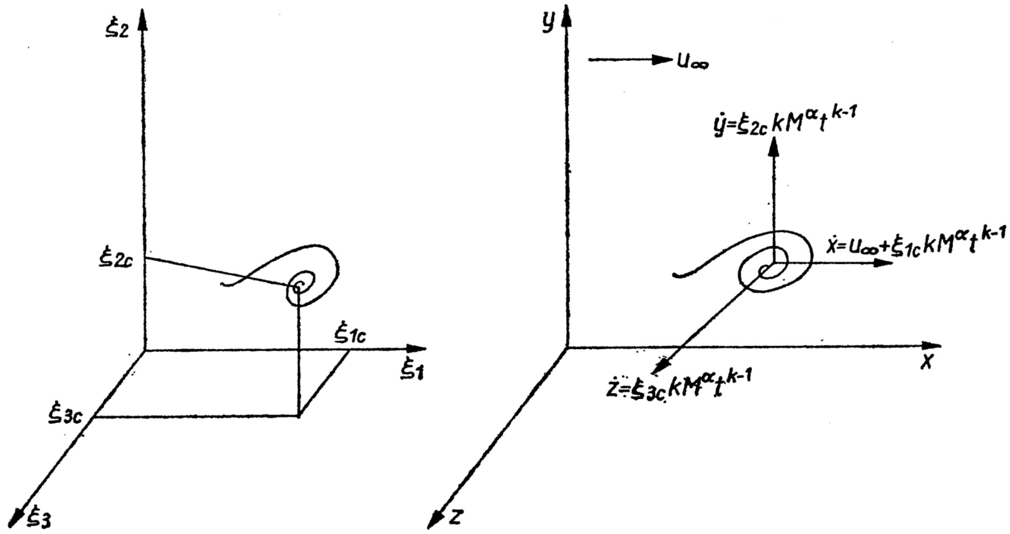

FIG. 1.

flow and ξ_2 and ξ_3 as cross-stream directions, then in physical coordinates the trajectory of the critical point is given by

$$x_c(t) = \xi_{1c} M^\alpha t^k + u_\infty t,$$
$$y_c(t) = \xi_{2c} M^\alpha t^k, \tag{2.6}$$
$$z_c(t) = \xi_{3c} M^\alpha t^k.$$

If we take $\delta = \sqrt{y_c^2 + z_c^2}$ as a cross-stream length scale and $u_0 = u_\infty - \dot{x}_c$ as a characteristic streamwise velocity, then two cases may be distinguished.

CASE 1. Jet-like flows ($u_\infty = 0$). Solving for t in terms of x_c we have $t \sim x_c^{1/k}$ and

$$\delta \sim x, \quad u_0 \sim x_c^{1-1/k}. \tag{2.7}$$

CASE 2. Wake-like flows ($u_\infty \neq 0$). Here we require $k \leqslant 1$ so that for large time $u_\infty t \gg \xi_{2c} M^\alpha t^k$. In this approximation the imposed free stream decouples streamwise convection from growth so that $x_c \sim t$ and

$$\delta \sim x_c^k, \quad u_0 \sim x_c^{k-1}. \tag{2.8}$$

If the parameter M has the units $L^m T^{-n}$, then we can solve for α and k in terms of m and n so that the similarity variable ξ_i is dimensionless. The result is

$$\alpha = 1/m, \quad k = n/m. \tag{2.9}$$

Using the relations (2.7), (2.8) and (2.9) plus the parameters which are used to characterize various turbulent shear flows we can construct Table 1. The exponents in x which

