MECH427/527 and AA 284a
Advanced Rocket Propulsion

Lecture 2
Thrust Equation, Nozzles and Definitions

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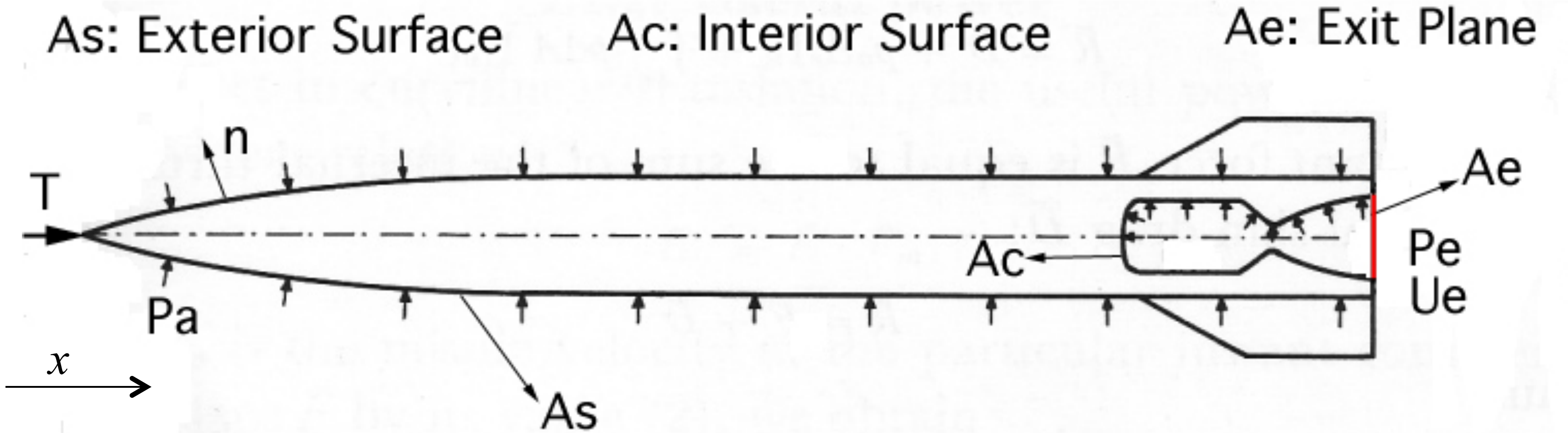
Derivation of the Static Thrust Expression



- Force balance in the *x* direction

$$T + \int_{A_s} P_a \hat{n}\, dA\Big|_x + \int_{A_c} \left(P\bar{\bar{I}} - \bar{\bar{\tau}}\right)\cdot \hat{n}\, dA\Big|_x = 0$$

- *Assumption 1:* Static firing/ External gas is at rest
- *Assumption 2:* No body forces acting on the rocket

$\bar{\bar{I}}$: *Unity Tensor*

$\bar{\bar{\tau}}$: *Stress Tensor*

P : Pressure

P_a : *Ambient* Pressure

P_e : *Exit* Pressure

U_e : *Exit Velocity*

T : *Thrust Force*

Derivation of the Static Thrust Expression

- First Integral:
  - Since A_s+A_c and A_e+A_c are closed surfaces and P_a is constant

$$\int_{A_s+A_c} P_a \hat{n}\ dA\Big|_x = 0 \qquad \int_{A_c+A_e} P_a \hat{n}\ dA\Big|_x = 0$$

  - These integrals can be separated and combined to yield the following simple expression for the first integral

$$\int_{A_s} P_a \hat{n}\ dA\Big|_x = P_a A_e$$

Derivation of the Static Thrust Expression

- Second Integral:
  - *Assumption 3:* No body forces on the working gas
  - Momentum equation:

$$\frac{\partial \rho\, \bar{u}}{\partial t} + \nabla \cdot \left(\rho\, \bar{u}\bar{u} + P\bar{\bar{I}} - \bar{\bar{\tau}} \right) = 0$$

