J. W. Miller. 20 1821

A

TREATISE

ON THE

MOTION OF ROCKETS:

TO WHICH IS ADDED,

AN ESSAY ON NAVAL GUNNERY,

IN

THEORY AND PRACTICE;

DESIGNED FOR THE USE OF THE

ARMY AND NAVY,

AND ALL PLACES OF

MILITARY, NAVAL, AND SCIENTIFIC INSTRUCTION.

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PREFACE.

It was not till the year 1810, when the Academy of Copenhagen proposed as a prize question, the curve that a rocket describes, when projected, in any oblique direction, in vacuo, that I was led to consider the theory of the motion of rockets in different mediums. Since that period, I have at different times published, in the Philosophical Journal, some short and incomplete papers on this subject; but finding that my enquiries would extend to a considerable length, and make a tolerable size treatise, which to military and other students would not be altogether useless, I resolved to arrange the matter which those contained with that of my other investigations, and publish them with another new theory on Naval Gunnery, in a volume collectively.

This, then, may be considered my apology for laying before the public the present work;—of the plan of arrangement of which, and of the principal articles which it contains, the following is a brief outline.

Previously to entering upon the theory of rockets, I have judged it not improper to lay down such parts of the doctrine concerning variable quantities, and of constant and variable forces, as are usually employed in the solution of mechanical problems, not merely for the ease and convenience of reference, but for the more important object of giving to the young student a clear notion of the meaning and application of those quantities; for it deserves to be remarked, that in most of our minor works on mechanics, which are usually put into the hands of beginners, they are not given in that eligible and practical form, or treated with that clearness and perspicuity, as immediately to satisfy the minds of learners in general of their nature; or of determining their precise values in the resolution of problems in which they may be concerned; a defect, let me add, that cannot be too much guarded against by writers of scientific and elementary treatises.

The first section on rockets, includes the theory of these bodies, considered as moving in a non-resisting medium. It commences with the proposition respecting the time of motion of a rocket in a vertical ascent, and the height to which it will rise before all its motion is destroyed by gravity; then follows the investigation of the curve that the body describes; then that of its velocity at any given instant of its flight; and lastly, that of the range of the rocket on the horizontal plane.

Section 2, embraces all the theory concerning the resistance to planes, cones, spheres, and cylinders, moving in fluids, that was necessary to establish the subsequent theory of rockets.—The investigations of the resistance to a cylinder moving in a fluid in any direction different from that of its axis are, I believe, new; no work with which I am acquainted containing a solution to this problem generally, but merely of the common particular case where the solid is supposed to move in the direction of its axis; and perhaps, the theory of the flight of rockets is one, out of but very few, in which the subject is at all applicable.

The third section, contains the theory of rockets in resisting mediums. First, the motion of the body in a vertical ascent in the atmosphere is considered, and not only the height to which it will rise before all its motion is destroyed is determined, but also the time of its ascent and descent; a problem of no small labour, even upon the hypotheses which I have assumed; then the proposition concerning its motion in a medium independent of gravity is resolved, and all the circumstances relating to it most fully developed; next that of the effects of the wind upon the rocket in deflecting it from the plane of projection; and finally, the computation of the errors of bomb-shells and cannon-balls in any given case and velocity of the wind.—In this section I do not pretend to have given a complete theory of rockets;—the numerous difficulties that attend the perfection of even what is here offered, lead me to doubt of this from the ablest hands. All I can say in its behalf is, that the several subjects of which it treats, are at once of a new and natural description, containing many facts of importance, investigated in such a manner, as, it is hoped, cannot fail to benefit the young student who is just entering upon such enquiries.

Section 4, relates to the motion of wheels, suspended on fixed horizontal axes, as impelled by the force of rockets attached to their circumferences. And in the following section is given such part of the theory of pendulums, abandoned to the action of these machines, as is most useful in practice; as the estimation of the arc through which the pendulum is urged by the rocket during the time of its combustion, from which, an easy and correct method is derived for finding the strength of its composition.

Next follows a complete essay on naval gunnery, as relating to the most effectual means of destroying the fleets of our enemies, when not far distant from the artillery. It rests on the problem, which determines the charge of gunpowder for any given piece of ordnance, to cause its shot to produce the greatest possible damage to any splintering object of given thickness; for it is well known that ships of war are built of wood of this nature—and as the issue of a contest greatly depends upon the damage done to the vessel, it follows, that those charges that will effect the most mischief possible, and in the shortest time, are the fittest to be used in all cases of actual service. It is a fact deserving observation, that with some charges, a complete broadside fired into the enemy's ship, would not in any material degree disable it for fighting; whilst with others, even half the number of guns would sink her on the first discharge; and surely, it is hence not unreasonable to infer, if the destruction of an enemy's vessel when in action be an object, to effect it by a few guns at one blow, is preferable to that from any distant cannonading, kept up perhaps for hours together, with frequent disadvantage to ourselves, in loss of men, injury to our ships, and unnecessary expenditure of ammunition.

But it may be asked, are not the charges here recommended generally used by our officers, and do they ever combat the enemy, except in unavoidable instances, but when they are nearly in contact with him? I reply that they do not; yet from the quantity of firing that sometimes takes place before the enemy is sunk or captured, it is to be suspected, that the charges employed, are not always the most efficacious; and I speak further from experience, for I have seen in his Majesty's dock-yard at Woolwich, prize men of war having many shot holes in them, almost wholly closed by the wood's own efforts, and that required nothing more than a small wooden peg, or a piece of cork, to stop them up perfectly. Whence it is evident, that the charges in those cases were much too great, and gave to the shot an improper force, insomuch, that no sensible effect was produced by them in disabling the ships for action.

In some sanguinary conflicts, recourse has been had to the double shotting of the guns, in order to produce more extensive damage to the enemy; thus, it has been observed, that in the glorious (and unparalleled important) battle of Trafalgar, the gallant Nelson bore down upon the enemy with his artillery double shotted, which he discharged into the Santissima Trinidada, (the Spanish admiral's flag ship,) as soon as he approached her within pistol shot. The effect was complete. It was not, however, altogether, in consequence of the guns being double shotted that the Santissima was at that blow so dreadfully disabled, but chiefly from the nicety of charge of gunpowder that was employed; for had not this been the case, although double or triple the number of shot should have pierced the side of the vessel, yet that circumstance would have added but little to its destruction, had they not passed through it with a proper motion.

Far be it from me to impeach the judgment of our officers in the distribution of charges that do not always produce the most desired effects; I am too well aware of the impossibility of this under the numerous opposing circumstances that attend a naval engagement; nor am I ignorant of the necessity of experiments to prove, that the charges which are here offered to their notice have any decided worth over those which they employ in the case of service for which these are calculated; but this I must say, that the standard experiments with which they are connected, were never more accurately made by any experimentalists in any age or country, and if my endeavours prove not for their benefit, I have still the satisfaction of having meant well towards them, and the honour of offering something for their censure, if not for their applause.

Lastly, in order to render this work as useful as possible, I have subjoined to it a table of hyperbolic logarithms, for all numbers from one to two thousand; most of the computations in the theory of rockets requiring the use of a table of such numbers.

In concluding this preface, I must observe, that in all my researches, I have strictly adhered to the fullest illustration of them by example, conceiving that, a theory is never so well felt or understood by a learner, as when the several subjects it considers are properly exemplified in numbers; it is also gratifying in many instances, to know the results under particular data, while at the same time it checks in most cases the correctness of the investigations.

Such, then, are the outlines of the present work, and such my motives for publishing it; I trust it will meet a fair examination—that it will prove useful to those for whom it is designed—and thus gratify my wishes, and realize my intentions.

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OF

VARIABLE QUANTITIES

DEFINITION AND NOTATION.

If A, B, C, &c. denote any variable quantities, and a, b, c, &c. other values thereof; and if their magnitudes be so dependent on each other that when A is increased or diminished to a; B, C, D, &c. become b, c, d, &c.: then if it be said that A varies directly as B, the assertion implies that A : a :: B : b. Or, $\frac{A}{a}=\frac{B}{b}$.

If A vary reciprocally as B, it denotes that A : a :: $\frac{1}{B} : \frac{1}{b}$. Or, $\frac{A}{a}=\frac{b}{B}$.

And if A vary as B and C directly, and D reciprocally, it signifies that A : a :: $\frac{BC}{D} : \frac{bc}{d}$. Or, $\frac{A}{a}=\frac{BCd}{bcD}$.

Also if the product of A and B vary as C directly, and D reciprocally; it implies that AB : ab :: $\frac{C}{D} : \frac{c}{d}$. Or, $\frac{AB}{ab}=\frac{Cd}{cD}$.

And on the contrary, if AB : ab :: $\frac{C}{D}$: $\frac{c}{d}$; then AB will vary as C directly and D inversely.

PROP. 1.

If any quantity A *vary as another* B; B *will also vary as* A.

For by *Def.* A : a :: B : b; or which is the same B : b :: A : a; therefore B varies as A also by *Definition.*

PROP. 2.

If one quantity A *vary as another* B, *and* B *as another* C, *and* C *as another* D; *the first* A *will vary as the last* D.

For A : a :: B : b :: C : c :: D : d; therefore seeing, that A : a :: D : d, it follows from *Definition* that A varies as D.

Cor.—If one quantity A vary as another B, and B reciprocally as another C; the first A will vary reciprocally as C.

For A : a :: B : b :: $\frac{1}{C}$: $\frac{1}{c}$; therefore A varies as $\frac{1}{C}$.

PROP. 3.

Either side of a general Proportion may be multiplied or divided by any given quantity.

Thus if A varying as B constitute any general proportion then A will vary as nB.

For A : a :: B : b :: nB : nb :: $\frac{B}{n}$: $\frac{b}{n}$; therefore A varies as nB, and also as $\frac{B}{n}$.

PROP. 4.

Any general proportion may be transformed into an equation, and the general value of each of the terms constituting it deter-

mined, by first multiplying one side of it by a proper homologous quantity.

If A vary as BC, then A $= n \times$ BC; where n is some given quantity composed of other values of A, B and C.

For since A varies as BC, therefore A : a :: BC : bc; and hence $\frac{A}{a} = \frac{BC}{bc}$; or A $=$ BC $\times \frac{a}{bc}$: therefore n in this instance is $= \frac{a}{bc}$. And B and C are found in the same manner.

Cor.—Hence, if in the solution of any problem the quantity required be expressed in a general proportion or be one term of the same; its general value will be had by referring all the variable quantities contained in the proportion to other known values thereof as standards, and finding the homologous multiplier as above.

PROP. 5.

If both sides of a general proportion be multiplied or divided by any variable quantity; the results will still constitute a general proportion.

If A vary as B, and C be any variable quantity, then AC will vary as BC; and $\frac{A}{C}$ as $\frac{B}{C}$.

For A : a :: B : b; and C : c :: C : c; therefore compoundedly AC : ac :: BC : bc; and hence AC varies as BC.

In the same manner it is proved that $\frac{A}{C}$ varies as $\frac{B}{C}$.

PROP. 6.

Any quantity which is proportional to any other quantity in a general proportion may be substituted for it in the general proportion.

If A vary as BC, and C vary as D; then will A vary as BD.

For since C varies as D, BC will vary as BD, *Prop.* 5. Hence, A varying as the former, will also vary as the latter by *Prop.* 2.

PROP. 7.

If the corresponding like sides of two or more general proportions be multiplied or divided by each other, the products and quotients will constitute two other general proportions.

If A vary as B, and C vary as D;

Then AC will vary as BD; and $\frac{A}{C}$ as $\frac{C}{D}$.

For $A : a :: B : b$

and $C : c :: D : d$;

Therefore $AC : ac :: BD : bd$; and consequently AC varies as BD.

In the same manner it is shewn that $\frac{A}{C}$ varies as $\frac{C}{D}$.

COR.—The equal powers or roots of the sides of a general proportion, constitute a general proportion.

If A vary as B, then $A^{\frac{n}{m}}$ will vary as $B^{\frac{n}{m}}$ where $\frac{n}{m}$ denotes any number whole or fractional.

For $A : a :: B : b$

and $A^{\frac{n}{m}} : a^{\frac{n}{m}} :: B^{\frac{n}{m}} : b^{\frac{n}{m}}$;

Therefore $A^{\frac{n}{m}}$ varies as $B^{\frac{n}{m}}$.

PROP. 8.

If any quantity A *vary as* B × C × D, &c; *and* C × D, &c. *be given,* A *will vary as* B; *and if* BC, &c. *be given* A *will vary as* D.

For A varying as BCD, it follows from *Prop.* 4, that if BC be given A will vary as D; and as B when CD is given. That is A : a :: D : d, or $\frac{A}{a} = \frac{D}{d}$ when BC is given; and $\frac{A}{a} = \frac{B}{b}$ when CD is given.

Note.—When any quantity is said to be given it is meant that the relation of it to some fixed quantity of the same kind considered as a standard is known, and with which it is always supposed to be connected; in like manner, when any quantity is sought, it is required to find the relation of this unknown quantity to some fixed standard of the same kind.

PROP. 9.

If any variable quantity A *depend on several other variable quantities* B, C; *and if when* B *is invariable* A *varies as* C, *and as* B *when* C *is invariable; then will* A *vary as* B × C *when both are variable.*

For when A becomes a, let B become b, and C become c according to *Definition.* And suppose, that had C continued constant A would have become $\acute{a}$, when B became b: then since by supposition A varies as B when C is constant, A : $\acute{a}$ :: B : b. But b continuing the same when $\acute{a}$ becomes a, C becomes c; and since A varies as C (by supposition) when B or b is constant, therefore $\acute{a}$: a :: C : c; and by compounding these two proportions, we have A$\acute{a}$: $\acute{a}a$ :: BC : bc, and by division A : a :: BC : bc. Hence A varies as BC.

COR.—If there be any number of quantities, and A varies as each of them when the rest are considered constant, it will vary as their product when all are variable.

ON MOTION, FORCES, &c.

DEFINITIONS.

1. *Matter*, is that of which all bodies are constituted.

2. *Body*, is the mass or quantity of matter in any material substance; and it is always proportional to its weight, whatever its figure may be.

3. Bodies are either hard, soft, or elastic. A *hard body*, is that which cannot be changed by any stroke. A *soft body*, is that which yields to any impression, but does not restore itself to its former figure. An *elastic body*, is that which after yielding to a stroke recovers its former shape; and is such that if it were let fall on a hard plane it would rise to precisely the same height from which it fell.

No bodies, either perfectly hard or perfectly elastic, such as are here defined are to be found in nature, but all partaking these properties in some intermediate degree.

4. *Density*, is the proportional weight, or quantity of matter in any body. So in two spheres, or cubes, of equal size or magnitude, if the one weighs 1*lb.* and the other 2*lb.*, then the density of the latter is double the density of the former; if it weigh 3*lb.* its density is triple; and so on.

5. *Motion*, is that state in which a body is, when passing from one place to another.

6. Motion is either uniform, accelerated, or retarded. *Uniform motion*, is that when a body describes equal spaces in equal successive portions of time. *Accelerated motion*,

is that when a body describes unequal *increasing* spaces in equal successive portions of time. *Retarded motion*, is that when a body describes unequal *decreasing* spaces in equal successive portions of time.

7. *Velocity*, is that quality of motion, by which a body passes over a certain space in a certain time.