Table 1

Class	Flow	Parameter	Dimensions	α	k	Exponent in x: δ	Exponent in x: u_0	Comment
Wake-Like	mixing layer	u_0	LT^{-1}	1	1	1	0	velocity difference u_1-u_2
	turbulent spot	u_0	LT^{-1}	1	1	1	0	free stream velocity u_∞
	plane wake (laminar)	$u_0\,\delta, \nu$	L^2T^{-1}	1/2	1/2	1/2	$-1/2$	2-*D* far wake approximation, viscosity
	plane wake	$u_0\,\delta$	L^2T^{-1}	1/2	1/2	1/2	$-1/2$	2-*D* far wake approximation
	round wake	$u_0\,\delta^2$	L^3T^{-1}	1/3	1/3	1/3	$-2/3$	3-*D* far wake approximation
	grid turbulence	$u_0^2\,\delta^5$	L^7T^{-2}	1/7	2/7	2/7	$-5/7$	initial period of decay, Loitsianskii invariant
Jet-Like	plane plume	$gu_0\,\delta$	L^3T^{-3}	1/3	1	1	0	2-*D* buoyancy flux, units of g LT^{-2}
	round plume	$gu_0\,\delta^2$	L^4T^{-3}	1/4	3/4	1	$-1/3$	3-*D* buoyancy flux, units of g LT^{-2}
	vortex sheet rollup	β	$L^{\frac{3}{2}}T^{-1}$	2/3	2/3	1	$-1/2$	rollup from an impulsively started sharp edge. β is the potential flow parameter (KADEN 1931)
	plane jet	$u_0^2\,\delta$	L^3T^{-2}	1/3	2/3	1	$-1/2$	2-*D* momentum flux
	round jet (laminar)	$u_0^2\,\delta^2, \nu$	L^4T^{-2}, L^2T^{-1}	1/2	1/2	1	-1	3-*D* momentum flux, viscosity
	round jet	$u_0^2\,\delta^2$	L^4T^{-2}	1/4	1/2	1	-1	3-*D* momentum flux
	radial jet (laminar)	$u_0^2\,\delta^2, \nu$	L^4T^{-2}, L^2T^{-1}	1/2	1/2	1	-1	3-*D* momentum flux, viscosity
	radial jet	$u_0\,\delta^2$	L^4T^{-2}	1/4	1/2	1	-1	3-*D* momentum flux
	line vortex (laminar)	$u_0\,\delta, \nu$	L^2T^{-1}	1/2	1/2	1	-1	circulation, viscosity
	line vortex	$u_0\,\delta$	L^2T^{-1}	1/2	1/2	1	-1	circulation
	vortex pair	$\omega\delta^3$	L^3T^{-1}	1/3	1/3	1	-2	2-*D* impulse, ω-vorticity
	vortex ring	$\omega\delta^4$	L^4T^{-1}	1/4	1/4	1	-3	3-*D* impulse, ω-vorticity

are a consequence of the relations (2.7) and (2.8) are the same as the usual values found in standard texts. The new element here is that these two columns can be generated from the single column of k values. Note the convenient ordering of the flows in terms of monotonically changing values of k. Note further that flows with $k = 1$ could be classed as wake-like or jet-like and that for this case the large time approximation is unnecessary.

It is clear from the above that each of the flows listed in Table 1 can be described by a single global time scale. This is precisely what is required by the form of Eqs. (2.2) where lengths in all three coordinate directions are normalized by the *same* power of time. Moreover, from the analysis of the trajectory of the critical point it is clear that growth rates of structural features are not distinct from their convection rates; they are connected through the constant k. They are part and parcel of the temporal evolutionary process by which the fluid responds to externally imposed forces. In this respect we are able to include in our analysis flows where the velocity field (relative to a certain frame of reference) is perfectly steady since time is always a parameter along particle paths and a global time scale or, to be more precise, a constant whose dimension are a length divided by some power of time, may often be defined whether the flow is steady or unsteady.

It is important that the flow be reducible to only one global parameter or, if two parameters are involved, that they have commensurable units. For example, the laminar round jet has two parameters; J with units (L^4T^{-2}) and ν with units (L^2T^{-1}), both of which lead to the same value $k = 1/2$. As a second example the plane turbulent jet has the single parameter J with units (L^3T^{-2}) which leads to $k = 2/3$. We can turn the previous analysis around and arrive at the same k from the usual growth and decay laws

$$y \sim x,$$
$$\frac{dx}{dt} \sim x^{-\frac{1}{2}},$$

which upon integration lead to

$$x \sim t^{2/3},$$
$$y \sim t^{2/3}.$$

Both of these flows evolve in the x and y directions with the same power of time and can be accommodated by the system (2.2).

A counter example would be the plane laminar jet with parameters $J/\sqrt{\nu}$ with units $(L^2T^{-3/2})$ and ν with units (L^2T^{-1}) which lead to $k = 3/4$ and $k = 1/2$, respectively. We can solve for streamwise and transverse time scales along particle paths from

$$y \sim x^{2/3},$$
$$\frac{dx}{dt} \sim x^{-1/3},$$

which lead to

$$x \sim t^{3/4},$$
$$y \sim t^{1/2}.$$

Here it is necessary to make a boundary layer approximation to accommodate a length in the streamwise direction which grows according to the inertial time scale and a length in the transverse direction which grows with the viscous time scale. The plane laminar jet can not be accommodated by the system (2.2).