$\bar{u}$: *Velocity Vector*

ρ : *Density*

  - *Assumption 4:* quasi-steady operation
  - Define control volume (*cv)* as volume covered by A_c+A_e
  - *Assumption 5:* cv is constant in time
  - Integral of the momentum eq. over the cv

$$\int_{cv} \nabla \cdot \left(\rho\, \bar{u}\bar{u} + P\bar{\bar{I}} - \bar{\bar{\tau}} \right) dv = 0$$

Derivation of the Static Thrust Expression

- Second Integral:
  - Use Gauss's Theorem to obtain

$$\int_{A_c} \left(\rho\, \bar{u}\bar{u} + P\bar{\bar{I}} - \bar{\bar{\tau}}\right) \cdot \hat{n}\; dA + \int_{A_e} \left(\rho\, \bar{u}\bar{u} + P\bar{\bar{I}} - \bar{\bar{\tau}}\right) \cdot \hat{n}\; dA = 0$$

  - With use of the no slip condition, this equation takes the following form in the *x*-direction

$$\int_{A_c} \left(P\bar{\bar{I}} - \bar{\bar{\tau}}\right) \cdot \hat{n}\; dA\Big|_x + \rho_e u_e^2 A_e + P_e A_e = 0 \quad \Longrightarrow \quad \int_{A_c} \left(P\bar{\bar{I}} - \bar{\bar{\tau}}\right) \cdot \hat{n}\; dA\Big|_x = -\rho_e u_e^2 A_e - P_e A_e$$

  - We have used the following assumptions
    - *Assumption 6:* Quasi 1D flow at the nozzle exit. Higher order averaging terms are ignored. Velocity parallel to *x*-axis at the exit plane
    - *Assumption 7:* $\int_{A_e} \bar{\bar{\tau}} \cdot \hat{n}\; dA\Big|_x \cong 0$
    - Average quantities have been introduced at the exit plane

$$P_e \equiv \int_{A_e} P\, dA\Big|_x \qquad u_e \equiv \int_{A_e} \bar{u} \cdot \hat{n}\; dA\Big|_x \qquad \rho_e \equiv \int_{A_e} \rho\; dA\Big|_x$$