8. *Force*, is that which causes a change in the state of motion or rest of a body.

9. *An Accelerative force*, is that which respects the communication of velocity only, any difference in the quantities of matter moved not being considered. It is proportional to the velocity which it generates in a given time.

10. *A Retardive force*, is that which relates to the destruction of velocity only; and it is as the velocity which it destroys in a given time.

11. *A Constant accelerative or retardive force*, is that by which equal velocities are generated or destroyed in equal successive portions of time.

12. *A Variable accelerative or retardive force*, is that by which unequal velocities are communicated or destroyed in equal successive portions of time.

13. *Momentum*, is the product of the mass of a body into its velocity. It is the same as quantity of motion.

14. *A Motive or moving force*, is that which relates to the communication of momentum; and it is as the momentum which it generates in a given time.

15. *A Resisting force*, is that which relates to the destruction of momentum; and it is as the momentum which it destroys in a given time.

16. *Gravity*, is that force by which a body endeavours

to descend towards the centre of the earth. It is called *absolute* gravity when the body is in empty space, or in vacuo; but *relative* gravity when immersed in a fluid.

17. *Specific gravity* of a body, is the proportional weight of a given magnitude of the matter of which it is composed. The specific gravity of a body is therefore proportional to its density.

18. *Centre of gravity* of a body, is that point which being supported, the body itself will rest in any position; no other force acting upon it but that of gravity.

The centre of gravity of a body, is considered to be the place of the body; since whatever supports this centre supports the body and bears all the weight of it.

19. *Inertia*, is that by which a body endeavours, as much as in it lies, to retain the state in which it is, whether of rest or motion, when any force is impressed upon it to cause a change. The inertia of a body is proportional to the quantity of matter contained in it, or to its weight.

20. *A Fluid*, is a body, the parts of which yield to the smallest force impressed, and by so yielding are easily moved among each other.

This is the definition of what is called a perfect fluid: if the fluid require some force to move its parts, it is imperfect, and so much so, in proportion to that force: such are perhaps all the fluids in nature with which we are acquainted.

21. *A Medium*, is any fluid through which a body passes in its motion towards any point. Thus the air or atmosphere is the medium in which birds and other animals move; and in which projectiles move; and water is the medium in which fishes move.

22. *A Non-resisting medium*, is one that affords no resist-

ance to bodies passing through it; and a *resisting medium*, is that in which the motion of bodies are retarded.

AXIOMS.

1. Every body will continue in its state of rest or uniform motion in a right line until it is compelled to change that state by the action of some external force.

2. Any change effected in the motion of a body is in the direction of the force impressed, and is proportional to it in quantity.

3. To every action there is always opposed an equal re-action; or the mutual actions of two bodies on each other are always equal and directed towards contrary parts.

Thus, in the communication of pressure upon any immoveable plane, whether arising from the protrusion, gravity, or impact of a body, the sense of the axiom is, that the resistance of the plane, and an opposite force equal to that producing the pressure, have each of them the same effect, as either of them only destroys the force of protrusion, gravity, or impact. In the communication of motion, by one body striking another, the axiom asserts that the momenta lost and gained by the bodies are equal, when estimated in opposite directions. In the communication of motion by unknown means, as by magnetism, or electricity; it affirms that the body attracting or repelling moves in an opposite direction to that of the body attracted or repelled, and with an equal momentum. Thus to propose an instance in the case of attraction:—when a loadstone and a piece of iron, equal in weight, float in water upon equal and similar pieces of cork, they are found to approach each other with equal velocities; and when they meet, or are kept asunder by any obstacle, they sustain each other by equal and opposite pressures.

ON THE GENERAL LAWS OF MOTION.

PROP. I.

ART. 1.—*The moving forces which communicate the same velocity in a given time to different bodies will be as the quantities of matter contained in those bodies.*

For suppose one body to contain ten times the quantity of matter of another. Then because that greater body may be divided into 10 masses, each equal in quantity of matter, to the less body; it is evident that whatever force be required to produce a certain velocity in the lighter body, that 10 of such forces will be required to impel the 10 bodies through the same space in the same time respectively, so that the velocity of all the bodies shall be equal at the end of that time; and it signifies not, with regard to the velocity, whether the bodies be separated or united, if the said 10 forces still act upon them.

COR.—Hence, because it is found by experiment, that all bodies whether heavy or light, great or small, near the earth's surface descend through equal spaces in equal times (the resistance of the air not being considered); it follows that the moving forces exerted by gravity on bodies are proportional to the quantities of matter contained in them.

PROP. II.

2. *The moving forces acting upon bodies and the momenta communicated to them in a given time, are proportional to the quantities of matter moved, and the velocities communicated jointly: or putting* M *and* m *for any two moving forces,* Q *and* q *the quantities of matter moved, and* V, v, *their velocities;* $\frac{M}{m} = \frac{Q}{q} \times \frac{V}{v}$.

For by the preceding proposition, when the velocity communicated in any given time is the same, the moving force is as the matter moved, or $\frac{M}{m} = \frac{Q}{q}$; and when the quantity of matter moved is the same, the moving force is as the velocity communicated in the same time (*Def.* 14, and *Prop.* 6, *Var. Quan.*); therefore, when neither the quantity of matter or velocity communicated in the same time is given $\frac{M}{m} = \frac{Q}{q} \times \frac{V}{v}$ by *Prop.* 9. *Var. Quan.*

PROP. III.

3. *The accelerating forces which communicate velocities to bodies, are as the moving forces directly, and the quantities of matter moved reciprocally: or putting* F *and* f *to denote any two accelerative forces, and retaining the letters for the other quantities in the last proposition;* $\frac{F}{f} = \frac{M}{m} \times \frac{q}{Q}$.

For by the last proposition the moving force is as the quantity of matter into the velocity generated in a given time; or $\frac{M}{m} = \frac{Q}{q} \times \frac{V}{v}$: and since the accelerative force is as the velocity, or $\frac{F}{f} = \frac{V}{v}$; we shall, substituting $\frac{F}{f}$ for $\frac{V}{v}$ in the above, have $\frac{M}{m} = \frac{Q}{q} \times \frac{F}{f}$; and hence $\frac{F}{f} = \frac{M}{m} \times \frac{q}{Q}$.

It may here be remarked once for all, that in the following propositions F and f are always understood to mean the accelerative forces, proportional to the velocities generated in a given time.

PROP. IV.

4. *In bodies moving uniformly, the spaces described are in the compound ratio of the velocities and times of their descriptions: or* $\frac{S}{s} = \frac{V}{v} \times \frac{T}{t}$.

For by the nature of uniform motion, the greater the velocity, the greater will be the space described in a given time; that is when the time is given the spaces will be as the velocities; or $\frac{S}{s} = \frac{V}{v}$. And if the velocity be given, the spaces will be as the times of describing them; that is, in a double time, a double space will be described; in a triple time, a triple space; and so on: or $\frac{S}{s} = \frac{T}{t}$. Therefore when neither the velocities or times are given, the spaces by *Prop.* 9, *Var. Quan.* will be as the velocities and times conjunctly: or $\frac{S}{s} = \frac{V}{v} \times \frac{T}{t}$.

COR. 1. $\frac{V}{v} = \frac{S}{s} \times \frac{t}{T}$. That is, the velocities of bodies moving uniformly, are as the spaces described directly, and times of describing them inversely.

COR. 2. $\frac{T}{t} = \frac{S}{s} \times \frac{v}{V}$. Or the times of bodies describing any spaces with uniform motions, are as the spaces directly, and velocities reciprocally.

SCHOLIUM.

This proposition is applicable to bodies of all kinds moving with uniform velocities over any kind of spaces; as the hands of a clock or watch round the dial-plate; the motion of sounds of all kinds, as those from the discharge of artillery, the roar of rockets, thunder, &c. also the sounds from woodmens' strokes, and of echoes, which are found by experiments to move uniformly.

PROP. V.

5. *The velocities generated in bodies by the action of constant forces, are as those forces and the times in which they act jointly: or* $\frac{V}{v} = \frac{F}{f} \times \frac{T}{t}$.

For when the times are the same, the velocities generated, will be as the forces of acceleration: the velocities being their natural and general effects; that is $\frac{V}{v} = \frac{F}{f}$. But the forces being the same, the velocities generated are as the times wherein the forces act; because when the force is given, equal velocities are generated in equal times (*Def.* 11.); and consequently the whole velocities acquired are as the times wherein the given force acts: that is when $F=f$, or $\frac{F}{f} = 1$, $\frac{V}{v} = \frac{T}{t}$. Therefore both times and forces being variable, the velocities generated will be as the forces and times of their acting conjunctly (*Prop.* 9, *Var. Quan.*); or $\frac{V}{v} = \frac{F}{f} \times \frac{T}{t}$.

PROP. VI.

6. *If a body from rest be impelled by any constant force acting upon it for a given time the space described will be to the space described in the same time by the body moving uniformly with the last acquired velocity, in the ratio of* 1 *to* 2.

For let the given time be divided into equal evanescent instants, the number of which is n; then the velocity generated being, by the foregoing proposition, as the time, and continuing uniform during any one instant, we shall have the space described in any proposed instant proportional to the number of instants comprehended in the time of motion; so that if during the first instant the space described be s, in the next instant the space described will be $2s$, in the third $3s$, and in the first three instants the space described will be $s + 2s + 3s = 6s$: so in the first n instants, the space described will $s + 2s + 3s + 4s +$ &c. to $ns = \frac{(n+1)\,ns}{2}$: and since by preceding *Prop.* the velocity last acquired is as the time (the force being given); and the space described by any *uniform* velocity, is as the time and velocity jointly (*Prop.* 4.); it follows that the space described by the last acquired velocity continued uniform for the time of the accelerated motion, will be as the square of that time. So that if s be the space uniformly described in the first instant of motion, n^2s will be the space described in n instants with the velocity last acquired. Wherefore the space described by acceleration from rest, is to the space described uniformly with the last acquired velocity, in the same time, as $\frac{(n+1)\,ns}{2}$ to n^2s; or as $n+1$ to $2n$: and since the force acts not by successive impulses, but by unceasing acceleration, the magnitude of each instant must be diminished, and consequently their number increased *sine limite;* therefore n being ultimately infinite, the last proportion of $n+1$ to $2n$ will become that of n to $2n$ or 1 to 2.

SCHOLIUM.

It is found by very nice experiments that the space through which a body descends near the earth's surface in 1 second is $16\frac{1}{12}$ feet; and in this descent it appears by the proposition that such a velocity is acquired as would carry the body uniformly over $32\frac{1}{6}$ feet, or twice that space in the same time, 1 second. Wherefore, if $16\frac{1}{12}$ feet be put $= g$, then will the velocity per second, generated by the constant accelerative force of gravity be $2g$; which may therefore be considered the measure of the intensity of that force, and a standard to which all other accelerative forces may be referred.

PROP. VII.

7. *The spaces which bodies describe from rest by the action of constant forces, are in a compound ratio of the velocities last acquired and times of motion:* or $\frac{S}{s} = \frac{V}{v} \times \frac{T}{t}$.

For by *Prop.* 4, the spaces described by the last acquired velocities continued uniform are as those velocities and the times of motion jointly: and the spaces described by the accelerating forces acting constantly for equal respective times being by the last proposition, half the former spaces; are also as the velocities last acquired and times of motion jointly: that is $\frac{S}{s} = \frac{V}{v} \times \frac{T}{t}$.

PROP. VIII.

8. *The spaces passed over by bodies urged by any constant forces, are as the forces and squares of the times jointly: or* $\frac{S}{s} = \frac{F}{f} \times \frac{T^2}{t^2}$.

For by the foregoing proposition the spaces described by bodies estimated from rest, are as the velocities last acquired and the times of motion jointly; or $\frac{S}{s} = \frac{V}{v} \times \frac{T}{t}$. Also by *Prop.* 5, $\frac{V}{v} = \frac{F}{f} \times \frac{T}{t}$: therefore by substitution we have $\frac{S}{s} = \frac{F}{f} \times \frac{T^2}{t^2}$.

PROP. IX.

9. *The constant forces, which accelerate bodies from rest, are as the squares of the velocities generated directly, and the spaces described inversely: or* $\frac{F}{f} = \frac{V^2}{v^2} \times \frac{s}{S}$.

For by *Prop.* 5, $\frac{F}{f} = \frac{V}{v} \times \frac{t}{T}$; and by *Prop.* 7, $\frac{t}{T} = \frac{V}{v} \times \frac{s}{S}$; therefore by substitution we have $\frac{F}{f} = \frac{V^2}{v^2} \times \frac{s}{S}$.

SCHOLIUM.

Whatever has been demonstrated concerning constant accelerative forces, holds equally true for uniform retardive forces; since it is evident, that whatever velocity is generated by the former in any time, the same forces would destroy in the same time if they acted in the manner of retardive forces. In like manner, if any moving force act upon a body constantly for any time, and generate a certain quantity of motion or momentum; the same force would, in the same time, destroy the same momentum if it acted as a resisting force. Thus if a body falling freely from rest near the earth's surface by the constant acceleration of gravity acquire in any time a certain velocity, the same velocity will be destroyed in the same time by the (now) retardive force, if the body be projected upwards with that velocity. In the former case v being the velocity acquired or last velocity, and in the latter the first, or initial velocity. And the same quantity of motion that was generated in the descent by gravity considered as a moving force, would be destroyed by the same gravity considered as a resisting force, in the same time in its ascent. Also, in all the intermediate points, the velocities and quantities of motion or momenta would be the same in both cases.

10. The various relations between constant forces, times, velocities, and spaces described, demonstrated in the foregoing propositions, and others immediately deduced from them, put down in order will be as follows.

IN CONSTANT FORCES.

1. $$\frac{s}{S} = \frac{tv}{TV} = \frac{t^2f}{T^2F} = \frac{v^2F}{V^2f}$$

2. $$\frac{v}{V} = \frac{ft}{FT} = \frac{sT}{St} = \left(\frac{fs}{FS}\right)^{\frac{1}{2}}$$

3. $\frac{t}{T} = \frac{Fv}{fV} = \frac{sV}{Sv} = \left(\frac{Fs}{fS}\right)^{\frac{1}{2}}$

4. $\frac{f}{F} = \frac{Tv}{tV} = \frac{T^2s}{t^2S} = \frac{v^2S}{V^2s}$

Hence, if the forces be referred to that of gravity at the earth's surface, or F be considered that force acting for 1 second, or corresponding time T, and be called 1; then since the space S described in that time is $16\frac{1}{12}$ feet (*Schol.* to *Prop. Art.* 6), and the velocity acquired (V) $32\frac{1}{6}$ feet; or $2g$ calling $16\frac{1}{12}$ feet g. Then the above formulæ in this case will become as under.

5. $s = \frac{1}{2}tv = gft^2 = \frac{v^2}{4gf}$

6. $v = \frac{2s}{t} = 2gft = (4gfs)^{\frac{1}{2}}$

7. $t = \frac{2s}{v} = \frac{v}{2gf} = \left(\frac{s}{gf}\right)^{\frac{1}{2}}$

8. $f = \frac{v}{2gt} = \frac{s}{gt^2} = \frac{v^2}{4gs}$

Hence also, from the equations $v = 2gft$ and $s = \frac{1}{2}tv$ for constant forces here deduced, may the following theorems expressive of the relation of the fluxions of the time, velocity, force and space described by bodies in motion when acted upon by any variable accelerating force be derived; considering the portion of time infinitely small, so that the force for that time may be considered constant. So,

IN VARIABLE FORCES.