Essential to all of the above is the assumption that for turbulent flows the large scale motions are independent of ν. This assumption is intimately connected with the dependence of the flow Reynolds number on time

$$\mathrm{Re} = \frac{u_0\,\delta}{\nu} \sim \frac{kM^{2\alpha}}{\nu}\,t^{2k-1}.$$

For flows with $k = 1/2$, inertial and viscous times scale together and the assumption is unnecessary. For flows with $k > 1/2$ the inertial time will dominate at all but the smallest scales. However, flows with $k < 1/2$ will tend to follow a viscous scale as time increases. In particular, the assumptions of this paper will break down in the case of the round turbulent wake where the large times required for similarity are inconsistent with Reynolds number independence of the large scales.

3. Invariance of the pattern of particle paths

Under the assumption of non-steady similarity, the global parameter M determines the appropriate value of k. Once k has been determined, the rate of convection and growth of structural features in the flow (critical points, turbulent interfaces, etc.) is determined. If we choose to move (non-uniformly) with a coordinate system which remains attached to some preferred feature, then, in the moving coordinate system

$$\begin{aligned} x_i' &= x_i + a_i M^\alpha t^k, \\ t' &= t, \\ u_i' &= u_i + a_i k M^\alpha t^{k-1}, \\ p' &= p, \\ \tau_{ij}' &= \tau_{ij}, \end{aligned} \tag{3.1}$$

and the similarity variables in moving coordinates are

$$\begin{aligned} \xi_i' &= \xi_i + a_i, \\ U_i'(\xi') &= U_i(\xi) + k a_i, \\ P'(\xi') &= P(\xi), \\ T_{ij}'(\xi') &= T_{ij}(\xi) \end{aligned} \tag{3.2}$$

where a_i is a dimensionless rate of motion in the x_i direction.

It is clear from the above that the *pattern* formed by the velocity vector field will depend on the a_i. This is true whether one plots the u_i field in physical coordinates or the U_i field in similarity coordinates. Similarly, the pattern of particle displacements, dx_i, in physical coordinates will depend on the a_i. However, the pattern of particle displacements in similarity coordinates, $d\xi_i$, is independent of the a_i. This follows from

$$U_i(\xi) - k\xi_i = U_i'(\xi') - k a_i - k\xi_i' + k a_i = U_i'(\xi') - k\xi_i'. \tag{3.3}$$

Equation (3.3) is an important result for it states that the location and character of a critical point in similarity coordinates is fixed by the dynamics governing the flow and by the choice of a value for k (which is a consequence of the units of M) and *not* by the incidental choice of speed for a moving observer. The pattern of particle paths is invariant under translations in the ξ_i.

4. Unsteady critical points in the plane

The following is a generalization to arbitrary values of k of the analysis in CANTWELL, COLES and DIMOTAKIS (1978) for $k = 1$. See PERRY and FAIRLIE (1974) for a similar analysis of critical points in steady flows.

In two dimensions, $(\xi_1, \xi_2) \leftrightarrow (\xi, \eta)$ and $(U_1, U_2) \leftrightarrow (U, V)$, Eqs. (2.4) are

$$\begin{aligned} \frac{d\xi}{d\tau} &= U(\xi, \eta) - k\xi \equiv F_1(\xi, \eta), \\ \frac{d\eta}{d\tau} &= V(\xi, \eta) - k\eta \equiv F_2(\xi, \eta). \end{aligned} \tag{4.1}$$

If the flow is regular in the neighborhood of a critical point (ξ_c, η_c), then near the point Eqs. (4.1) can be linearized as

$$\begin{bmatrix} \dfrac{d\xi}{d\tau} \\ \dfrac{d\eta}{d\tau} \end{bmatrix} = \begin{bmatrix} \left.\dfrac{\partial F_1}{\partial \xi}\right|_{\xi_c,\eta_c} & \left.\dfrac{\partial F_1}{\partial \eta}\right|_{\xi_c,\eta_c} \\ \left.\dfrac{\partial F_2}{\partial \xi}\right|_{\xi_c,\eta_c} & \left.\dfrac{\partial F_2}{\partial \eta}\right|_{\xi_c,\eta_c} \end{bmatrix} \begin{bmatrix} \xi-\xi_c \\ \eta-\eta_c \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \xi-\xi_c \\ \eta-\eta_c \end{bmatrix} = [A] \begin{bmatrix} \xi-\xi_c \\ \eta-\eta_c \end{bmatrix}. \tag{4.2}$$