Derivation of the Static Thrust Expression

$\dot{m} : Mass\ Flow\ Rate$

- Combined to obtain the thrust force

$$T = \rho_e u_e^2 A_e + (P_e - P_a) A_e$$

- Introduce the mass flow rate: $\dot{m} = \rho_e u_e A_e$

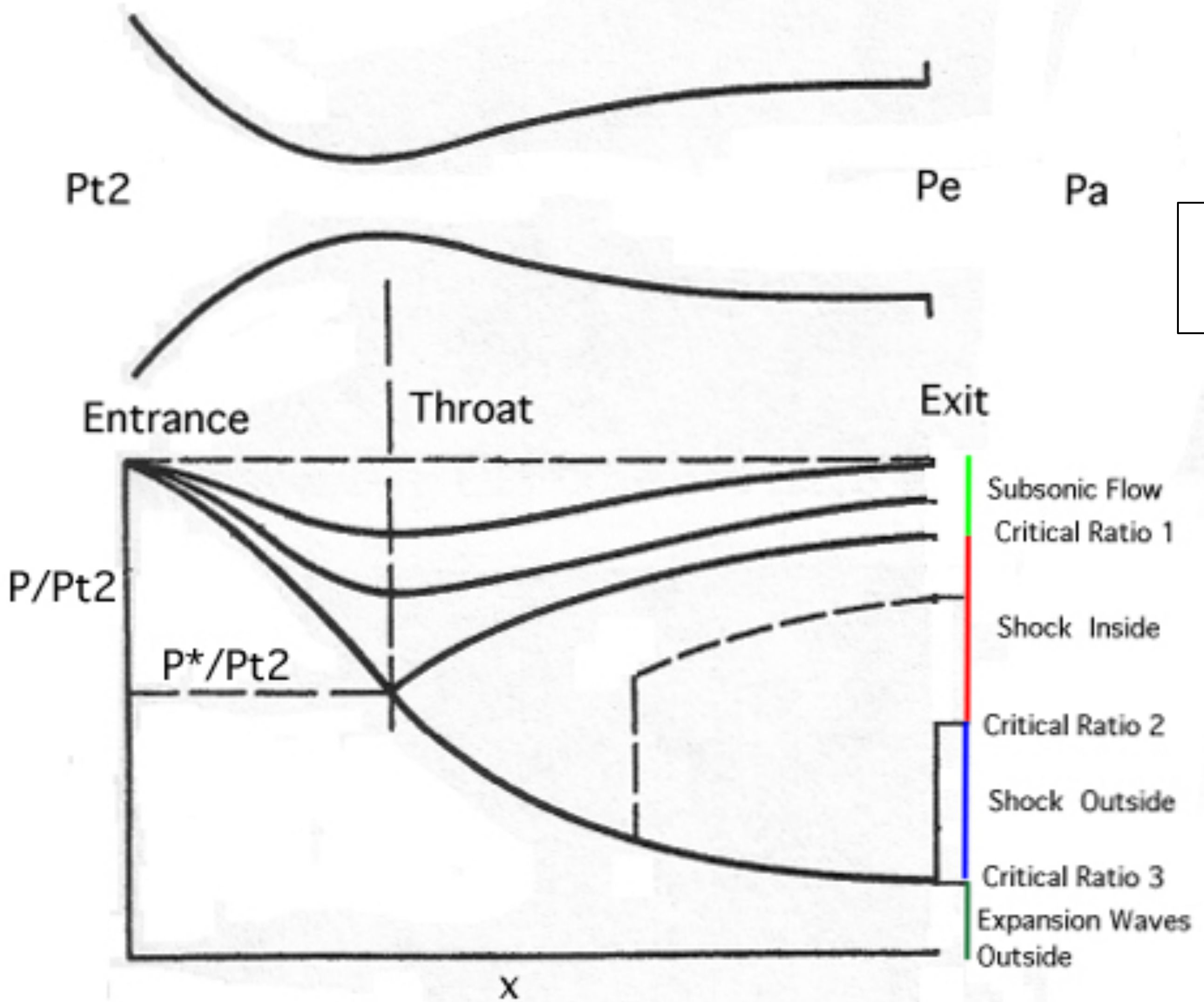
$$T = \dot{m}\, u_e + (P_e - P_a) A_e$$

- Two terms can be combined by introducing the effective exhaust velocity, V_e

$$T = \dot{m} V_e$$

- Maximum thrust for unit mass flow rate requires
  - High exit velocity
  - High exit pressure
- This cannot be realized. Compromise -> optimal expansion

Convergent Divergent Nozzle Design Issues



P_{t2} : *Chamber Stagnation Pressure*

P^* : Pressure @ $M = 1$

Convergent Divergent Nozzle Design Issues

- For isentropic flow - perfect gas-no chemical rxns

1. Mass flow rate relation for chocked flow

$$\dot{m} = \frac{A_t P_{t2}}{\sqrt{R T_{t2}}} \left[\gamma \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{\gamma-1}} \right]^{1/2}$$

T_{t2} : *Chamber Stagnation Temperature*
R : *Gas Cons.*
R_u : *Universal Gas Cons.*

2. Area relation (Nozzle area ratio-pressure ratio relation)

$$\frac{1}{\varepsilon} \equiv \frac{A_t}{A_e} = \left(\frac{\gamma+1}{2} \right)^{\frac{1}{\gamma-1}} \left(\frac{P_e}{P_{t2}} \right)^{\frac{1}{\gamma}} \left\{ \left(\frac{\gamma+1}{\gamma-1} \right) \left[1 - \left(\frac{P_e}{P_{t2}} \right)^{\frac{\gamma-1}{\gamma}} \right] \right\}^{\frac{1}{2}}$$

3. Velocity relation

$$u_e = \left\{ \frac{2\gamma}{\gamma-1} \frac{R_u T_{t2}}{M_w} \left[1 - \left(\frac{P_e}{P_{t2}} \right)^{\frac{\gamma-1}{\gamma}} \right] \right\}^{\frac{1}{2}}$$

Mw : *Molecular Weight*
A_t : *Nozzle Throat Area*
ε : *Nozzle Area Ratio*
γ : *Ratio of Specific Heats*

- Solve for pressure ratio from 2 and evaluate the velocity using 3.