9. $\dot{s} = v\dot{t} = \frac{v\dot{v}}{2gf}$

$$10.\quad \dot{v} = 2gf\dot{t} = \frac{2gf\dot{s}}{v}$$

$$11.\quad \dot{t} = \frac{\dot{s}}{v} = \frac{\dot{v}}{2gf}$$

$$12.\quad f = \frac{v\dot{v}}{2g\dot{s}} = \frac{\dot{v}}{2g\dot{t}}.$$

For v being $= 2gft$, we shall, (f being constant for the infinitely small time $\dot{t}$,) have $\dot{v} = 2gf\dot{t}$; also $s = \frac{1}{2}tv$,* — should be "t" therefore $\dot{s} = \frac{1}{2}t\dot{v} + \frac{1}{2}v\dot{t} =$ (substituting the values of ~~v~~ and $\dot{v}$ above) $v\dot{t}$. Whence all the equations in the above table are readily deduced.

If a motive force happen to be concerned in the problem or investigation, the accelerative force (f) in the above theorems will be had by dividing the motive force by the quantity of matter moved. For by *Prop.* 3. we have $\frac{f}{F} = \frac{m}{M} \times \frac{Q}{q}$; where taking F, M, and Q each equal to 1 (to which the corresponding terms f, m, and q will each be expressed proportionally), the equation will be $f = \frac{m}{q}$.

It is to be observed that the above theorems hold equally true for constant, and variable retardive forces.

Note.—The utility and convenience of these theorems will abundantly appear in the following work.

PROP. X.

11. *The weights or quantities of matter in all bodies are in the compound ratio of their magnitudes and densities; or* $\frac{Q}{q} = \frac{C}{c} \times \frac{N}{n}$: *where* C, c, *denote the magnitudes or capacities of the bodies, and* N, n, *their respective densities.*

For by *Def.* 4. when the magnitude is constant, the quantity of matter is as the density; or $\frac{Q}{q} = \frac{N}{n}$. And the density being constant, the quantity of matter will evidently be as the magnitude; that is $\frac{Q}{q} = \frac{C}{c}$. Hence, when neither the magnitude or density is constant, the quantity of matter is as the magnitude and density compoundedly; or $\frac{Q}{q} = \frac{C}{c} \times \frac{N}{n}$: *Prop.* 9. *Var. Quan.*

Cor. 1.—In spheres, the quantities of matter are in the joint ratio of the cubes of their diameters and densities, or $\frac{Q}{q} = \frac{D^3}{d^3} \times \frac{N}{n}$. And in all similar bodies the masses are jointly as the cubes of their like linear sides and densities.

For the magnitudes of all similar bodies are as the cubes of their like sides.

Cor. 2.—The quantities of matter in spheres, are as the cubes of their diameters and specific gravities; or $\frac{Q}{q} = \frac{D^3}{d^3} \times \frac{G}{g}$: where G, g, are the respective specific gravities of Q and q. And in all similar bodies the quantities of matter are as the cubes of their like linear dimensions and specific gravities.

For by *Def.* 17, the specific gravities of bodies are as the densities of the same; or $\frac{N}{n} = \frac{G}{g}$; wherefore, substituting $\frac{G}{g}$ for $\frac{N}{n}$ in the preceding corollary, it is $\frac{Q}{q} = \frac{D^3}{d^3} \times \frac{G}{g}$.

Cor. 3. -Hence also the weights of spheres are as the cubes of their diameters and specific gravities jointly; or $\frac{W}{w} = \frac{D^3}{d^3} \times \frac{G}{g}$.

For the weights of bodies are as the quantities of matter contained in them, or $\frac{W}{w} = \frac{Q}{q}$; therefore, $\frac{W}{w} = \frac{D^3}{d^3} \times \frac{G}{g}$.

12. Let G denote the specific gravity of water, then since it is found that 1 cubic foot of water weighs just 1000 ounces avoirdupoise, let G represent 1000; in which case we may not only exhibit the specific gravity of any other body in numbers compared with this as a standard, but also the weight of 1 cubic foot of the same; and hence the weight of a greater bulk of the same matter will be had by common proportion. Since $\frac{W}{w} = \frac{D^3}{d^3} \times \frac{G}{g}$, we shall, taking a sphere of water of 1 cubic foot content, and assuming G = 1000, have $\frac{1}{w} = \frac{1.2407^3}{d^3} \times \frac{1}{g}$; and $g = \frac{1.2407^3 \times w}{d^3}$. Therefore,

1. $$g = \frac{1.2407^3 w}{d^3}$$

2. $$d = 1.2407\left(\frac{w}{g}\right)^{\frac{1}{3}}$$

3. $$w = \frac{gd^3}{1.2407^3}$$

which theorems will give either the specific gravity of any sphere of matter, the diameter of the same, or its weight in ounces, when the other two quantities are known.

Ex. 1.

Let it be required to find the specific gravity of cast iron; a ball of the same metal of 4 inches diameter weighing 9lbs.

By substituting for d and w, in the first of the above formulæ the values here given, we shall have the specific gravity $g = \frac{1.2407^3 \times w}{d^3} = \frac{1.2407^3 \times 144}{\frac{1}{27}} = 7420.2668$, which is also the weight of a cubic foot of the same metal in ounces.

Ex. 2.

Required the weight of a leaden ball of 6.6 inches diameter.

The specific gravity of lead, compared with that of water here denoted by 1000 is 11325.

Hence $w = \frac{gd^3}{1.2407^3} = \frac{11325 \times \cdot 55^3}{1.2407^3} = 985.9227$ oz, or 61.62lbs the weight required.

Ex. 3.

Required the diameter of a 42lb. iron ball, the specific gravity of which is 7425 as expressed in the following table of specific gravities.

Here using the second of the foregoing theorems, we have $d = 1.2407 \times (\frac{w}{g})^{\frac{1}{3}} = 1.2407 \times (\frac{672}{7425})^{\frac{1}{3}} = .557049$ feet, or 6.684588 inches.

TABLE.

Of the specific gravities of bodies as compared with that of water denoted by 1000.

Lead - - - - - - - -	11325
Gun-metal - - - - - -	8784
Cast-brass - - - - - - -	8000
Iron - - - - - - - -	7645
Cast-iron - - - - - - -	7425
Clay - - - - - - - -	2160
Brick - - - - - - - -	2000

Chalk - - - - - - - - -	2784
Clay - - - - - - - -	2160
Common earth - - - -	1984
Sand - - - - - - - -	1520
Hard stone - - - - - -	2700
Flint - - - - - - - -	2570
Common stone - - - - -	2520
Nitre - - - - - - - -	1900
Native sulphur - - - -	2033
Solid gunpowder - - - -	1745
Gunpowder close shaken -	937
Do. in a loose heap - - -	836
Sea water - - - - - -	1030
Common water - - - -	1000
Oak - - - - - - - -	925
Elm - - - - - - - -	600
Ash - - - - - - - -	800
Beech - - - - - - - -	852
Male Fir - - - - - - -	550
Female do. - - - - - -	498
Hazel - - - - - - - -	600
Mahogany - - - - - -	1063
Maple - - - - - - - -	750
Poplar - - - - - - - -	383
Walnut - - - - - - -	671
Dutch Yew - - - - - -	788
Spanish do. - - - - - -	807
Air at a mean state - - - -	$1\frac{2}{9}$

Note.—The numbers in this table express also the respective weights of a cubic foot of each substance in avoirdupoise ounces.

ON THE MOTION, &c. OF ROCKETS.

DEFINITION.

13. ROCKET, in Pyrotechnics, is a machine, the form of the body of which is cylindrical, and its head conical. Its inside is filled with very inflammable materials; on the combustion of which the body is impelled forward with a continued acceleration.

14. On the combustion of the composition of a rocket, an elastic fluid is generated, the full force of which is exerted in the first instant alike in all directions, whether there be any other substance for it to act against or not. Hence, if in a vacuum, the combustion took place as freely as in common air, the force of a laminum of the composition in its transformed state (equal to the initial strength of the same into the rocket's base), would be that which constantly acted upon the rocket during the time of its burning. For it is only the first force of the gas in this case that has effect upon the body to move it, it being the very next succeeding instant so greatly diminished from the extreme velocity with which it rushes into the vacuum, that it affords, comparatively speaking, no resistance whatever to the fluid next generated, whereby more motion to the rocket would be communicated *.

* Supposing the elastic force of the gas from the rocket composition to be 1000 times as great as the elastic force of the atmosphere at the earth's surface; it will be found by accurate computation that the velocity with which it would rush into a vacuum is nearly at the rate of 8 miles per second!

Each laminum of gas as it is produced, acts upon and fires at the same time the following laminum of composition; when the produce of this exerts its force upon, and converts into fluid in the same manner the next contiguous laminum of matter, which acts upon and fires the next, and so on continually, till the whole body of the rocket is consumed. If the rocket burns in a medium, then, as there is a body reacting against the fluid that rushes from the rocket, there is not so instantaneous a dissipation of the force of the latter the moment after it is generated; but a time of its action upon the rocket which is greater or less according as the surrounding medium is more or less dense and elastic. In this case, therefore, more motion is communicated to the body than in the former, and but for the resistance to the forepart of the rocket it would move farther in a medium than in a vacuum. A gun recoils farther when fired with powder and ball, than when it is charged only with powder; from the same cause of a longer action of the fluid against the breech of it.

15. To estimate the quantity of action of the fluid at any given instant after its production, would be to find with what force and velocity it then expanded itself, which if not greater than the velocity with which the rocket moved, it would have no effect whatever upon the rocket, and in any other case it will act only with their difference.

In the following theory of the motion of these machines, I have considered the first force only of every laminum of composition (indefinitely thin) to have effect, or the rocket to be urged during the time of its burning by this force acting constantly for that time; and it is imagined that the results determined from this supposition will not be found to differ very sensibly from those derived from experiments; the exact strength of the rocket composition being here supposed.

OF

THE THEORY OF THE MOTION OF ROCKETS

IN NONRESISTING AND RESISTING MEDIUMS.

16. To establish a theory of rockets that shall be consonant to the real phenomena from practice, or at all useful in it, it is necessary that the exact strength of the rocket composition be given. Such important datum, for any particular description of rockets, I have not been able, for want of experiments, to ascertain; but it is presumed that the force of the composition of those now used by the English in bombardment, &c. cannot, from their immense powers, differ very materially from half that of gunpowder; which is supposed to be nearly 2000 times as great when converted into fluid, as the elastic force of the atmosphere*.

* If this supposed strength of the rocket matter, for the nature of those for which it is assigned, or for any other species of rockets be not correct, it will only be necessary, when the real force of it for any proposed description shall have been determined, to substitute it for s in the several investigations that follow to get the true values of the results there deduced; for s being a constant quantity will not at all affect the steps of those investigations. I have merely assumed the above for the numerical illustration of my theory.

I have taken the initial force of gunpowder what Dr. Hutton imagines it must be from the various nice experiments and accurate computations which he has made to ascertain this important point.

It will be, therefore, with this assumed power of the composition, and the supposition that the lamina of it fire uniformly and burn parallel to the rocket's base, that I shall proceed to the investigation of the several effects of these machines; the nature and times of their motion in different mediums; their powers at any given instant, &c.—For all these are very interesting and important particulars for rocket artillerists to know, to whom the management of them generally devolves, and whose immediate concern it is to make themselves acquainted with every fact which the theory as well as the practice of throwing rockets may discover to them.

SECTION I.

ON THE MOTION OF ROCKETS IN A NONRESISTING MEDIUM.

PROP. I.

17. *The strength or first force of the gas from the inflamed composition of a rocket being given; as also the weight of the quantity of composition the rocket contains, together with the time of its burning, and the weight and dimensions of the*

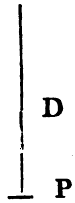
Perhaps no person ever came nearer the truth of the thing than Dr. Hutton. Robins computed the force at just half what Dr. Hutton makes it; but it was independent of particulars which the enquiry evidently involved, and which would materially have affected his conclusion had they been considered. These particulars have been pointed out by Dr. Hutton in his edition of that author's distinguished work, entitled "New Principles of Gunnery;" and also by Euler in his excellent and learned Comment on the same performance; and it is to these works I refer the reader for every information he may require on the subject.

rocket; to find the height it will ascend if projected perpendicularly, and also the velocity acquired at the end of that time; the laminæ of the composition being supposed to fire uniformly, and to burn parallel to the rocket's base.

Put w = weight of the case of the rocket and head

c = weight of the whole quantity of matter with which it is filled

a = time in which the same is consuming itself uniformly



n = 230 ozs. the medium pressure of the atmosphere on 1 square inch

s = 1000 times the pressure of the atmosphere; or force of the inflamed composition

d = diameter of the rocket's base

x = PD the space the rocket describes in the time t, and

v = the acquired velocity in that time. Then,

ed^2 is equal to the area of the rocket's base (e being ·7854 the area of a circle the diameter of which is 1), and ned^2 the pressure of the atmosphere on a surface $= ed^2$. Hence $sned^2$ is the constant impelling force of the composition.

Now the weight of the quantity of rocket matter that is fired or consumed in the time t is $\frac{ct}{a}$, therefore $c - \frac{ct}{a}$ is the weight of the part unconsumed, which added to w gives $w + c - \frac{ct}{a} = m - \frac{ct}{a}$ (by putting $m = w + c$) for the weight of the whole mass at the end of the time t, or when the rocket has ascended to D, and so far as weight resists the motion of the rocket, this must be deducted from the impelling force. Hence $sned^2-(m-\frac{ct}{a})$ is the *motive* force of the rocket at D, and $\frac{sned^2-(m-\frac{ct}{a})}{m-\frac{ct}{a}} = \frac{asned^2}{am-ct}-1$ the *accelerative* force.

By theorem 10. of variable forces we have generally $\dot{v}= 2gf\dot{t}$ (where f denotes the accelerative force and $g = 16\frac{1}{12}$ ft). Therefore $\dot{v}=\frac{2agsned^2\dot{t}}{am-ct}-2g\dot{t}$; the fluent of which is $v=-\frac{2agsned^2}{c}\times$ hyp. log. $(\frac{am}{c}-t)-2gt$.

Now when $t=o$, $v=o$; therefore the fluent corrected will be

$$v=\frac{2gasned^2}{c}(\text{hyp. log. }\frac{am}{c}-\text{hyp. log. }\frac{am-ct}{c})-2gt$$

$$=\frac{2agsned^2}{c}\text{ hyp. log. }\frac{am}{am-ct}-2gt;$$

which, when t becomes a is

$$v=\frac{2agsned^2}{c}\text{ hyp. log. }\frac{m}{m-c}-2ag;\text{ or,}$$

because $m=w+c$, it will be

$$v=\frac{2agsned^2}{c}\text{ hyp. log. }\frac{w+c}{w}-2ag;$$

which therefore is the velocity of the rocket when all the matter of inflammability in its body is just consumed.

For an example in numbers, suppose the weight, dimensions, &c. to be as below; namely,

$s = 1000$

$n = 230$ ozs.