The velocity, vorticity and rate of strain fields in the neighborhood of the point are

$$\begin{aligned} u &= u_\infty + M^\alpha t^{k-1}\big(k\xi + a(\xi-\xi_c) + b(\eta-\eta_c)\big), \\ v &= M^\alpha t^{k-1}\big(c(\xi-\xi_c) + d(\eta-\eta_c) + k\eta\big), \\ \omega &= t^{-1}(c-b), \\ s_{ij} &= \begin{pmatrix} t^{-1}(a+k) & \dfrac{t^{-1}}{2}(b+c) \\ \dfrac{t^{-1}}{2}(b+c) & t^{-1}(d+k) \end{pmatrix}. \end{aligned} \tag{4.3}$$

The fluid velocities at the critical point are

$$\begin{aligned} u_c &= u_\infty + kM^\alpha t^{k-1}\xi_c, \\ v_c &= kM^\alpha t^{k-1}\eta_c. \end{aligned} \tag{4.4}$$

In this way the similarity coordinates of a critical point are directly related to the velocity history of the fluid at the point.

The character of the critical point is determined by the negative of the trace $p = -(a+d)$ and determinant $q = ad-bc$ of the matrix A. From continuity

$$p = 2k. \tag{4.5}$$

The presence of dissipation in real flows insures length scales which increase with time and for this reason negative values of k do not occur. The determinant of A is related to the vorticity and strain fields in the neighborhood of the critical point by

$$q = \Omega^2 - S^2 + k^2, \tag{4.6}$$

where $\Omega \equiv \omega t/2$ and $S \equiv t(-\mathrm{Det}\, s_{ij})^{\frac{1}{2}}$. The eigenvalues of A are

$$\begin{aligned} \lambda_1 &= -k-(k^2-q)^{\frac{1}{2}}, \\ \lambda_2 &= -k+(k^2-q)^{\frac{1}{2}} \end{aligned} \tag{4.7}$$

and are real only if $S \geqslant |\Omega|$. If $S = |\Omega|$, then $\lambda_1 = \lambda_2 = -k$. We can conveniently display the various possibilities for the critical points in a plot of p (or k) versus q (Fig. 2).