Maximum Thrust Condition

- Thrust equation:

$$T = \dot{m}\, u_e + \left(P_e - P_a\right) A_e$$

- At fixed flow rate, chamber and atmospheric pressures, the variation in thrust can be written as

$$dT = \dot{m}\, du_e + \left(P_e - P_a\right) dA_e + A_e\, dP_e$$

- Momentum equation in 1D

$$A \rho u\, du + A\, dP = 0 \qquad \dot{m}\, du = -A\, dP$$

- Substitute in the differential expression for thrust

$$dT = \left(P_e - P_a\right) dA_e \qquad \frac{dT}{dA_e} = \left(P_e - P_a\right)$$

- Maximum thrust is obtained for a perfectly expanded nozzle

$$P_e = P_a$$

Other Nozzle Design Issues

A) Equilibrium:

- Mechanical:
  - Needed to define an equilibrium pressure
  - Very fast compared to the other time scales
- Thermal:
  - Relaxation times associated with the internal degrees freedom of the gas
  - Rotational relaxation time is fast compared to the other times
  - Vibrational relaxation time is slow compare to the rotational. Can be important in rocket applications.
  - Calorically perfect gas assumption breaks down.
- Chemical:
  - Finite time chemical kinetics (changing temperature and pressure)
  - Three cases are commonly considered:
    - Fast kinetics relative to residence time- Shifting equilibrium (Chemical composition of the gases match the local equilibrium determined by the local pressure and temperature in the nozzle)
    - Slow kinetics relative to residence time- Frozen equilibrium (Chemical composition of the gases is assumed to be fixed)
    - Nonequilibrium kinetics (Nozzle flow equations are solved simultaneously with the chemical kinetics equations)

Other Nozzle Design Issues

B) Calorically perfect gas

- Typically not valid. Chemical composition shifts, temperature changes and vibrational non-equilibrium.

C) Effects of Friction

- Favorable pressure gradient unless shock waves are located inside the nozzle.
- The effects can be list as
  - Nozzle area distribution changes due to displacement layer thickness
  - Direct effect of the skin friction force
  - Shock boundary layer interaction-shock induced separation.
  - Viscous effects are small and generally ignored for shock free nozzles

D) Multi Phase Flow Losses

E) Effects of 3D flow field

- Velocity at the exit plane is not parallel to the nozzle axis, because of the conical flow field.