$w =$ 18 lbs. = 288 ozs.
$c =$ 10 lbs. = 160 ozs.
$a =$ 3 sec.
$d =$ 3 in. = $\frac{1}{4}$ ft.
$g =$ 16 ft.
$e =$ ·7854

Then the above expression for v, namely $\frac{2agsned^2}{c} \times$ hyp. log. $\frac{w+c}{w} - 2ag = \frac{2 \times 3 \times 16 \times 1000 \times 230 \times ·7854 \times \frac{1}{16}}{160} \times$ hyp. log. $\frac{448}{288} - 96 = 6774·075 \times$ hyp. log. $\frac{14}{9} - 96 = 2992·9895 - 96 = 2896·9895$ feet, the velocity of the rocket per second at the instant of exhaustion of the composition.

To find the space x, we have by theorem 9th, variable forces $\dot{x} = v\dot{t} = b\dot{t} \times$ hyp. log. $\frac{am}{am-ct} - 2gt\dot{t}$ (where b represents the fraction $\frac{2agsned^2}{c}$) $= b\dot{t}$ hyp. log. $am - b\dot{t}$ hyp. log. $(am - ct) - 2gt\dot{t}$.

Now the fluent of the former part of this is evidently bt hyp. log. am, and the fluent of $\dot{t}$ hyp. log. $(am - ct)$ $= t$. hyp. log. $(am - ct) +$ fluent of $\frac{ct\dot{t}}{am-ct} = t$. hyp. log. $(am - ct) - t - \frac{am}{c}$. hyp. log. $(am - ct) = (t - \frac{am}{c})$. hyp. log. $(am - ct) - t = -\frac{1}{c}(am - ct)$. hyp. log. $(am - ct) - t$. So that the whole fluent will be $x = bt$. hyp. log. $am + \frac{b}{c}$

$(am-ct)$. hyp. log. $(am-ct)+bt-gt^2$; which when $x=o$, and $t=o$ is $\frac{bam}{c}$. hyp.log. am. Hence the fluent corrected is

$$x=(bt-\frac{bam}{c})\text{hyp.log.}am+\frac{b}{c}(am-ct).\text{ hyp. log. }(am-ct)$$

$+bt-gt^2$, and in the case where $t=a$ it is $x=(\frac{abc-abm}{c})$

$$\text{hyp. log. }am+\frac{ab}{c}(m-c.)\text{ . hyp. log. }(am-ac)+ab-a^2g$$

$$=(c-m)\text{. hyp. log. }am+(m-c)\text{ . hyp. log. }(am-ac)+c$$

$$-\frac{acg}{b}=\frac{ab}{c}\Big((m-c).\text{ (hyp. log. }(am-ac)-\text{hyp. log. }am)+$$

$$c-\frac{acg}{b}\Big)=\frac{ab}{c}+\Big((m-c).\text{ hyp. log.}\frac{m-c}{m}+c-\frac{acg}{b}\Big).$$

This in numbers is $=127{\cdot}0139\ (288+160.$ hyp. log. $\frac{9}{14}-1{\cdot}133734)=4015{\cdot}9827735$ ft. the space the rocket ascends through during the 3 seconds it is on fire.

18. Since we have found the velocity at the end of this space to be 2896·9195 feet per second, we shall, on the supposition that the retardive force of gravity remains constant from D have, by the theory of uniform forces $\frac{v^2}{4gf}=\frac{(2896{\cdot}9895)^2}{64\times{\cdot}9993709}=131261{\cdot}131$ feet for the height to which the rocket will farther ascend; which being added to that just determined 4015·9827735 ft. gives 135277·1137735 feet for the whole height of the rocket above the surface of the earth when it has just lost all its motion, which is nearly equal to 27 miles.

But if the height to which it will farther rise be demanded on the true principle, that gravity varies inversely as the square of the distance from the earth's centre; Then,

Putting $r=$ CL the rad. of the earth
$a=$ CD the distance of the point to which the rocket has already ascended from the centre C
$x=$ CI any variable distance from C
$v=$ velocity at I
and $c=$ velocity at D $=2896·9895$ ft.

Then $x^2 : r^2 :: 1 : \frac{r^2}{x^2}$ the retardive force of gravity at I when that at the surface L is considered as unity.

Hence $-v\dot{v}=2gf\dot{x}=\frac{2gr\dot{x}}{x^2}$ (the negative sign being used because the velocity decreases) whose fluent is $v^2=\frac{4gr^2}{x}$, which, when $x=a$, and $v=c$, is $c^2=\frac{4gr^2}{a}$; therefore the fluent corrected will be $v^2=c^2+\frac{4gr^2(a-x)}{ax}$: So that when $v=o$, we shall have $c^2+\frac{4gr^2(a-x)}{ax}=o$, and $x=\frac{4agr^2}{4gr^2-ac^2}=$ (taking the earth's radius at 3979 miles) 21145143·65521 feet, the whole height of the rocket from the centre of the earth, and consequently 21145143·65521 $-r=$ 136023·65521 feet is the whole height from the surface. Whence also the height to which the rocket rises from the point where the impelling force of the composition ceases or is destroyed is 132007·67221 feet.

Hence it appears, that, in consequence of the diminution of the force of retardation from gravity upwards according to the inverse square of the distance from the earth's centre, the rocket will ascend nearly 746·54121 feet higher from a point 4230·609 feet above the earth's surface with a velocity of 2896·9895 feet per second, than it would do if the same force as at the point D had continued constant, or had continued to act upon the body always with the same intensity. Hence also, if the rocket had a velocity of 2896·9895 feet per second upwards when at a height from the earth's surface $=\frac{4gr^2}{c^2}-r$, it would never return, but continue to move for ever, or fly off to an infinite distance. For the expression for x is $\frac{4agr^2}{4gr^2-ac^2}$, or $x=\frac{4agr^2}{4gr^2-ac^2}$, where it is evident that on ac^2 becoming $=4gr^2$, x will be infinite; and therefore to find a, put $4gr^2-ac^2=0$ and reduce the equation.

19. Whence, having the height from which the body must fall to acquire a velocity, which, being added to that of 2896·9895 feet per second, shall cause it to move perpetually when projected with the velocity of their sum; we can readily determine what that velocity is; and it being a very curious fact to know, we will therefore give a solution to the problem in this place.

Put $d=\frac{4gr^2}{c^2}=$ CI the given height from the centre C

$x=$ CD, any variable height from the same point greater than the rad. CL

$r=$ CL

Then $\frac{r^2}{x^2}$ is the accelerative force of gravity at D when that at the surface is 1. Therefore $v\dot{v}=-2gf\dot{x}$; and the fluent of the same is $v^2=\frac{4gr^2}{x}$; which when properly corrected is $v^2 = 4gr^2\left(\frac{1}{x} - \frac{1}{d}\right) =$ (when $x = r$) $4gr^2 \times \left(\frac{1}{r} - \frac{1}{d}\right) = 4gr^2\left(\frac{d-r}{dr}\right) = \left(\text{because } d = \frac{4gr^2}{c^2}\right) 4gr^2 \times \frac{4gr - c^2}{4gr^2} = 4gr - c^2$. Therefore the velocity acquired in descending through $d - r$ is $v = (4gr - c)^{\frac{1}{2}} = 36553{\cdot}3482$ feet per second; which, added to the given velocity 2896·9482 feet per second, gives 39450·2377 feet, or 7·471768 miles for the velocity of projection to cause a body to move to an infinite distance.

PROP. 2.

20. *To find the period of the rocket's motion; or the time from its first going off to that of its return to the earth.*

This is equal to the time of its ascent and of its descent.—To find the time of the rocket's ascent from the point where it first ceases burning.

Put r = CL the radius of the earth

a = CD the height of the rocket from the centre C at the end of its burning

d = CS the distance of the limit of the rocket's ascent from the same point

x = CI any variable distance from C greater than CD

v = velocity at I

t = time of its motion from D to I

c = velocity at D at the end of its burning

g = 16 feet

Then, since we have found the general value of $v = \left\{c^2 + \frac{4gr^2(a-x)}{ax}\right\}^{\frac{1}{2}}$ (See preceding *Prop.*); we shall have $\dot{t} = \frac{\dot{x}}{v} = \frac{\dot{x}}{\left\{\frac{ac^2x + 4gr^2(a-x)}{ax}\right\}^{\frac{1}{2}}}$ = (putting $b = 4gr^2 - ac^2$ and $k = 4agr^2$) $\frac{a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x}}{(k - bx)^{\frac{1}{2}}} = \frac{a^{\frac{1}{2}} x\dot{x}}{(kx - bx^2)^{\frac{1}{2}}} = \left(\frac{a}{b}\right)^{\frac{1}{2}} \times \frac{x\dot{x}}{(dx - x^2)^{\frac{1}{2}}}$, $\frac{k}{b}\left(= \frac{4agr^2}{4gr^2 - ac^2}\right)$ being $= d$. Hence $t = \left(\frac{a}{b}\right)^{\frac{1}{2}} \left\{ \text{cir. arc to rad. } \tfrac{1}{2}d \text{ and versed sine } x - (dx - x^2)^{\frac{1}{2}} \right\}$; which, on correction will, in the extreme case where $x = d$, be $t = \left(\frac{a}{b}\right)^{\frac{1}{2}} \left\{ (ad - a^2)^{\frac{1}{2}} + \text{arc to rad. } \tfrac{1}{2}d \text{ and versed sine } (d-a) \right\}$; as will be evident by conceiving a semicircle described on CS as a diameter.

For an example. Let it be the same rocket as in the example to the foregoing proposition. Then we shall have

r = 3979 miles, or 21009120 feet.
a = 21013135·6 feet.
d = 21145143·655 feet.
c = 2896·9895 feet.
g = 16 feet.
$b = 4gr^2 - ac^2$ = 28072165812115919 feet.

Whence $t = \left(\frac{a}{b}\right)^{\frac{1}{2}} \left\{ (ad - a^2)^{\frac{1}{2}} + \text{arc to rad. } \tfrac{1}{2}d \text{ and versed sine } (d - a) \right\}$ = 45·55647 + 45·7666 = 91·32307 seconds; and consequently the whole time of the rocket's ascent is 94·32307 seconds.

Now to determine the time of its descent. Let as before

r = CL the rad. of the earth. *(See preceding figure.)*
d = CS the extreme height of the rocket from the centre C.
x = CD any var. dist. from C.
v = vel. of the rocket at I.
t = time of falling to that point.
g = 16 feet.

We have already found the general value for v under these circumstances. *(See last Problem.)* $= \left\{4gr^2 \times \left(\frac{d-x}{dx}\right)\right\}^{\frac{1}{2}}$ or $\frac{8r}{d^{\frac{1}{2}}}\left(\frac{d-x}{x}\right)^{\frac{1}{2}}$. Therefore $\dot{t} = \frac{-\dot{x}}{v} = \frac{d^{\frac{1}{2}}}{8r} \times \frac{-x\dot{x}}{(dx-x^2)^{\frac{1}{2}}}$ and $t = \frac{d^{\frac{1}{2}}}{8r}\left\{(dx-x^2)^{\frac{1}{2}} - \text{cir. arc to rad. } \tfrac{1}{2}d \text{ and vers. sin. } x\right\}$; and the correct fluential equation is $t = \frac{d^{\frac{1}{2}}}{8r}\left\{(dx-x^2)^{\frac{1}{2}} + \text{cir. arc to rad. } \tfrac{1}{2}d \text{ and ver. sin. } (d-x)\right\}$: whence in the case where $x=r$, it is $t = \frac{d^{\frac{1}{2}}}{8r}\left\{(dr-r^2)^{\frac{1}{2}} + \text{cir. arc to rad. } \tfrac{1}{2}d \text{ and vers. sin. } (d-r)\right\}$

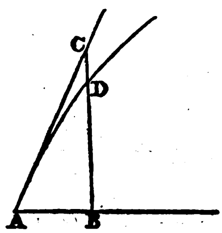
This in numbers is equal to 46·448185 + 46·250625 = 92·69881 seconds, whence the whole time of the rocket's motion is 187·02188 seconds, or 3 min. 7 sec.

Cor. When h $(=4gr^2-ac^2)=0$, the first value of t above is infinite as is evident by inspection.

PROP. III.

21. *To determine the path of a rocket near the earth's surface, neglecting the resistance of the atmosphere.*

If during the time the rocket was on fire, the weight of the whole mass remained constant, the path of the rocket would, by mechanics, be a straight line: but this not being the case on account of the continual wasting of the matter which feeds the flame of the rocket, the accelerative force of the body will be different at every instant; and therefore, since the accelerative force of gravity (as we will suppose) is constant to the height to which rockets generally ascend, the route of the rocket will consequently be a curvilinear one.



Let AC be the first direction of the rocket, and AD the curve in which it moves, and draw CDB perpendicular to the horizontal line AB. Now the path of the rocket will be determined by finding the relation between AC and CD. Let us then suppose gravity not to act, and that the rocket arrives at the point C, in the line AC, in the time t. For although the contrary be the case, yet gravity does not hinder the rocket from arriving at the line CB, parallel to the direction in which that force is exerted, in the *same* time that it would have done by the single action of its own impelling force. Therefore, put AC$=x$; and we shall have (*Prop.* 1.) $x = \left(bt - \frac{abm}{c}\right)$ hyp. log. $am + \frac{b}{c}(am - ct)$ hyp. log. $(am - ct) + bt$.

This expression for x being in terms of logarithms and other quantities; the general value of t in terms of x (which is what we want to find), is not immediately to be obtained; therefore some other expression must be sought. Now under the present case, $\dot{x} = b\dot{t}$ hyp. log. $\frac{am}{am - ct}$ (*Prop.* 1.); the fluent of which may be had by finding the log. of $\frac{am}{am-ct}$; which is done by first putting it into fluxions and then finding its fluent in a series. Thus, the fluxion of the log. $\frac{am}{am-ct}$ being $\frac{c\dot{t}}{am-ct}$, we shall by expanding the fraction and taking the fluent of each term have, for the log. $\frac{am}{am-ct}$ the series $\frac{c}{am} \times (t + \frac{ct^2}{2am} + \frac{c^2t^3}{3a^2m^2} + \frac{c^3t^4}{4a^3m^3} + \frac{c^4t^5}{5a^4m^4} + \text{\&c.})$. Hence the above fluxional expression becomes $\dot{x} = \frac{bc}{am} \times (t\dot{t} + \frac{ct^2\dot{t}}{2am} + \frac{c^2t^3\dot{t}}{3a^2m^2} + \frac{c^3t^4\dot{t}}{4a^3m^3} + \frac{c^4t^5\dot{t}}{5a^4m^4} + \text{\&c.})$; whose fluent is $x = \frac{bc}{2am}(t^2 + \frac{ct^3}{3am} + \frac{c^2t^4}{6a^2m^2} + \frac{c^3t^5}{10a^3m^3} + \frac{c^4t^6}{15a^4m^4} + \text{\&c.}$ wanting no correction. Or, multiplying by $\frac{2am}{bc}$ (= suppose Q) and calling the coefficients of the several terms of the series A, B, C, &c.; it will be $Qx = t^2 + At^3 + Bt^4 + Ct^5 + Dt^6 + \text{\&c.})$; which reverted into a series of x, is $t = (Qx)^{\frac{1}{2}} - \frac{A}{2}Qx + \frac{5A^2-4B}{8}(Qx)^{\frac{3}{2}} + \frac{3AB-2A^3-C}{2}Q^2x^2 + \text{\&c.} = Q^{\frac{1}{2}}(x^{\frac{1}{2}} - \frac{A}{2}Q^{\frac{1}{2}}x + \frac{5A^2-4B}{8}Qx^{\frac{3}{2}} + \frac{3AB-2A^3-C}{2}Q^{\frac{3}{2}}x^2 + \text{\&c.})$; the time of describing the distance x, along AC, from the commencement of motion.