The separatrices through a saddle or node have slopes

$$m_{1,2} = \frac{\lambda_{1,2}-a}{b} = \frac{c}{\lambda_{1,2}-d} \tag{4.8}$$

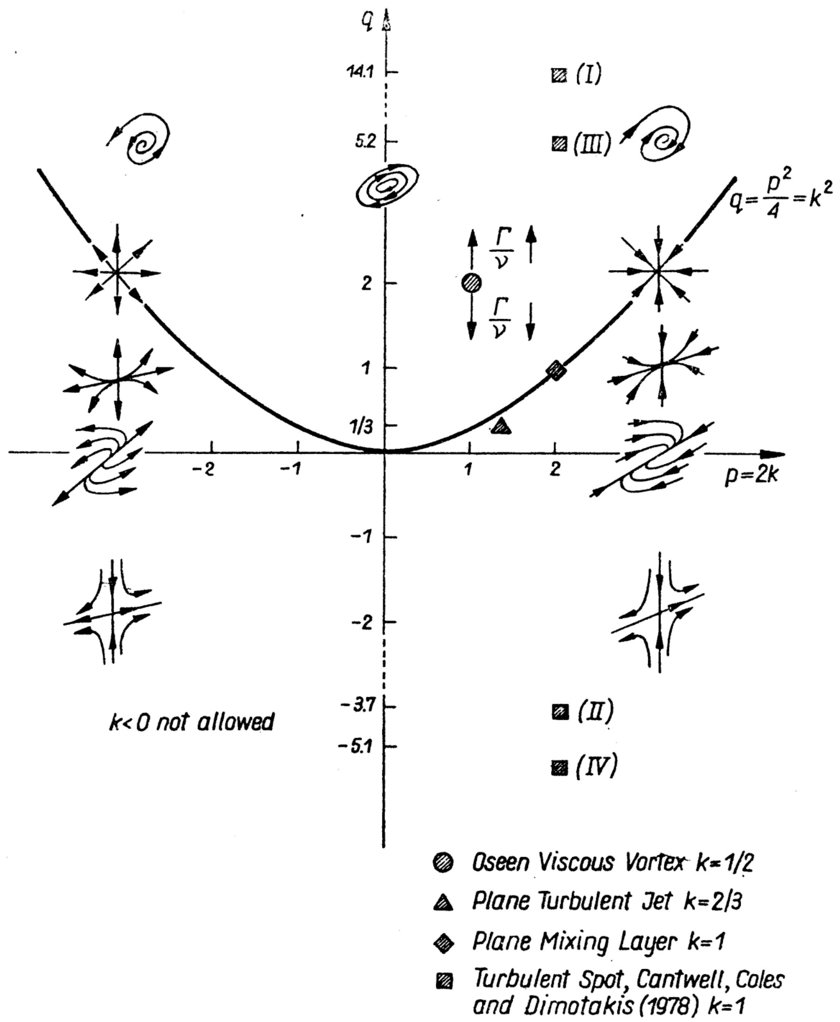

FIG. 2.

and the included angle, θ, between the separatrices is given by

$$\cos\theta = \frac{\frac{1}{2}(b-c)}{\left[-(k+a)(k+d)+\left(\frac{b+c}{2}\right)^2\right]^{\frac{1}{2}}} = -\frac{\Omega}{S}.$$

The angle between the separatrices through a saddle reflects a balance between rotational motion, Ω, and straining motion S. If $|\Omega| < S$, the fluid exhibits saddle behavior. As the ratio increases through one ($q > k^2$), the angle passes to zero; the vorticity dominates and the fluid begins to exhibit focal behavior. If the vorticity is zero, the angle is 90°.

We can illustrate some of the ideas just presented with some simple examples.

EXAMPLE 1. The viscous Oseen vortex. In cylindrical coordinates

$$v_\theta = \frac{\Gamma}{2\pi r}(1-e^{-r^2/4\nu t});\ v_r = 0,$$

where Γ is the circulation of the vortex. We can put this in the form of the system (2.1) by multiplying and dividing by $M^{\frac{1}{2}}t^{\frac{1}{2}}$ where $M = 4\nu$. If we define $R = r/\sqrt{4\nu t}$, the particle path equations become

$$\frac{dR}{d\tau} = -\frac{R}{2}; \quad \frac{d\theta}{d\tau} = \frac{\Gamma}{8\pi\nu}\frac{(1-R^2)}{R^2}$$

near the critical point $R = 0$

$$\frac{dR}{d\tau} = -\frac{R}{2}; \quad \frac{d\theta}{d\tau} = \frac{\Gamma}{8\pi\nu}.$$

In Cartesian coordinates with

$$\xi = x/\sqrt{4\nu t} \quad \text{and} \quad \eta = y/\sqrt{4\nu t},$$

$$\begin{bmatrix} \dfrac{d\xi}{d\tau} \\ \dfrac{d\eta}{d\tau} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{2} & -\dfrac{\Gamma}{8\pi\nu} \\ \dfrac{\Gamma}{8\pi\nu} & -\dfrac{1}{2} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

The critical point is a stable focus with $q = \left(\dfrac{\Gamma}{8\pi\nu}\right)^2 + \dfrac{1}{4}$ (Fig. 3).

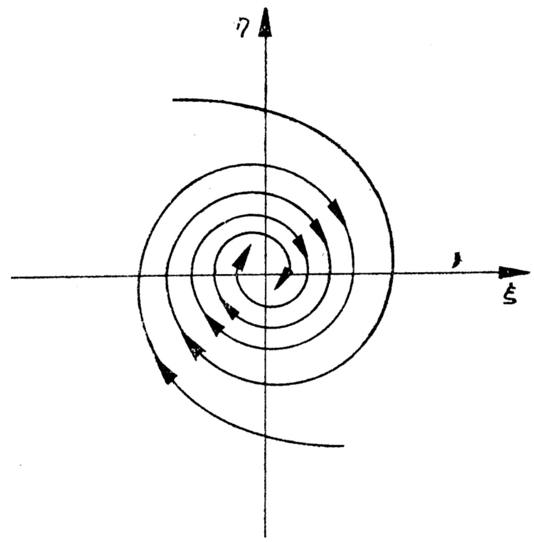

FIG. 3.