Other Nozzle Design Issues

– Types of nozzle geometries

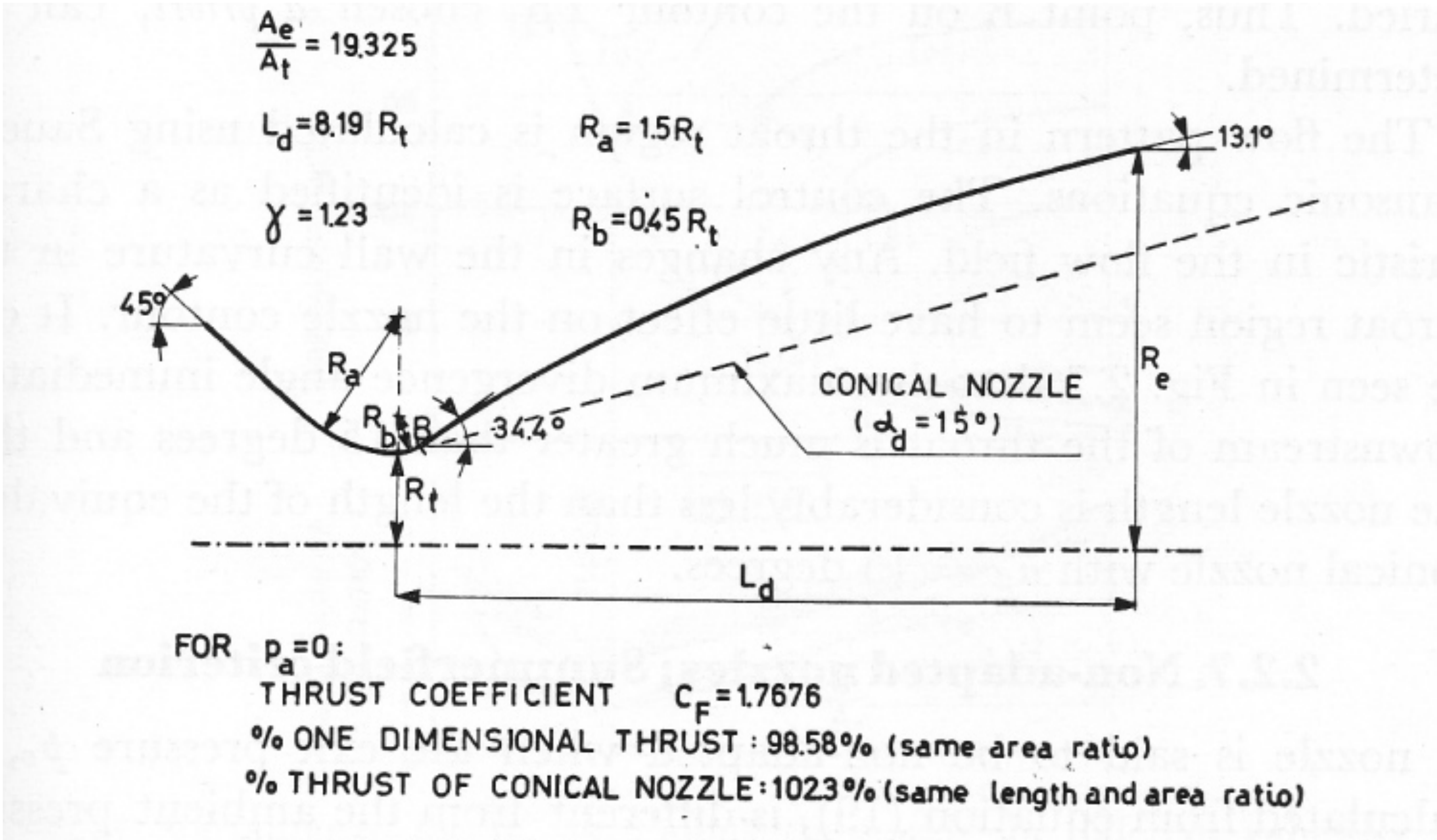
- Conical nozzle
  - Simple design and construction
  - Typical divergence angle 15 degrees (~2% Isp loss)
  - 3D thrust correction can be significant