Now CD (y) being the distance descended by gravity in the same time; we therefore get $\frac{1}{4}y^{\frac{1}{2}}$ (omitting the $\frac{1}{12}$) for the time of the rocket's describing CD by the force of gravity: and consequently $\frac{1}{4} y^{\frac{1}{2}} = Q^{\frac{1}{2}} \times (x^{\frac{1}{2}} - \frac{A}{2} Q^{\frac{1}{2}} x + \frac{5A^2 - 4B}{8} Qx^{\frac{2}{3}} + \&c.)$

Hence, knowing the equation which subsists between AC and CD, the track which the rocket describes may be drawn; for it will only be necessary to give some value to x in order to determine the corresponding value of y; and to lay off this upon CD drawn perpendicular to AB, and thus finding several points of the curve, the curve itself may be described.

We have here supposed gravity to act in parallel lines, which is not strictly true; but the distance to which a rocket ranges on the earth's surface being very small compared with its circumference, the error arising from the contrary supposition will not in any material degree affect our conclusions.

PROP. IV.

22. *To find the velocity of the rocket in the curve at any given instant.*

In the preceding diagram let AC $= x$, and AD $= z$ being the space described by the rocket in the time t: then calling the velocity at C ($= b \times$ hyp. log. $\frac{am}{am - ct}$ (*Prop.* 1.) V; the velocity at D, in the curve, will be expressed generally by $\frac{\dot{z}V}{\dot{x}}$, following from the laws for the resolution of motion. Now by the theory of falling bodies in vacuo CD $= gt^2$: and putting k and l for the natural sine and co-sine (to rad. 1.) of the angle CAB of projection; we shall have AB $= lx$, CB $= kx$, and DB (the ordinate of the curve) $= k\dot{x} - gt^2$. Therefore $\dot{z} = \left\{(k\dot{x} - 2gt\dot{t})^2 + l^2\dot{x}^2\right\}^{\frac{1}{2}}$; and $v = \frac{\dot{z}\mathrm{v}}{\dot{x}} = \frac{\left\{l^2\dot{x}^2 + (k\dot{x} - 2gt\dot{t})^2\right\}^{\frac{1}{2}}}{\dot{x}} \times \mathrm{v}$.

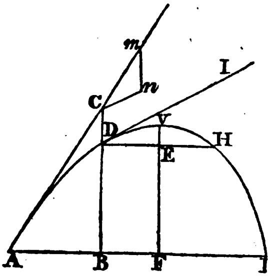
Again, by the theory of variable motions $\dot{x} = \mathrm{v}\dot{t}$. Consequently $v = \frac{\left\{l^2\mathrm{v}^2\dot{t}^2 + (k\mathrm{v}\dot{t} - 2gt\dot{t})^2\right\}^{\frac{1}{2}}}{\mathrm{v}\dot{t}} \times \mathrm{v} = \left\{\mathrm{v}^2l^2 + (k\mathrm{v} - 2gt)^2\right\}^{\frac{1}{2}} = \left\{l^2b^2 \text{ hyp. log.}^2 \frac{am}{am - ct} + (kb.\text{ hyp. log. } \frac{am}{am - ct} - 2gt)^2\right\}^{\frac{1}{2}}$, the velocity of the rocket at D; which wants no correction, because when $v = 0$, $t = 0$, and the whole vanishes: therefore $v = \left\{l^2b^2 \text{ hyp. log.}^2 \frac{am}{am - ct} + (kb \text{ hyp. log.} \frac{am}{am - ct} - 2gt)^2\right\}^{\frac{1}{2}}$

When the angle of projection is 90°, $l = 0$, and $k = 1$: therefore v in this case will be $b \times$ hyp. log. $\frac{am}{am - ct} - 2gt$; as determined in *Prop.* 1: and when $k = 0$, or the action of gravity is 0, the velocity of the rocket in its rectilinear path is $b \times$ hyp. log. $\frac{am}{am - ct}$; which agrees with what has already been observed.

When the angle of elevation is 30°, $k = \frac{1}{2}$ and $l = (\frac{3}{4})^{\frac{1}{2}}$: $\because v = \left\{\frac{3}{4}b^2 \text{ hyp. log.}^2 \frac{am}{am - ct} + (\frac{1}{2}b \text{ hyp. log.} \frac{am}{am - ct} - 2gt)^2\right\}^{\frac{1}{2}}$. And when the angle of elevation is 60°, then k being $= (\frac{3}{4})^{\frac{1}{2}}$, and $l = \frac{1}{2}$; $v = \left\{\frac{1}{4}b^2 \text{ hyp. log.}^2 \frac{am}{am - ct} + \left(\frac{3^{\frac{1}{2}}b}{2} \text{ hyp. log.} \frac{am}{am - ct} - 2gt\right)^2\right\}^{\frac{1}{2}}$.

PROP. V.

23. *To find the horizontal range of the rocket, having the angle of elevation of the engine, and the time the rocket is on fire given.*



Let D be the place of the rocket when all the matter it contains is just exhausted; and Cm and Cn the measures of the velocities of the rocket in the directions AC and DI, the latter of which is a tangent to the curve at D: then by trig. sin. ∠Cnm ($=n$CB$=$ IDB) $= \frac{Cm}{Cn}$. sin. ∠Cmn $= \frac{Cm}{Cn}$. co-sin. of the angle of elevation of the engine $= \frac{velocity\ at\ C}{velocity\ at\ D}$. co-sin. of the ∠CAB.

Whence calling the velocities at C and D, V and v (computed from the 3*rd Prop.*), we have sin. ∠IDB $= \frac{V}{v}$. co-sin. ∠CAB. And since we have found the ∠IDB, it will be easy to determine that part of the range denoted by BL. For the curve from D being a parabola DH $= \frac{suv^2}{g}$, and VE $= \frac{s^2v^2}{4g}$ (from the laws of projectiles in vacuo); where s and u represent the sin. and co-sin. of the ∠IDH = ∠IDB − 90°; consequently VF = VE + EF = VE + DB = $\frac{s^2v^2}{4g} + kx - gt^2$; whereof x is given by the first proposition.

Again, by the nature of the parabola, VE : VF :: EH^2 :

$FL^2 = \frac{u^2v^2}{g} \cdot \left(\frac{s^2v^2}{4g} + kx - gt^2\right)$; and $FL = \frac{uv}{4} \cdot \left(\frac{s^2v^2}{4g} + kx - gt^2\right)^{\frac{1}{2}}$. Whence $AL = \frac{uv}{4}\left(\frac{s^2v^2}{4g} + kx - gt^2\right)^{\frac{1}{2}} + \frac{suv^2}{2g} + lx$, the entire range of the rocket, which was required.

For an example in numbers: suppose the engine from whence the rocket is thrown to make an angle with the horizon $= 45°$: and let all other things remain as in the first proposition. Then v, the velocity of the rocket in the curve at the end of its burning $= \left\{ l^2b^2 \text{ hyp. log.}^2 \frac{m}{m-c} + (kb \text{ hyp. log. } \frac{m}{m-c} - 6g)^2 \right\}^{\frac{1}{2}} = (4479024 + +4080400)^{\frac{1}{2}} = 2925{\cdot}6$; and sine angle $IDB = \frac{V}{v} \times$ co-sin. $\angle CAB = \frac{2993}{2925{\cdot}6}$. co-sin. $\angle CAB = 134° \; 6' \; 38''$. Whence $\angle IDH = 44° \; 6' \; 38''$; the nat. sin. and co-sin. of which are ·6960172 and ·7180251 $= s$ and u respectively: and the values of the letters in the above expression for the range collectively are as under.

s	=	·696
u	=	·718
v	=	2925·6
k	=	·7071
l	=	·7071
x	=	4159·6
g	=	16
t	=	3

Whence the range itself is readily found equal to 273116·29 feet, or 51·72657 miles.

EXAMPLES FOR PRACTICE.

EXAMPLE I.

Given the diameter of a cylindrical rocket 4 inches, the length of the case 2 feet, and the weight of the case $8\frac{1}{2}$ *lb.* to find to what height the rocket will rise in a vertical projection*.

EXAMPLE II.

All things remaining as in the foregoing example, to determine the time in which the rocket will lose all its motion upwards; or before it will begin to descend.

EXAMPLE III.

The same data being retained, to find the period of the rocket's return to the earth from the first moment of projection.

EXAMPLE IV.

Having given the diameter of a rocket equal to 7 inches, and its length $2\frac{1}{2}$ feet; also the weight of the case of the rocket 13 *lbs.* and the angle of projection 30°; to find the range of the rocket on the horizontal plane.

EXAMPLE V.

Let the same rocket be supposed to contain a ball (of the same diameter) at the end of it; and to be impelled after the consuming of the wild-fire by the explosion of a charge of gunpowder that fills the last 3 inches

* The weight of the composition of the rocket, and the time of its burning, may be had, by reference to these given in the example at Art. 17.

of the case of the rocket; to find the range of the shot on the horizontal plane.

EXAMPLE VI.

All things remaining as in the 4th example; to find the velocity with which the rocket is moving at the end of 4 seconds.

EXAMPLE VII.

To find the height of the same rocket from the earth at any given instant; as at the end of 5 seconds.

EXAMPLE VIII.

Required the time of flight of the same rocket on the horizontal plane.

EXAMPLE IX.

The weight of the case of a rocket is 10*lb.* its length $2\frac{1}{4}$ feet, and the diameter of its base 6 inches: What will be the oblique range and the time of flight of this rocket, reckoning from the point where it ceases burning to the point where it falls upon the horizontal plane?

SECTION II.

ON THE RESISTANCE TO BODIES MOVING IN FLUIDS WITH GIVEN VELOCITIES.

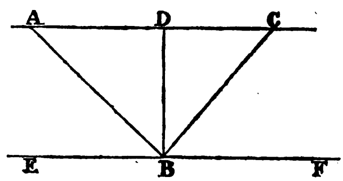
24. As frequent mention will be made in what follows on the theory of Rockets concerning the resistance that planes, cones, spheres, and cylinders suffer when moving in given directions in fluids; it will here be proper to lay down such matter on this head as will suffice for our

further enquiries on that subject; especially as no book extant (with which at least I am acquainted) contains the principal part of the information that will be required, to which reference could otherwise be made.

PROP. VI.

25. *To determine the resistance a plane meets with from a fluid, in which it moves, in an inclined position, with a given velocity.*

It is universally allowed, and indeed it is evident, that the resistance to a body moving through an infinitely compressed fluid at rest (such as is here supposed), is the same in effect as the force of the fluid in motion with equal velocity, on the body at rest: therefore, as it will be somewhat more convenient to consider the fluid in motion, and the body quiescent, I shall pursue the several investigations in this section upon this hypothesis.



Let AB be the given position of the plane; and CA the direction of the fluid moving against it. Draw BC perpendicular to AB, and let BD be perpendicular to line AC; also draw EBF parallel to AC.

Let AC denote the *full* force of the fluid against AB; or the force with which the plane would be struck thereby, if it were perpendicular to the direction of the fluid's motion. Then this being resolved into the two forces AB, CB, the former AB being parallel to the plane has no effect to move it in any direction whatever, but only the force CB in direction CB, perpendicular to AB; which is to the whole force CA as sine angle A to rad. (1); and this force CB to urge the plane AB in the direction CA is as CD, which is to the force CB as sin. ∠CBD, or sine angle A to rad. (1): CD therefore being that part only of the full force CA which has efficacy in moving the plane in the direction of the fluid, and in proportion to the whole force CA as $\sin.^2$ ∠A to 1; the full force of the fluid on the plane will be diminished from the obliquity of the impact in the ratio of 1 to the square of the sine of the angle of incidence. But the whole force will be further diminished in the ratio of 1 to sin. ∠A, on account of no more fluid striking the plane AB than what passes between the parallels AC and EF, or that meet the vertical section BD, which is to AC as sin. ∠A to rad. (1); and therefore, on both these accounts, the full force of the fluid on AC will be diminished in the ratio of 1 to the cube of the sine of the angle of incidence.

Let A = the area of the given plane.
f = the sine ∠A to rad. 1.
V = velocity of the (supposed) moving fluid.
n = density of the fluid.

Then by the nature of fluids, the force with which any one in motion strikes a plane perpendicularly, being equal to the weight of a column of such fluid, the base of which is equal to that of the given plane, and altitude the height through which a body must fall to acquire the velocity of its motion; the full force of the fluid on the plane, denoted above by the line AC, will be $= A \times n \times \frac{V^2}{4g}$ (where $g = 16\frac{1}{12}$). And therefore, as $1 : \sin.^3$ ∠A $(f^3) :: \frac{AnV^2}{4g} : \frac{AnV^2f^3}{4g}$ the absolute force of the fluid on the plane AB, in direction CA, when the sine of the angle of incidence is f. Hence, conversely, the real resistance to the plane is $\frac{\text{A}n\text{v}^2 f^3}{4g}$, as was required.

26. If AB represent a line the length of which is L, and f be the sine of the angle of incidence, or angle at which the line is inclined to the direction of its motion; then the resistance to the line estimated in the directly opposite direction to that of its motion will be $\frac{\text{L}n\text{v}^2 f^3}{4g}$.

27. And if a cylinder, the radius of the base of which is r, move in a fluid in the direction of its axis with velocity v; then the end of the cylinder opposing in this case the full inertia of the fluid; the real resistance to the cylinder will be $\frac{pr^2 n\text{v}^2}{4g}$; p being $= 3{\cdot}1416$ and n the density of the medium as before.

28. Also if a cone move in a fluid in the direction of its axis with its vertex foremost; the resistance it suffers will be $\frac{pr^2 n\text{v}^2 f^2}{4g}$; r being the radius of its base, v the velocity of motion, and f the sine of the angle of incidence of the reacting fluid against the solid.

For here, *as* many particles strike the surface of the solid as would meet the base; and therefore the full force of the fluid against the base can only be diminished in the ratio of 1 to sin.^{2} of the angle of incidence (supposing throughout rad. 1.); or of the angle which the slant side of the cone makes with the axis, which is equal to it.

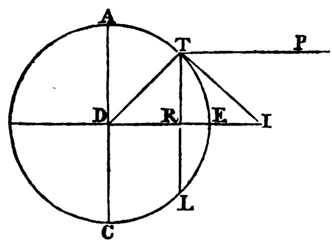
29. And if r be the radius of a circular plane moving obliquely in a fluid, and the sine of the angle of incidence, or angle at which the plane is inclined to the direction of its motion, be f; the resistance opposed to the plane

in the directly contrary direction to that in which it moves will be $\frac{pr^2nv^2f^3}{4g}$.

Thus much concerning the resistance to planes, cones, and cylinders, when these move in the direction of their axes in fluids: I shall now proceed to determine the resistance to a sphere, or any segment of a sphere moving in the direction of the versed sine.