EXAMPLE 2. The plane mixing layer. Choose a stream function of the form

$$\psi = \left(\frac{u_1+u_2}{2}\right)y + \left(\frac{u_1-u_2}{2\sigma}\right)\times F(\sigma y/x),$$

where u_1 and u_2 are the velocities on the high and low speed sides and σ is a dimensionless rate of spread parameter. The function F is defined so that $F'(\infty) = 1$ and $F'(-\infty) = -1$. As in Example 1 we can recast this steady flow in unsteady terms with $M = u_1-u_2 = \Delta u$, $\alpha = 1$ and $k = 1$ so that

$$\psi = \Delta u^2 t\left\{\frac{\eta}{2\lambda} + \frac{\xi}{2\sigma}F(\sigma\eta/\xi)\right\},$$

where $\lambda = \dfrac{u_1 - u_2}{u_1 - u_2}$, $\xi = x/\Delta ut$ and $\eta = y/\Delta ut$. Using this form the particle path equations become

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$$\frac{d\xi}{d\tau} = \frac{1}{2\lambda} + \frac{F''}{2} - \xi,$$

$$\frac{d\eta}{d\tau} = -\frac{F}{2\sigma} + \frac{\eta F'}{2\xi} - \eta$$

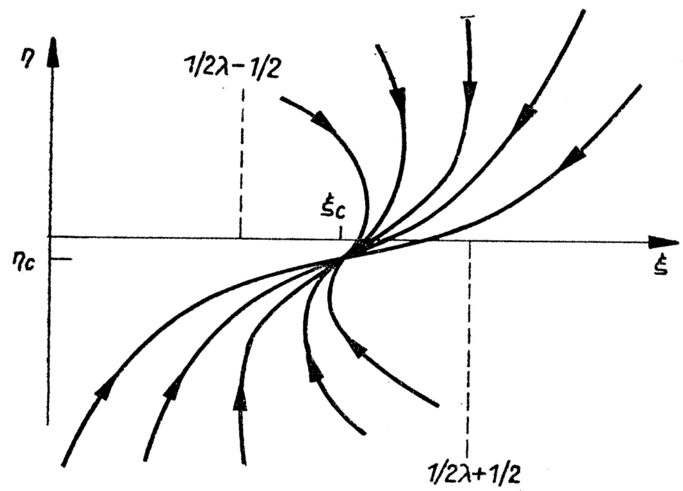

FIG. 4.

with a critical point whose coordinates (ξ_c, η_c) trace out the dividing stream line of the layer ($\eta_c/\xi_c = V/U$). In the neighborhood of the critical point the matrix of slopes becomes

$$\begin{bmatrix} \dfrac{d\xi}{d\tau} \\[2ex] \dfrac{d\eta}{d\tau} \end{bmatrix} = \begin{bmatrix} -\dfrac{\sigma\eta}{2\xi^2}F'' - 1 & \dfrac{\sigma}{2\xi}F'' \\[2ex] -\dfrac{\sigma\eta^2}{2\xi^2}F'' & \dfrac{\sigma\eta}{2\xi^2}F'' - 1 \end{bmatrix}_{\xi_c, \eta_c} \begin{bmatrix} \xi - \xi_c \\[2ex] \eta - \eta_c \end{bmatrix}.$$

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The vorticity at (ξ_c, η_c) is

$$\omega = \frac{1}{t}\left(-\frac{\sigma}{2\xi_c}\right)\left(\frac{\eta_c^2}{\xi_c^2} + 1\right) F''\left(\sigma\frac{\eta_c}{\xi_c}\right).$$

For any F which depends only on the ratio η/ξ the critical point is a star with $q = 1$ (Fig. 4). If, for example, we choose $F = \ln 2\cosh\left(\dfrac{\sigma y}{x}\right)$ which assumes $v(\pm\infty) = 0$, then the equations for ξ_c and η_c become

$$\sigma\frac{\eta_c}{\xi_c} = -\lambda \ln\left(2\cosh\left(\sigma\frac{\eta_c}{\xi_c}\right)\right),$$

$$\xi_c = \frac{1}{2\lambda} + \frac{1}{2}\tanh\left(\sigma\frac{\eta_c}{\xi_c}\right).$$

For the case $u_2/u_1 = 1/\sqrt{7}$ the critical point occurs at $(\xi_c, \eta_c) = (0.945, -0.32/\sigma)$ which corresponds to a dividing streamline at $\sigma y_c/x_c = -0.34$. Notice that we could have treated this flow as wake-like with $u_\infty = \dfrac{u_1+u_2}{2}$.