$$T = \dot{m}\ u_e \frac{1+\cos(\alpha)}{2} + (P_e - P_a)A_e$$

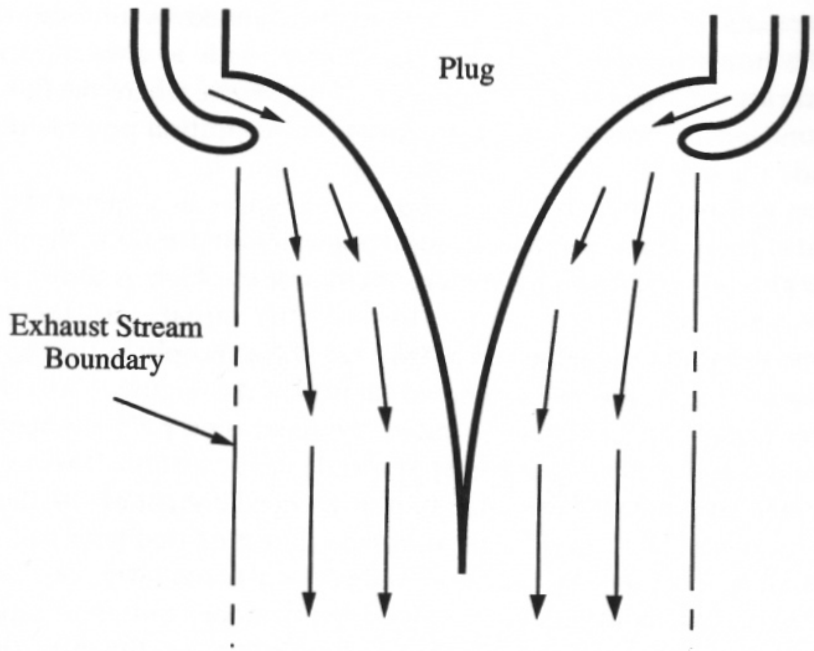
α : Nozzle Cone Angle

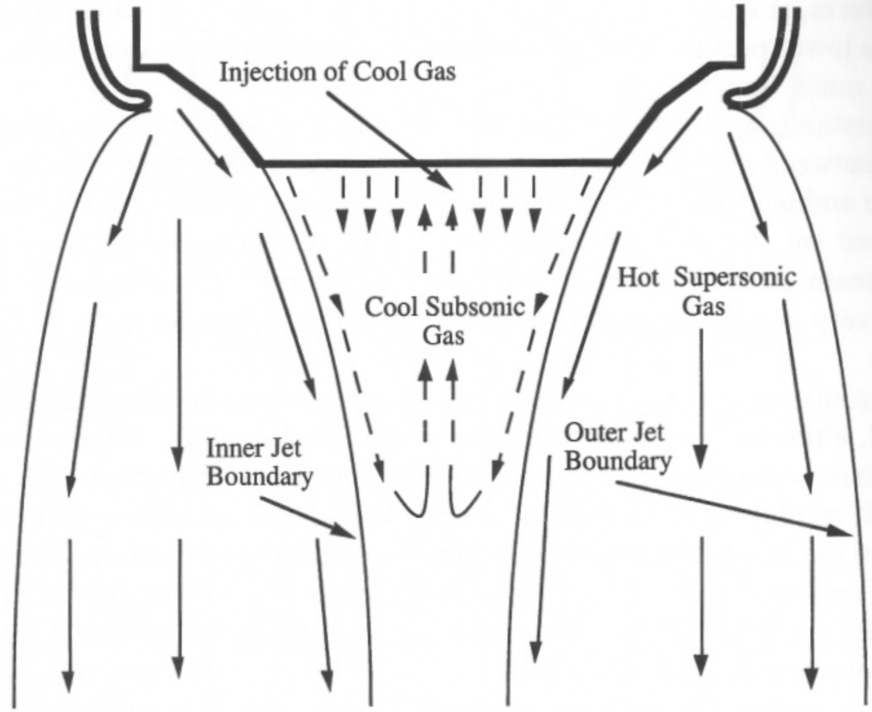
- Perfect nozzle
  - Method of characteristics to minimize 3D losses
  - Perfect nozzle is too long
- Optimum nozzle (Bell shaped nozzles)
  - Balance length/weight with the 3D flow losses
- Plug nozzle and Aerospike nozzle
  - Good performance over a wide range of back pressures

Optimum Bell Nozzle - Example



Plug and Aerospike Nozzles





Definitions-Thrust Coefficient

- Thrust equation:

$$T = \dot{m}\, u_e + \left(P_e - P_a\right) A_e$$

- Thrust coefficient:

$$C_F \equiv \frac{T}{A_t P_{t2}} = \frac{\dot{m}\, u_e}{A_t P_{t2}} + \left(\frac{P_e}{P_{t2}} - \frac{P_a}{P_{t2}}\right)\frac{A_e}{A_t}$$

- For isentropic flow and calorically perfect gas in the nozzle the thrust coefficient can be written as

$$C_F = \left\{\left(\frac{2\gamma^2}{\gamma - 1}\right)\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1 - \left(\frac{P_e}{P_{t2}}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} + \left(\frac{P_e}{P_{t2}} - \frac{P_a}{P_{t2}}\right)\frac{A_e}{A_t}$$

Definitions - c* Equation

- Mass flow equation (choked and isentropic flow of a calorically perfect gas in the convergent section of the nozzle):

$$\dot{m} = \frac{A_t P_{t2}}{\sqrt{R T_{t2}}} \left[\gamma \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{\gamma-1}} \right]^{1/2}$$