PROP. VII.

30. *To determine the resistance to a sphere or a cylinder, with a hemispheric end, moving in a fluid with a given velocity, in the direction of its axis.*



Let ATECA be any section of the sphere through the axis DE, in the direction in which the solid moves. Draw TI a tangent to any point of the curve as T, meeting the axis produced in I, and draw also TR perpendicular to DE, and join DT.

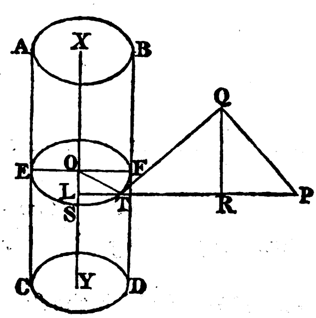
Put DR $= x$, TR $= y$, ET $= z$, and DT $= r$. Then the sine (f) of the angle of incidence PTI or its equal angle DTR $= \frac{\text{DR}}{\text{DT}} = \frac{x}{r}$. Now $2py\dot{z}$ is the fluxion of the surface of the spherical zone generated by AT, and $\frac{nv^2f^3}{4g} \times 2py\dot{z}$ (*Prop.* 6.), the fluxion of the force of resistance on the same; where $2py\dot{z}$ denotes the same quantity here that A does in that proposition. But $f^3 = \frac{x^3}{r^3}$; and

$\dot{z} = \frac{r\dot{x}}{y}$. Therefore the fluxion of the force $(= \frac{nv^2 f^3}{4g} \times 2py\dot{z}) = \frac{pnv^2x^3\dot{x}}{2gr^2}$; the fluent of which is $\frac{pnv^2x^4}{8gr^2}$, the resistance to the sphere as far as relates to the action of the fluid against the surface of the said spherical zone ATLC. And when $x = r$ the expression becomes $\frac{pnv^2r^2}{8g}$ which is therefore the whole resistance to the sphere AE CA, or cylinder, the end of which is the hemisphere AEC, and the direction of whose axis is that of DE.

The resistance to the spherical segment TEL, when moving in the direction RE, is hence determined to be $\frac{pnv^2}{8gr^2} \times (r^4 - x^4) = \frac{pnv^2y^4}{8gr^2}$; where y is the radius of its base, and r the rad. of the sphere of which TEL is the segment.

PROP. VIII.

31. *To determine the resistance a cylinder meets with in a fluid when moving in a direction perpendicular to its axis with a given velocity.*



Let ABCD be the cylinder, and ETF any section parallel to the base. Let a particle strike this section at T in the direction PT, perpendicular, by supposition, to BD; and draw TO to the centre

O: draw also the tangent TQ to the circle ETF or cylinder at T, upon which let fall the perpendicular PQ, and let fall the perpendicular QR upon TP.

Let ST be denoted by z, and TQ represent the fluxion of $z = \dot z$; then it is evident by bare inspection of the figure (where TP may represent the *full* force of the fluid against TQ, &c.), and from *Art.* 25, that $\frac{nV^2\dot z}{4g} \times \sin.^3 \angle PTQ$ will be the real force that urges QT in the direction PT; and consequently the fluxion also of the force of the fluid against the circular arc to move it in the same direction.

Put ST $= x$,
LT $= y$,
OT $= r$,
and $f =$ the sine of the angle PTQ.

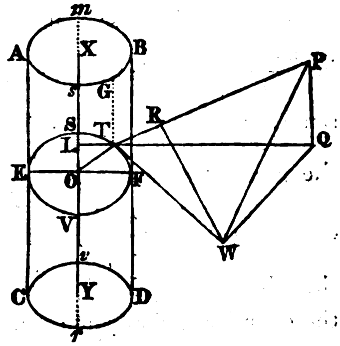
Then $\dot z = (\dot x^2 + \dot y^2)^{\frac{1}{2}}$; and $y = (2rx - x^2)^{\frac{1}{2}}$ by the property of the circle: consequently $\dot y = \frac{r\dot x - x\dot x}{(2rx - x^2)^{\frac{1}{2}}}$, and $\dot z = (\dot x^2 + \dot y^2)^{\frac{1}{2}} = \frac{r\dot x}{(2rx - x^2)^{\frac{1}{2}}}$. Also by reason of similar triangles, $\frac{QP}{TP} = \frac{LT}{OT} = \frac{y}{r}$: whence f being $= \frac{QP}{TP}$ will also be equal to $\frac{y}{r}$. Therefore by substitution the fluxion of the force of the fluid on ST $= \frac{nV^2 f^3 \dot z}{4g} = \frac{nV^2}{4g} \times \frac{y^3}{r^3} \times \frac{r\dot x}{(2rx - x^2)^{\frac{1}{2}}} = \frac{nV^2}{4g} \times \frac{(2rx - x^2)^{\frac{3}{2}}}{r^3} \times \frac{r\dot x}{(2rx - x^2)^{\frac{1}{2}}} = \frac{nV^2}{4gr^2}$ $(2rx\dot x - x^2\dot x)$, of which the fluent is $\frac{nV^2}{4gr^2}\left(\frac{3rx^2 - x^3}{3}\right)$, wanting no correction; so that when $x = 2r$, the fluent will be $\frac{nv^2r}{3g}$; which is the effective force of the fluid on the semicircumference of a section of the cylinder parallel to the base. Consequently $\frac{nv^2r}{3g}$ into the height of the cylinder $(h) = \frac{nv^2rh}{3g}$, will be the force of the fluid on the whole semicylindric surface; or the resistance that the cylinder suffers when it moves in a direction perpendicular to its axis with the velocity v.

Cor.—Because it is found, that a sphere, the radius of which is r, moving in a fluid of the density n, with the velocity v, is $\frac{pnv^2r^2}{8g}$; we shall have the resistance of the sphere to the resistance of its circumscribing cylinder as $\frac{pnv^2r^2}{8g}$ to $\frac{2nv^2r^2}{3g}$, or as 1 to $\frac{16}{3p}$ (where $p = 3{\cdot}1416$); the latter therefore being resisted more than the former by about ·69829 of the former. Whence, the resistance to a sphere being given, the resistance to its circumscribing cylinder will be had by multiplying the former by 1·69829.

PROB. IX.

32. *To determine the same as in the last, when the cylinder moves in any direction oblique to its axis.*

Let TP in the following diagram be the direction of the cylinder moving in the fluid, or PT that of the fluid against the cylinder.



At any point T in the circumference of the section EFT (parallel to the base CD), draw the tangent TW; also let LTQ be perpendicular to the diameter VOS, which is at right-angles to the axis XY; and draw PQ, QW, and WR perpendicular to TQ, TW and TP respectively. Join PW, which will evidently be perpendicular to TW.

Now because of the oblique motion of the cylinder in the fluid, the full resistance to the same will, on this account, be diminished in the ratio of 1 to the cube of the sine of the angle of incidence (*Art.* 25). Or, supposing the fluid to move against the cylinder at rest, its full force against the cylinder, from the obliquity of the direction of the impact with regard to the *position* of the cylinder, will be diminished in the ratio of 1 to sin.^{3} of the angle PTG of incidence. Let FT $= z$ and $\dot{z}$ represent the fluxion of z. Let the full force of the fluid striking $\dot{z}$ as above diminished ($= \frac{nv^2f^3\dot{z}}{4g}$, calling sin. ∠PTG, f) be denoted by TP; then resolving this force into the two forces TW, PW; and the latter of these into the two PR, WR; the former only PR, which has effect in moving the solid in the direction PT, will be to the whole force TP as sin.^{2} ∠P TW to 1 (rad. being 1), or as sin.^{2} of its supplement to 1; and the force TP being also further diminished in the ratio of 1 to sin. ∠PTW, on account of the number of particles striking $\dot{z}$, being so diminished (from the obliquity of the line $\dot{x}$ with regard to PT); and therefore the real force upon $\dot{x}$ to urge it in the direction PT, from the consideration of both the oblique motion of the fluid, and the oblique surface of the cylinder, will be $\frac{nv^2 f^3 \dot{x}}{4g} \times \sin.^3 \angle PTW$; which is also the fluxion of the force of the fluid on the arc FT.

Put $r =$ OT, $x =$ OL, and $y =$ TL. Then by reason of the similitude of the triangles OLT, QTW, we obviously obtain the sine of the angle TQW ($=$ LTO) $= \frac{x}{r}$. Call TP unity, and we get TQ $= f$; also sin. $\angle$TQW being expressed by $\frac{x}{r}$, by Trig. TW $= \frac{fx}{r}$. Hence in the right-angled triangle TPW, PW $= (\text{TP}^2 - \text{TW}^2)^{\frac{1}{2}} = \left(1 - \frac{f^2 x^2}{r^2}\right)^{\frac{1}{2}} = \frac{(r^2 - f^2 x^2)^{\frac{1}{2}}}{r}$; which in the present case is equal to the sine of the angle PTW. Therefore by substitution, the fluxion of the force of the fluid on FT will be $\frac{nv^2 f^3}{4gr^2} \cdot \frac{\dot{x}(r^2 - f^2 x^2)^{\frac{3}{2}}}{(r^2 - x^2)^{\frac{1}{2}}}$; the fluent of which is $\frac{nv^2 f^3}{4gr^2} \cdot \left\{ r^2 x - \frac{3f^2 - 1}{6} x^3 + \frac{3(f^2 - 1)^2}{40 r^2} x^5 + \frac{(f^2 + 5) \cdot (f^2 - 1)^2}{112 r^4} x^7 + \&c. \right\}$: which on x becoming r is

$$\frac{nv^2 f^3 r}{4g} \cdot \left\{ 1 - \frac{3f^2 - 1}{6} + \frac{3(f^2 - 1)^2}{40} + \frac{(f^2 + 5) \cdot (f^2 - 1)^2}{112} + \&c. \right\}.$$

This therefore is the effective force of the fluid on the quadrantal arch FTS. Hence the force on the whole semicylindric surface mD$v r$Bs is

$$\frac{nv^2rhf^3}{2g}\left\{1-\frac{3f^2-1}{6}+\frac{3(f^2-1)^2}{40}+\frac{(f^2+5).(f^2-1)^2}{112}+\&c.\right\},$$ which is also the resistance to the cylinder when this moves in the fluid at rest, so far as relates to the surface $mDvrBs$ only.

Now the resistance arising from the fluid against the top $AsBm$ is $\frac{nv^2pr^2}{4g}$. co-sin.^{3} $\angle PTW$ (*Art.* 25.). Hence the whole resistance to the cylinder is

$$\frac{nv^2rhf^2}{4g}\left\{1-\frac{3f^2-1}{6}+\frac{3(f^2-1)^2}{40}+\frac{(f^2+5).(f^2-1)^2}{112}+\&c.\right\}+\frac{nv^2pr^2}{4g}.\ (1-f^2)^{\frac{3}{2}}.$$

Cor.—When the angle TPQ is 90°, or the solid moves in a direction perpendicular to its axis; then f becoming 1, all the terms of the above series except the first two will vanish (each and all of them containing the factor f^2-1), and the resistance will be $\frac{nv^2rh}{2g}\left(1-\frac{3-1}{6}\right)$ $=\frac{nv^2rh}{3g}$ as determined in *Prop. Art.* 31.

EXAMPLES FOR PRACTICE.

EXAMPLE I.

A cylinder, the radius of the base of which is 8 inches, is terminated by a cone whose base is the same as that of the cylinder, and altitude 17 inches; what will be the resistance to this cylinder, moving in the atmosphere in the direction of its axis, with a velocity of 1200 feet per second?

EXAMPLE II.

What will be the resistance to a cylinder, whose diameter is 3 ft. and length 17 ft. moving in water in a direction perpendicular to its axis with a velocity of 2 ft. per second?

EXAMPLE III.

The velocity of the wind is 88 feet per second: required its force to upset the monument of London, the radius of the base of which is 7·5 feet, and its height 202 feet, being that of an upright cylinder.

EXAMPLE IV.

The radius of the base of a cylinder is 11 inches; and its height 7 feet; what will be the resistance to this cylinder moving in air in a direction inclined to that of its axis in an angle of 54° with a velocity of 1500 feet per second?

EXAMPLE V.

The resistance to a sphere is 54*lbs.* when moving with a certain velocity in a certain medium: required the resistance to its circumscribing cylinder moving with the same velocity in the same medium perpendicular to its axis.

EXAMPLE VI.

The velocity of the wind is 50 miles per hour: required its force against a cylinder of 3 inches in radius and 50 inches in height, standing inclined to the horizon in an angle of 30°.

EXAMPLE VII.

Given the base of a cylinder, to determine its height; so that the resistance to the cylinder when it moves in the direction of its axis, may be equal to the resistance when the direction of its motion is perpendicular to the axis: the velocity being given.

SECTION III.

ON THE MOTION OF ROCKETS IN RESISTING MEDIUMS.

PROP. IX.

33. *The time of burning, &c. of a rocket being given; to find the height to which it will rise in the atmosphere in a vertical ascent; and also the velocity acquired at the end of that time; the resistance being as the square of the velocity directly.*

Put w = weight of the case of the rocket and head,
c = weight of the whole quantity of matter with which it is filled,
a = time in which the same is consuming itself uniformly,
n = 230 ozs.
s = 1000,
d = diameter of the rocket's base,
x = PD, the space the rocket describes in the time t,
v = the acquired velocity in that time,
R = the resistance of the air to the rocket when moving with a velocity of b feet per second.

Then $b^2 : v^2 :: R : \frac{Rv^2}{b^2}$ the resistance at D; and consequently $sned^2 - \left(m - \frac{ct}{a}\right) - \frac{Rv^2}{b^2}$ (see *Prop.* 1.) will be the motive force of the rocket at D in this case; and $\frac{(sned^2b^2 - Rv^2)a}{(am - ct)b^2} - 1$ the accelerative force. Therefore $\dot{v} = 2gf\dot{t} = \frac{(sned^2b^2 - Rv^2)\,2ga\dot{t}}{(am - ct)\,b^2} - 2g\dot{t}$; or putting

$2ag \times sned^2b^2 = h$, $2ag\text{R} = k$, $amb^2 = l$, and $cb^2 = p$, we shall have $\dot{v} = \dfrac{h\dot{t} - kv^2\dot{t}}{l - pt} - 2g\dot{t}$; and $l\dot{v} - pt\dot{v} = h\dot{t} - kv^2\dot{t} - 2gl\dot{t} + 2gpt\dot{t}$; and further, putting $h - 2gl = q$ to render the expression as simple as possible, it will be $l\dot{v} - pt\dot{v} - q\dot{t} + kv^2\dot{t} - 2gpt\dot{t} = 0$; whence v may be determined in terms of t as follows:

Assume $v = \text{A}t + \text{B}t^2 + \text{C}t^3 + \text{D}t^4 + \text{E}t^5 + \&c.$: then making $\dot{t} = 1$; we have $\dot{v} = \text{A} + 2\text{B}t + 3\text{C}t^2 + 4\text{D}t^3 + 5\text{E}t^4 + \&c.$: and substituting these in the given equation it becomes as follows:

$$\left.\begin{matrix} l\text{A} + 2l\text{B} \\ - p\text{A} \\ -q \\ \\ -2gp \end{matrix}\right\} t \quad \left.\begin{matrix} + 3l\text{C} \\ - 2p\text{B} \\ \\ + k\text{A}^2 \end{matrix}\right\} t^2 \quad \left.\begin{matrix} + 4l\text{D} \\ - 3p\text{C} \\ \\ + 2k\text{AB} \end{matrix}\right\} t^3 \quad \left.\begin{matrix} + 5l\text{E} \\ - 4p\text{D} \\ \\ + 2k\text{AC} \\ + k\text{B}^2 \end{matrix}\right\} t^4 = 0.$$

Whence equating the co-efficients of the homologous terms to find the quantities A, B, C, &c. they become

$$\text{A} = \frac{q}{l};\ \text{B} = \frac{pq + 2gpl}{2l^2};\ \text{C} = \frac{p^2q + 2gp^2l - kq^2}{3l^3};$$

$$\text{D} = \frac{p^3q + 2gp^3l - 2kpq^2 - 2gkpql}{4l^4};$$

$$\text{E} = \frac{12p^4q + 24gp^4l - 35kp^2q^2 - 52gkp^2ql - 12g^2p^2l^2k + 8k^2q^3}{60l^5}$$

&c. &c. &c.