EXAMPLE 3. The plane turbulent jet. Choose a stream function of the form

$$\psi = \left(\frac{Jx}{\sigma}\right)^{\frac{1}{2}} F\left(\sigma \frac{y}{x}\right),$$

where J is the momentum flux of the jet with units L^3T^{-2} and σ is a dimensionless rate of spread parameter. The function F is defined so that $F(0) = 0$, $F'(0) = 1$ and $F''(0) = 0$. We can recast this stream function into unsteady form with $M = J$, $\alpha = 1/3$ and $k = 2/3$ so that

$$\psi = J^{2/3}t^{1/3}\left\{\sigma^{-\frac{1}{2}}\xi^{\frac{1}{2}}F\left(\sigma\frac{\eta}{\xi}\right)\right\},$$

where $\xi = x/J^{1/3}t^{2/3}$, $\eta = y/J^{1/3}t^{2/3}$. Using this form the particle path equations become

$$\frac{d\xi}{d\tau} = \sigma^{\frac{1}{2}}\xi^{-\frac{1}{2}}F' - \frac{2}{3}\xi,$$

$$\frac{d\eta}{d\tau} = \sigma^{\frac{1}{2}}\eta\xi^{-3/2}F' - \frac{\sigma^{-\frac{1}{2}}}{2}\xi^{-\frac{1}{2}}F - \frac{2}{3}\eta.$$

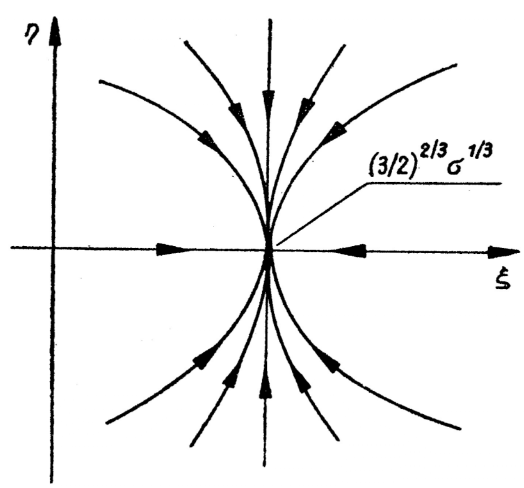

FIG. 5.

A critical point of the above equations occurs at $(\xi_c, \eta_c) = ((3/2)^{2/3}\sigma^{1/3}, 0)$. The matrix of slopes in the neighborhood of (ξ_c, η_c) is

$$\begin{bmatrix} \dfrac{d\xi}{d\tau} \\[2mm] \dfrac{d\eta}{d\tau} \end{bmatrix} = \begin{bmatrix} -\dfrac{\sigma^{1/2}}{2}\xi_c^{-3/2} - \dfrac{2}{3} & 0 \\[2mm] 0 & \dfrac{1}{2}\sigma^{1/2}\xi_c^{-3/2} - \dfrac{2}{3} \end{bmatrix} \begin{bmatrix} \xi - \left(\dfrac{3}{2}\right)^{2/3}\sigma^{1/3} \\[2mm] \eta \end{bmatrix}.$$

Upon substitution of ξ_c the determinant

$$q = -\frac{1}{9} + \left(\frac{2}{3}\right)^2.$$

The critical pointis a node with $q = \frac{1}{3}$. The angle between the separatrices is 90° (Fig. 5).

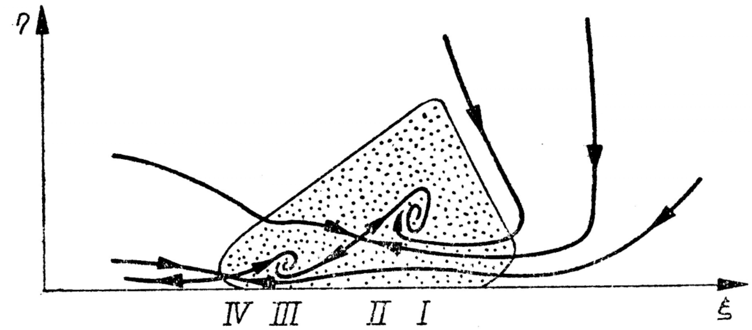

FIG. 6.

CANTWELL, COLES and DIMOTAKIS (1978) identified four critical points in the turbulent spot by fitting $\psi = u_\infty^2 tg(x/u_\infty t, y/u_\infty t)$ to measurements on the plane of symmetry. There were two saddles and two foci shown schematically in Fig. 6. Values of q at each critical point are plotted in Fig. 2.

6. Concluding remarks

The consequences of the assumption of non-steady similarity include a reduction in the number of independent variables by one and a connection between characteristic lengths and velocities in a turbulent flow via the parameter k. Perhaps most important is the fact that structural features of the velocity field are brought out automatically without reference to an observer who translates with any particular feature.