- Definition of c*:

$$c^* \equiv \frac{P_{t2} A_t}{\dot{m}}$$

- c* can be expressed in terms of the operational parameters as

$$c^* = \left[\frac{1}{\gamma} \left(\frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{\gamma-1}} \frac{R_u T_{t2}}{M_w} \right]^{1/2}$$

Definitions – Specific Impulse and Impulse density

- Combine the definitions of the thrust coefficient and c* to express the thrust

$$T = \dot{m}\, c^* C_F$$

Energy of the gases In the chamber (c^*) — Flow in the nozzle (C_F)

- Think of nozzle as a thrust amplifier and C_F as the gain
- Specific Impulse: Thrust per unit mass expelled

$$I_{sp} \equiv \frac{T}{\dot{m}\, g_o} = \frac{c^* C_F}{g_o}$$

g_o : *Gravitational Cons. on Earth*
ρ_p : *Propellant Density*

- Impulse Density: Thrust per unit volume of propellant expelled

$$\delta \equiv \frac{T}{\dot{V}_p} = \frac{T}{\dot{m}\, g_o}\frac{\dot{m}\, g_o}{\dot{V}_p} = I_{sp}\rho_p \qquad \dot{V}_p \equiv \dot{m}/\rho_p$$

Definitions – Total Impulse, Average Thrust, Delivered Isp

- The total impulse is defined as

$$I_{tot} \equiv \int_0^{t_b} T dt = \int_0^{t_b} \dot{m} \, I_{sp} g_o dt$$

- Average thrust

$$\bar{T} \equiv \frac{1}{t_b} \int_0^{t_b} T dt = \frac{I_{tot}}{t_b}$$

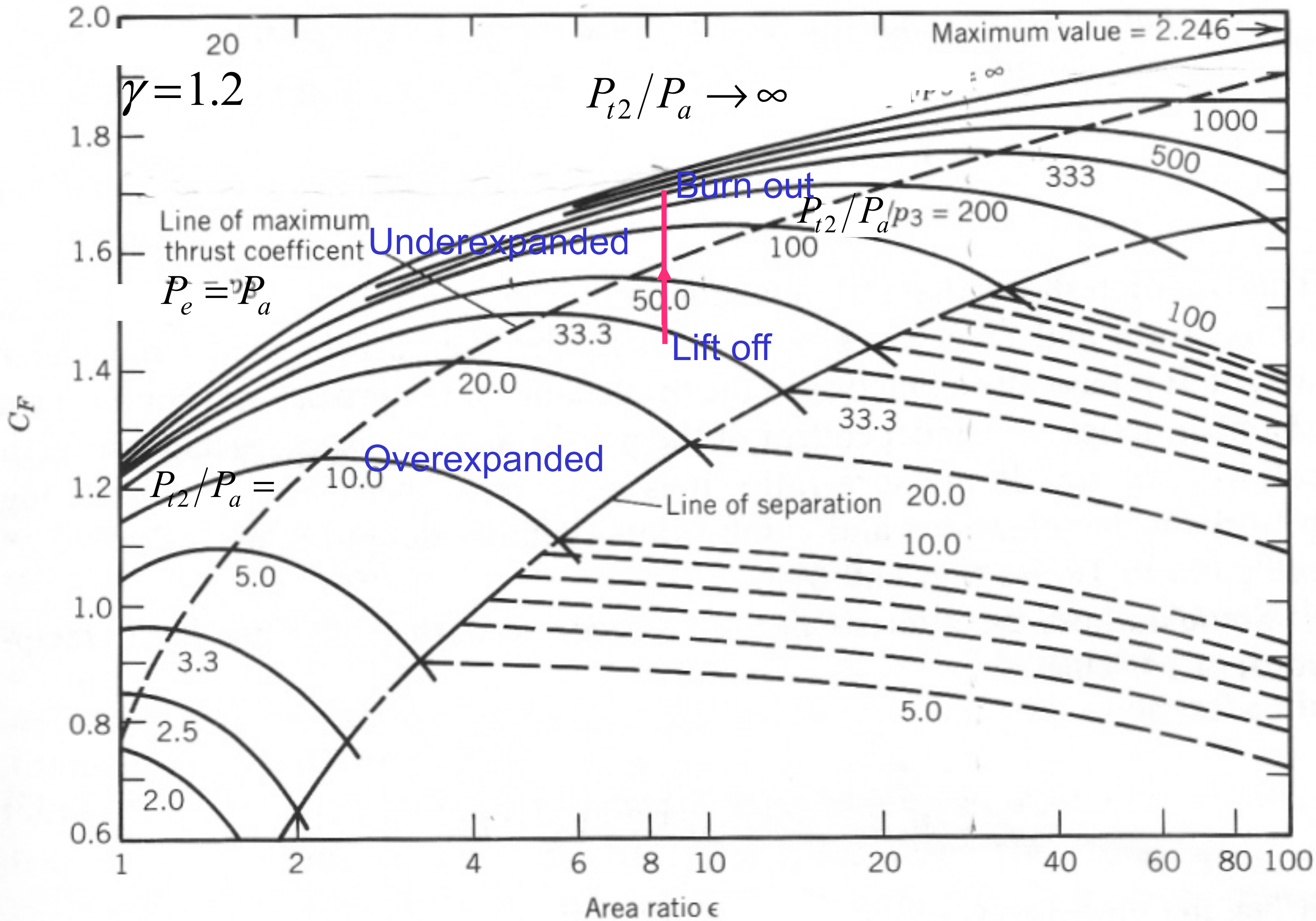
t_b : Total Burn Time

M_p : Total Propellant Mass

- Delivered I_{sp}

$$\left(I_{sp}\right)_{del} = \frac{I_{tot}}{M_p g_o}$$

Thrust Coefficient Curves (from Sutton)



Observations on the Thrust Coefficient Curves

- For the specific value of $\gamma = 1.2$
- C_F varies from 0.75 to 2.246
  - Assuming chocked flow
  - C_F is order 1
- For the case of convergent nozzle $\varepsilon = 1$
  - For $P_{t2}/P_a < 2.3$ subsonic flow in the nozzle
  - C_F varies from 0.75 (optimal) to 1.25
- For convergent divergent nozzles $\varepsilon > 1$