Therefore the fluent required is $v = \dfrac{q}{l}t + \dfrac{pq + 2gpl}{2l^2}t^2 + \dfrac{p^2q + 2gp^2l - kq^2}{3l^3}t^3 + \dfrac{p^3q + 2gp^3l - 2kpq^2 - 2gkpql}{4l^4}t^4 + \&c. =$ (in the ultimate case where $t = a$)

$$\frac{1}{448}\left\{ q + \frac{ap(q + 2gl)}{2l} + \frac{a^2}{3l^2}\left\{ p^2(q + 2gl) - kq^2 \right\} + \right.$$

$\frac{a^3}{4l^3}\left\{p^3(q+2gl)-2kpq(q+gl)\right\}+\&c.\Big\}$; the velocity as required by the proposition.

Now to determine what this velocity is, we must first find the value of R for the given case of velocity b. Now under the conditions, that the particles of the medium are perfectly nonelastic, and that the medium is infinitely compressed and affords no resistance to the motion of the rocket but what arises from the inertia of its particles, (which is the ground of our hypotheses concerning the law of resistance), we shall, putting r for the radius of the rocket's base, or of the head of the rocket; $f=$ the sine of the angle, which the slant side of the head (supposing it conical), makes with the axis; $p=3{\cdot}1416$; s = the specific gravity of the medium, which is here considered as the atmosphere; and $g=16$ feet, (omitting the $\frac{1}{12}$) have $R=\frac{psr^2b^2f^2}{4g}$. (*Art.* 28.)

Let $b=1$, in order to render the expression as simple as possible; and the angle, the sine of which is f, 30 degrees; then $f=\cdot 5$ or $\frac{1}{2}$ (to rad. 1.): and taking the specific gravity of air at a medium, or $s=1\frac{2}{9}$, R will be found $=\cdot 0002343$ ounces; which is the absolute resistance the rocket suffers when moving with a velocity of 1 foot per second. Hence in numbers we shall have $v=\frac{1}{448}(1040832+193542+5616-9792-3896)=2733$ ft. when the first five terms only of the series are taken; a number quite sufficient for our further enquiries.

As to the space described by the rocket it is $x=$ fluent $\dot{vt}=\frac{q}{2l}t^2+\frac{pq+2gpl}{6l^2}t^3+\frac{p^2q+2gp^2l-kq^2}{12l^3}t^4+$ &c. $=$ (when $t=a$) $\frac{a^2}{2l}\Big\{q+\frac{ap}{3l}(q+2gl)+\frac{a^2}{6l^2}$

$$\left\{ p^2(q+2gl)-kq^2 \right\} + \frac{a^3}{10l^3} \left\{ p^3(q+2gl) - 2kpq(q+gl) \right\}$$

$$+ \&c. \Big\} = \frac{3}{896}(1040832 + 129028 + 2808 - 3916 - 649)$$

$= 3910$ feet; the height of the rocket at the end of its burning.

From the numbers here brought out, the above series is shewn to be of a remarkable nature; and such, it is presumed, as very seldom occurs in practice. We observe the first three terms to be positive, and to decrease with common regularity; when a sudden violation of law takes place, and the fourth term becomes negative, and much greater than that which immediately precedes it. The fifth term being also negative and not uncommon with regard to the fourth, we may conclude perhaps (as the finding and working out more terms to give certainty to the thing is extremely laborious), that the series will now observe a proper law; in which case a very few feet more would be added to the foregoing velocity by the summation of a great number of its terms. Indeed it can be shewn that it is very nearly equal to the truth by reference to the similar result obtained in the 7th proposition, and the destruction of velocity by the retardive force of gravity in the time of the rocket's burning.

34. To find how far the rocket will farther ascend with its acquired velocity.

Let x = any variable distance from the point to which the rocket has already ascended,

v = the velocity at that point,

a = 2733 feet the acquired velocity.

Then $\frac{Rv^2}{b^2}$ will be the resistance of the medium to the rocket when moving with velocity v; or putting $b = 1$ as before, Rv^2 will express that resistance. Hence $\frac{w+Rv^2}{w}$ will be the retardive force to the rocket; and consequently $\dot{x} = \frac{-v\dot{v}}{2gf} = \frac{-w}{2g} \cdot \frac{v\dot{v}}{w+Rv^2}$; the fluent of which is $\frac{-w}{4gR} \cdot$ hyp. log. $(w+Rv^2)$.

Now $x = 0$ when $v = a$; therefore the fluent corrected is

$$x = \frac{w}{4gR}\left\{\text{hyp. log. } (w + Ra^2) - \text{hyp. log. } (w+Rv^2)\right\};$$

which in the extreme case where $v = 0$, is

$$x = \frac{w}{4gR}\text{ hyp. log. }\frac{w + Ra^2}{w}.$$

In numbers, this expression will be found equal to 7914·3 feet; which added to 3910 feet the space before ascended, gives 11824·3 feet for the height to which the rocket will rise before all its motion is destroyed, which is rather more than $2\frac{1}{5}$ miles.

Since $\frac{w}{4gR} \cdot$ hyp. log. $\frac{w + Ra^2}{w + Rv^2} = x$; we shall have hyp. log. $\frac{w+Ra^2}{w+Rv^2} = \frac{4gRx}{w}$; and putting $c = 2{\cdot}718282$ the number whose hyp. log. is unity, $\frac{w + Ra^2}{w + Rv^2} = c^{\frac{4gRx}{w}}$. Whence v is found equal to

$$\frac{\left\{w\left(c^{\frac{4gRx}{w}} + 1\right) + Ra^2\right\}^{\frac{1}{2}}}{(Rc)^{\frac{1}{2}}};$$

the velocity of the rocket corresponding to the space ascended x.

35. To determine the time of motion of the rocket through the above space. We have found the retardive force to the rocket moving with velocity v to be $\frac{w + \text{R}v^2}{w}$.

Therefore $\dot{t} = \frac{-\dot{v}}{2gf} = \frac{-\dot{v}w}{2g(w + \text{R}v^2)} = \frac{-w}{2g\text{R}} \cdot \frac{\dot{v}}{\frac{w}{\text{R}} + v^2}$

the fluent of which is

$$t = \frac{-w}{2g\text{R}}\left(\frac{\text{R}}{w}\right)^{\frac{1}{2}}. \text{ cir. arc to rad. 1, and tan. } \frac{v}{\left(\frac{w}{\text{R}}\right)^{\frac{1}{2}}};$$

$$= \frac{-1}{2g}\left(\frac{w}{\text{R}}\right)^{\frac{1}{2}}. \text{ cir. arc to rad. 1, and tan. } \frac{v}{\left(\frac{w}{\text{R}}\right)^{\frac{1}{2}}}$$

which corrected is

$$t = \frac{1}{2g}\left(\frac{w}{\text{R}}\right)^{\frac{1}{2}} \left\{ \text{arc to rad. 1, and tan. } \frac{a}{\left(\frac{w}{\text{R}}\right)^{\frac{1}{2}}} - \right.$$

$$\left. \text{arc to rad. 1, and tan. } \frac{v}{\left(\frac{w}{\text{R}}\right)^{\frac{1}{2}}} \right\}:$$

whence, in the case where v vanishes, we shall have

$$t = \frac{1}{2g}\left(\frac{w}{\text{R}}\right)^{\frac{1}{2}}. \text{ cir. arc to rad. 1, and tangent } \frac{a}{\left(\frac{w}{\text{R}}\right)^{\frac{1}{2}}};$$

which in numbers (retaining the same values of a, R, &c. as before) $= 9\cdot74834 \times 1\cdot457 = 14\cdot2$ seconds or $14\frac{1}{5}$ seconds.

Hence the whole time of the rocket's ascent is $17\frac{1}{5}$ seconds.

36. But to determine what time will elapse from the rocket's first going off to its return to the earth; we must find how long it will be in descending from the

whole height to which it has risen. To this end it will be first necessary to enquire what velocity will make the resistance of the medium to be an exact counterbalance to gravity; and thence cause the motion of the rocket to become uniform.

Now $w - Rv^2$ being in this case the moving force; $\frac{w - Rv^2}{w}$ will be the accelerative force; which when the body moves uniformly, is nothing. Therefore putting $\frac{w - Rv^2}{w} = 0$, and reducing the equation we shall have $v = \left(\frac{w}{R}\right)^{\frac{1}{2}}$ for the velocity of the rocket when the resistance will be equal to the force of gravity; or when the motion of the machine becomes equable.

By the theory of variable motions,

$$\dot{t} = \frac{\dot{v}}{2fg} = \frac{w\dot{v}}{2g(w - Rv^2)} = \frac{w}{2gR} \cdot \frac{\dot{v}}{\frac{w}{R} - v^2};$$

whereof the fluent is

$$t = \frac{w}{2gR} \cdot \frac{1}{2\left(\frac{w}{R}\right)^{\frac{1}{2}}} \cdot \text{hyp. log.} \frac{\left(\frac{w}{R}\right)^{\frac{1}{2}} + v}{\left(\frac{w}{R}\right)^{\frac{1}{2}} - v}$$

$$= \frac{1}{4g}\left(\frac{w}{R}\right)^{\frac{1}{2}} \cdot \text{hyp. log.} \frac{\left(\frac{w}{R}\right)^{\frac{1}{2}} + v}{\left(\frac{w}{R}\right)^{\frac{1}{2}} - v}.$$

Now when $t = 0$, $v = 0$, and the whole vanishes. Therefore in that case of the fluent where $v = \left(\frac{w}{R}\right)^{\frac{1}{2}}$, we shall have,

$$t = \frac{1}{4g}\left(\frac{w}{R}\right)^{\frac{1}{2}} \cdot \text{hyp. log.} \frac{\left(\frac{w}{R}\right)^{\frac{1}{2}} + \left(\frac{w}{R}\right)^{\frac{1}{2}}}{\left(\frac{w}{R}\right)^{\frac{1}{2}} - \left(\frac{w}{R}\right)^{\frac{1}{2}}}$$

equal to infinity; which shews that the rocket can never acquire the exact velocity $\left(\frac{w}{R}\right)^{\frac{1}{2}}$, but in an infinite time.

To find t therefore, we must first determine what velocity the rocket will acquire in descending the space a; which being substituted in the expression for t, the value of this will then be obtained.

$$\text{Now } \dot{x} = \frac{v\dot{v}}{2gf} = \frac{wv\dot{v}}{2g(w - Rv^2)} = \frac{w}{2g} \cdot \frac{v\dot{v}}{w - Rv^2}$$

the fluent of which corrected, is

$$x = \frac{w}{4gR} \cdot \text{hyp. log.} \frac{w}{w - Rv^2}.$$

Let $c = 2{\cdot}718282$, the number whose hyp. log. is unity.

$$\text{Then } c^{x} = \left(\frac{w}{w - Rv^2}\right)^{\frac{w}{4gR}},$$

$$\text{and } v = \frac{\left\{w\left(c^{\frac{4gRx}{w}} - 1\right)\right\}^{\frac{1}{2}}}{R^{\frac{1}{2}} c^{\frac{2gRx}{w}}};$$

In which, writing 11824·3 for x, and the several numeral values for w, R, &c.; v will be found equal to

$$\frac{\left\{288\left(2{\cdot}71828^{6{\cdot}1565} - 1\right)\right\}^{\frac{1}{2}}}{{\cdot}0153 \times 2{\cdot}71828^{3{\cdot}07825}}$$

= 350·2 feet. Whence,

$$t = \frac{1}{4g}\left(\frac{w}{R}\right)^{\frac{1}{2}} \text{. hyp. log. } \frac{\left(\frac{w}{R}\right)^{\frac{1}{2}} + v}{\left(\frac{w}{R}\right)^{\frac{1}{2}} - v} = 48''.2984$$

And consequently the time that elapses from the going off of the rocket to its return to the earth, is 65''.498, or 1 min. $5''\frac{1}{2}$ nearly.

37. In the solution to this problem, the density of the medium (that of our atmosphere) is supposed to be the same throughout the rocket's ascent; and the force of gravity also uniform. Now neither of these suppositions strictly obtains; the former varying in such manner that when the heights increase in arithmetical progression, the densities decrease in geometrical progression; and the latter varies as the inverse square of the distance from the earth's centre. Unless, therefore, the decrease of the force of gravity balances in a great measure the decrease of density of the medium, the rocket's height will be affected from such circumstance; and will be somewhat greater than what we have above determined it.

In the same solution also, the resistance of the air to the motion of the rocket is supposed to vary directly as the square of the velocity; an hypothesis which experiments disprove when applied to military projectiles with cannon balls. But it is to be apprehended, that in the motion of rockets, the deviation from this law is scarcely to be regarded; since what takes place in the flight of shot and shells to violate it, is in a great measure obviated in the rockets, by the extreme heat of the flame that rushes from them; which rarifying the ambient air promotes the motion of the particles striking the head of the rocket, towards its hinder parts; and since it is only the immediate motions of such particles backwards that can cause the law to obtain (for it would obtain precisely, if, after the impact of the particles they had no power to impel others lying before them, but either glided off from the surface struck, or had their force annihilated by it at the moment of striking), it is to be expected that the conclusions here brought out, which are grounded on this law of resistance, will be found to agree pretty correctly with the results determined from experiment.

But if they should not, let then the law of resistance be as the nth power of the velocity, and the method of solution will remain precisely the same as before. For it is only the fourth equation in the preceding process, namely, $kv^n \dot{t} =$ &c. that will vary or become affected by any deviation from the law we have assumed; and therefore when this shall have been settled by experiment (the only way in which it ever can be settled), and the absolute resistance determined in any one case of velocity, and the real strength of the rocket composition ascertained; then, and *not till then*, shall we be able to offer any *unerring* rules to the military practitioner.

PROP. XI.