An operational property of the systems (2.2) and (2.3) is that the entire flow is contained within a finite domain near the origin in similarity coordinates. This is clear from the particle path equations (2.4) where the U_i are limited by the velocity of the fastest moving particles in the flow. Far away from the origin the particle displacements are dominated by the velocity induced by the receding observer $k\xi_i$. This feature may make the systems (2.2) and (2.3) particularly well suited to computational methods.

The examples of the turbulent shear layer and turbulent plane jet are included here to demonstrate that such flows can be formulated as unsteady but similar in the sense of the system (2.2). Normally, these flows are measured by a laboratory observer who takes a long time average at a fixed position in physical coordinates (x, y) with the function $F(\sigma y/x)$ the empirically measured result. However, from the results of Sect. 2, it is clear that another kind of average is possible. Operationally this would be a long time average referred to a receding observer who looks at the flow quite literally through the zoom lens of a camera(1). The rate of zoom is adjusted to match the value of M and the

(1) The zoom lens analogy was first suggested by D. COLES (private communication).

averaging time of the experiment is limited by the physical size of the apparatus which contains the flow.

Turbulent flows are normally thought of as having infinite extent in space and time. Here we have taken an alternate view in which the flow is thought of as having been started at some initial time. Fluctuations in this evolving flow field are assumed to follow the same time scale as the coherent motion and are averaged by an observer who recedes out of the plane of motion at a rate which is determined by the global parameter which governs the flow in question. The receding observer fixes the slopes of particle trajectories by merely adding a virtual velocity vector $k\xi_i$ to each point in his field of view. It seems reasonable to conjecture that experimental solutions based on through-the-zoom-lens averaging or theoretical solutions of the system (2.3) will exhibit a dependence on ξ and η which is much more complex than the simple ratio η/ξ (cf. the turbulent spot). Such solutions are likely to contain critical points and in this regard the system (2.3) along with some appropriate model for the T_{ij} is a promising candidate in the search for large eddy solutions of the equations of motion.

References

1. P. M. BEVILAQUA and P. S. LYKOUDIS, *Mechanism of entrainment in turbulent wakes*, AIAA J., **9**, 1657–1659, 1971.
2. G. BIRKHOFF, *Hydrodynamics*, Princeton Univ. Press, 1950.
3. G. L. BROWN and A. ROSHKO, *On density effects and large structure in turbulent mixing layers*, J. Fluid Mech., **64**, 775–816, 1974.
4. B. J. CANTWELL, *Similarity transformations for the two-dimensional unsteady stream function equation*, J. Fluid Mech., **85**, 257–271, 1978.
5. B. J. CANTWELL, D. E. COLES and P. E. DIMOTAKIS, *Structure and entrainment in the plane of symmetry of a turbulent spot*, J. Fluid Mech., **87**, p. 641, 1978.
6. J. C. CROW and F. H. CHAMPAGNE, *Orderly structure in jet turbulence*, J. Fluid Mech., **48**, 547–591, 1971.
7. J. C. R. HUNT, C. J. ABELL, J. A. PETERKA and H. WOO, *Kinematical studies of the flows around free or surface-mounted obstacles; applying topology to flow visualization*, J. Fluid Mech., **86**, 179–200, 1978.
8. H. KADEN, *Aufwicklung einer unstabilen Unstetigkeitsflache*, Ing. Arch., **2**, 2, 140–168, 1931.
9. M. J. LIGHTHILL, *In laminar boundary layers* (L. ROSENHEAD, ed.), 48–88, Oxford Univ. Press Clarendon, London and New York 1963.
10. K. OSWATITSCH, *Die Ablösungsbedingung von Grenzschichten, Grenzschicht Forschung* (H. GOERTLER, ed.), p. 357, Springer Verlag, Berlin and New York 1958.
11. A. E. PERRY and B. D. FAIRLIE, *Critical points in flow patterns*, IUTAM-IUGG Symp. on Turbulent Diffusion in Environmental Pollution, Charlottesville, Va. (Adv. in Geophysics **18 B**, 1974), 299–315, 1973.
12. S. TANEDA, *Downstream development of the wakes behind cylinders*, J. Phys. Soc. of Japan, **14**, 843–848, 1959.
13. J. S. TURNER, *The flow into an expanding spherical vortex*, J. Fluid Mech., **18**, 195–208, 1964.

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Received August 10, 1978.