  - For a given pressure ratio P_{t2}/P_a there exists an optimum area ratio that maximizes the thrust coefficient
  - Right of the optimum is overexpanded
    - Shock structure is initially outside the nozzle
    - For larger area ratios shock moves inside the nozzle and the boundary separation takes place
    - This has a positive impact on the C_F

Observations on the Thrust Coefficient Curves

  - Left of the optimal is underexpanded
    - No separation in this case
- Optimal area ratio increases with increasing pressure ratio
  - Upper stages have large nozzle area ratios (i.e.70)
  - Booster stages have low area ratios (i.e. 10)
- Vacuum Isp and sea level Isp values can be quite different
  - For example:
    - Vacuum value: 1.9
    - Sea Level value: 1.2 (58% reduction in thrust)
- All curves are enveloped by the vacuum line. Vacuum Isp is always the largest value.
- All of the qualitative observations are valid for other gamma values.

Shock Induced Flow Separation in Nozzles

- Summerfield Criterion:
  - Danger of separation is present if

$$P_{t2} / P_a > 16 \qquad \alpha_e \approx 15^o$$

  - Calculate P_e from isentropic flow equations
  - Separation is likely if

$$P_e / P_a < \left(P_e / P_a\right)_{cr} \cong 0.40$$

  - If $\alpha_e > 15^o$ $\left(P_e / P_a\right)_{cr}$ can be lower
  - For large nozzles modern data suggests

$$P_e / P_a < \left(P_e / P_a\right)_{cr} \cong 0.286$$

- With separation flow ignores the divergent section beyond the starting point of the oblique shock wave
- This limits the drop in the thrust coefficient
- Conical nozzles operate better at low P_e/P_a ratios for which the separation is expected.
  - Nozzle exit divergence angle determines the stability of the separation zone. As the angle increases, the stability of the separation zone improves