38. *To determine whether the motion of a rocket ascending vertically in the atmosphere can ever become uniform; the law of resistance being directly as the square of the velocity, as before.*

When the motion of a body becomes uniform, or the velocity a maximum, the accelerative force is then nothing: therefore putting $\frac{(sned^2b^2 - \text{R}v^2)a}{(am - ct).b^2} - 1$ the accelerative force (see the last *Prop.*) $= 0$, and reducing

the equation, we have $v = b . \left(\frac{sned^2a - am + ct}{Ra}\right)^{\frac{1}{2}}$.
Whence it appears, that the velocity, and consequently the motion of the rocket can never become equable; being in terms of t, the time of its burning; but will be greater and greater unto the end of the time t, when the velocity will continually decrease till the whole is destroyed by the retardive force of gravity. And it is moreover evident, that the motion of a rocket can never become uniform under any law of resistance whatever.

PROP. XII.

39. *All things remaining as in the* 10*th Proposition, to find the velocity and space described by the rocket, when it is influenced only by the impelling force of the composition and the resistance of the medium.*

Here, gravity not acting, the accelerative force of the rocket at the end of the time t will be $\frac{(sned^2b^2 - Rv^2)a}{(am - ct)\, b^2}$ as determined in *Prop.* 9. Therefore $\dot{v} = 2gf\dot{t} = \frac{(sned^2b^2 - Rv^2).2ag\dot{t}}{(am - ct).b^2}$ = (putting $h = 2agsned^2b^2$, $k = 2agR$, $l = amb^2$, and $p = cb^2$) $\frac{h\dot{t} - kv^2\dot{t}}{l - pt}$; and $\frac{\dot{v}}{h - kv^2} = \frac{\dot{t}}{l - pt}$, whereof the fluent is $\frac{1}{2.(hk)^{\frac{1}{2}}}$. hyp. log. $\frac{\left(\frac{h}{k}\right)^{\frac{1}{2}} + v}{\left(\frac{h}{k}\right)^{\frac{1}{2}} - v} = -\frac{1}{p}$. hyp. log. $\left(\frac{l}{p} - t\right)$; which, when $v=0$, and $t=0$, is $0 = -\frac{1}{p}$. hyp. log. $\frac{l}{p}$: therefore the correct fluent is

$$\frac{1}{2(hk)^{\frac{1}{2}}} \cdot \text{hyp. log.} \frac{\left(\frac{h}{k}\right)^{\frac{1}{2}} + v}{\left(\frac{h}{k}\right)^{\frac{1}{2}} - v} = \frac{1}{p} \cdot \left\{\text{hyp. log.} \frac{l}{p}\right.$$

$$\left. - \text{hyp. log.}\left(\frac{l}{p} - t\right)\right\} = \frac{1}{p} \cdot \text{hyp. log.} \frac{l}{l-pt}$$: and hence by the nature of logarithms

$$\left(\frac{\left(\frac{h}{k}\right)^{\frac{1}{2}} + v}{\left(\frac{h}{k}\right)^{\frac{1}{2}} - v}\right)^{\frac{p}{2(hk)^{\frac{1}{2}}}} = \frac{l}{l-pt}$$: or, putting $\left(\frac{h}{k}\right)^{\frac{1}{2}} = j$, and $\frac{(hk)^{\frac{1}{2}}}{p} = w$, we shall have $\frac{j+v}{j-v} = \frac{l^w}{(l-pt)^w}$; and by reducing this equa. $v = \frac{jl^w - j(l-pt)^w}{l^w + (l-pt)^w}$; which, when $t = a$, is $v = \frac{jl^w - j(l-pt)^w}{l^w + (l-pt)^w}$, the velocity of the rocket when it just ceases burning. Or, restoring the values of j, w, l, h, &c., the velocity of the rocket in this case will be expressed by

$$\frac{db.\left(\frac{sne}{R}\right)^{\frac{1}{2}} \cdot \left\{(amb^2)^{\frac{4agd(sneR)^{\frac{1}{2}}}{cb}} - (amb^2 - acb^2)^{\frac{4agd(sneR)^{\frac{1}{2}}}{cb}}\right\}}{(amb^2)^{\frac{4agd(sneR)^{\frac{1}{2}}}{cb}} + (amb^2 - acb^2)^{\frac{4agd(sneR)^{\frac{1}{2}}}{cb}}};$$

or taking $R = \cdot0002343$, and $b = 1$, as in *Prop.* 9, it is

$$\frac{d\left(\frac{sne}{\cdot0002343}\right)^{\frac{1}{2}} \cdot \left\{(am)^{\frac{4agd}{c}(\cdot0002343sne)^{\frac{1}{2}}} - (am - ac)^{\frac{4adg}{c}(\cdot0002343sne)^{\frac{1}{2}}}\right\}}{(am)^{\frac{4agd}{c}(\cdot0002343sne)^{\frac{1}{2}}} + (am - ac)^{\frac{4agd}{c}(\cdot0002343sne)^{\frac{1}{2}}}};$$

and substituting the values for a, c, d, &c., which are as follow: namely,

$$s = 1000.$$
$$n = 230 \text{ ozs.}$$
$$w = 18 \text{ lbs.} = 288 \text{ ozs.}$$
$$c = 10 \text{ lbs.} = 160 \text{ ozs.}$$
$$m = w + c = 448 \text{ ozs.}$$
$$a = 3 \text{ sec.}$$
$$d = \tfrac{1}{4} \text{ ft.}$$
$$g = 16 \text{ ft.}$$
$$e = \cdot 7854.$$

it is $$v = \frac{6941{\cdot}575\left(1344^{1{\cdot}95171} - 864^{1{\cdot}95171}\right)}{1344^{1{\cdot}95171} + 864^{1{\cdot}95171}}$$

$$= \frac{6941{\cdot}575 \times 737094}{1814186} = 2820{\cdot}325$$ feet; which is therefore the greatest velocity the rocket can acquire, and which it does acquire at the end of its burning.

It is somewhat remarkable, that the whole resistance of the air to the rocket, on the supposition that gravity does not act, should so nearly approximate to the effect of this force (reckoned as constant), when there is no consideration of any resistance from the former; the deviation causing no more than ($2896{\cdot}9895 - 2820{\cdot}325 =$) $76{\cdot}6645$ feet per second difference in the greatest velocity of the rocket on the side of gravity.

To find the space described. By theorem the 10th of variable motions $\dot{x} = v\dot{t} = \dfrac{jl^w\dot{t} - j\dot{t}\,(l-pt)^w}{l^w + (l-pt)^w} = j\dot{t} - \dfrac{2j\dot{t}\,(l-pt)^w}{l^w + (l-pt)^w}$. Put $l - pt = \tau$; then $\dot{\tau} = -p\dot{t}$, and $\dot{t}$

$= \frac{-\dot{T}}{p}$. Whence $\dot{x} = \frac{-j\dot{T}}{p} + \frac{2j}{p} \cdot \frac{T^w \dot{T}}{l^w + T^w}$ = (by expanding $\frac{T^w \dot{T}}{l^w + T^w}$ in a series) $-\frac{j\dot{T}}{p} + \frac{2j}{p} - \left(\frac{T^w \dot{T}}{l^w} - \frac{T^{2w}\dot{T}}{l^{2w}} + \frac{T^{3w}\dot{T}}{l^{3w}} - \frac{T^{4w}\dot{T}}{l^{4w}} + \&c.\right)$; the fluent of which is $x = \frac{-jT}{p} + \frac{2j}{p}\left(\frac{T^{w+1}}{(w+1)\,l^w} - \frac{T^{2w+1}}{(2w+1)\,l^{2w}} + \frac{T^{3w+1}}{(3w+1)\,l^{3w}} - \frac{T^{4w+1}}{(4w+1).l^{4w}} + \&c.\right) = \frac{-jT}{p} + \frac{2jT^{w+1}}{pl^w} \cdot \left(\frac{1}{w+1} - \frac{T^w}{(2w+1)l^w} + \frac{T^{2w}}{(3w+1)\,l^{2w}} - \frac{T^{3w}}{(3w+1)l^{3w}} + \&c.\right) = \frac{j}{p} \cdot \left\{-(l-pt) + \frac{2(l-pt)^{w+1}}{l^w} \times \left(\frac{1}{w+1} - \frac{(l-pt)^w}{(2w+1).\,l^w} + \frac{(l-pt)^{2w}}{(3w+1)\,l^{2w}} - \frac{(l-pt)^{3w}}{(4w+1).\,l^{3w}} + \&c.\right)\right\}$; and the fluent corrected is $x = \frac{j}{p}\left\{l - 2l \cdot \left(\frac{1}{w+1} - \frac{1}{2w+1} + \frac{1}{3w+1} - \frac{1}{4w+1} + \&c.\right)\right\} + \frac{j}{p} \cdot \left\{-(l-pt) + \frac{2(l-pt)^{w+1}}{l^w} \times \left(\frac{1}{w+1} - \frac{(l-pt)^w}{(2w+1).\,l^w} + \frac{(l-pt)^{2w}}{(3w+1)\,l^{2w}} - \frac{(l-pt)^{3w}}{(4w+1).\,l^{3w}} + \&c.\right)\right\}$ = (when $t = a$) $j\left\{a + \frac{2\,(l-ap)^w}{l^w} \cdot \left(\frac{1}{w+1} - \frac{(l-ap)^w}{(2w+1).\,l^w} + \frac{(l-ap)^{2w}}{(3w+1).\,l^{2w}} - \frac{(l-ap)^{3w}}{(3w+1).\,l^{3w}} + \&c.\right) - \frac{2l}{p} \cdot \left(\frac{1}{w+1} - \frac{1}{2w+1}\right.\right.$

$+ \frac{1}{3w+1} - \&c. \Big) \Big\}$; for the space described by the rocket at the end of the time t.

40. Now to determine how far the rocket will farther move before its motion is wholly destroyed. Put a = the velocity at the end of its burning = 2820·325 feet per second, and v any variable velocity corresponding to the space x; w = weight of the rocket = 448 ozs., and R = ·0002343 ounces, the resistance of the medium to the rocket when moving with a velocity of 1 foot per second. Then Rv^2 will be the resistance to velocity v, and $\frac{Rv^2}{w}$ the force by which the rocket is retarded by the fluid. Hence $\dot{x} = \frac{-v\dot{v}}{2fg} = -\frac{w\dot{v}}{2gRv}$, and $x = \frac{-w}{2gR}$. hyp. log. v; and the fluent corrected $x = \frac{w}{2gR}$. hyp. log. a. Which by substitution of numbers is = 21672 feet.

Hence, it appears, that after the burning of the rocket ceases, it will move to a distance of 21672 feet, or somewhat more than $4\frac{1}{10}$ miles, before all its motion is destroyed, when it will remain at rest in the medium, there being no force to influence it in any manner or direction whatever, and having no power to create motion in itself.

41. As to the time that the rocket would be in moving through this space, it will be had as follows. The same substitution as above being retained, the general fluxional expression for the time $(\dot{t})$ namely $\frac{-\dot{v}}{2gf}$ will be found =

$\frac{-\dot{v}}{2g\text{R}v^2} = \frac{-1}{2g\text{R}} \cdot \frac{\dot{v}}{v^2}$ (substituting $\frac{\text{R}v^2}{w}$ for f as before)

the fluent of which is $t = \frac{1}{2g\text{R}v}$. Now when $t = 0$, $v = a$, therefore the correct fluent of the time is $t = \frac{1}{2g\text{R}v} - \frac{1}{2g\text{R}a}$ which, on v becoming nothing, will be infinite. So that it appears, that the rocket will not describe the above space but in an infinite time.

Suppose $v = 1$ foot; then $t = \frac{a-1}{2g\text{R}a} = 133{\cdot}344$ seconds or 2 min. 13 seconds. That is, the rocket will only have been in motion 2 min. 13 sec. after it has acquired the greatest velocity from its burning, before the celerity of its motion will be reduced to 1 foot per second; and yet, notwithstanding this great annihilation of velocity in so short a time, the remaining small part will not in any finite time be destroyed, though we know the limit at which the rocket would attain a state of quiescence.

And from the result here determined, we conclude, that into whatever medium a body is projected with any given velocity, great or small, it will in no finite time lose all its motion. So that, if the planetary bodies were moving in a resisting medium, and gravity should suddenly be destroyed, the bodies would all pursue rectilinear paths (that would be tangents to their orbits) to certain finite distances, which would not be wholly described by them but in infinite times.

PROP. XIII.

PROP. XIII.

42. *Given the time that elapses from the first going off of a rocket to its return to the earth, considering it to have ascended vertically; and the velocity or force of the wind; to find at what distance from the point of projection the rocket will fall.*

Before entering upon the solution of this problem, it will be proper to make a few preliminary observations. In the first place, then, we are to consider, that when a body from rest is put into motion by a fluid, it can never acquire a velocity greater than that with which the fluid moves; that when it has acquired that velocity, it will be relatively at rest, or move uniformly and in common with the fluid with its velocity. And in all other cases the velocity with which the fluid strikes the body to accelerate its motion, will be equal to the difference of the given velocity of the fluid and the velocity acquired by the body. Thus a vessel abandoned to, or influenced only by a current, can never acquire a velocity greater than that with which the current moves; nor indeed *exactly* equal to it in any finite time, as shall be hereafter shewn; and in any intermediate state the current will act upon the body only with the difference of its velocity and the acquired velocity of the body. If another force as that of the wind conspire with that of the stream, the body may acquire a greater velocity than the stream; that is to say if the velocity of the former be greater than that of the latter; but it can never arrive at a velocity equal to that of the wind, on account of the resistance that will be opposed to its motion after it has attained a greater velocity than that of the stream. Therefore, in the case before us, the rocket in its sideral motion will never arrive at a velocity greater than that of the wind, nor precisely equal to it in any finite time; and consequently will suffer no resistance from the medium in its deflection from the original line of projection.

Again, the direction of motion of the wind being horizontal, the action of the same upon the rocket will be at right-angles to its axis, provided there be no rotation of the rocket throughout its motion, which we will suppose there is not. Therefore the force of the wind to move the rocket in its own direction in the first instance will be $\frac{nV^2rh}{3g}$, as determined in *Prop. Art.* 31; and at any other instant, calling the velocity acquired v, it will be $\frac{nrh}{3g}(V-v)^2$, the force varying as the square of the velocity directly.

Let $R = \frac{nrh}{3g}$,

w = weight of the case of the rocket, considered as merely cylindrical,

c = weight of the matter contained in it,

$m = w+c$ the weight of both the case and the composition,

a = time of the rocket's burning,

v = velocity of the rocket in its sideral motion at the end of the time t.

Then $R(V-v)^2$ being the impelling force of the wind, and $\frac{am-ct}{a}$ (See *Prop. Art.* 17.) the weight of the mass at the end of the time t; $\frac{aR(V-v)^2}{am-ct}$ will be the accelerative force of the rocket at the end of that time.

Now $\dot{v} = 2gft = \frac{2ag\text{R}\,(\text{V}-v)^2}{am-ct}$; and $\frac{\dot{v}}{(\text{V}-v)^2} =$

$\frac{2ag\text{R}\dot{t}}{am-ct}$; the fluent of which, is

$$\frac{1}{\text{V}-v} = -\frac{2ag\text{R}}{c}\text{ hyp. log. }(am-ct),$$

which corrected, is

$\frac{v}{\text{V}^2-\text{V}v} = \frac{2ag\text{R}}{c}\text{ hyp. log. }\frac{am}{am-ct}$, and hence

$$v = \frac{\frac{2ag\text{RV}^2}{c}\text{ hyp. log. }\frac{am}{am-ct}}{\frac{2ag\text{RV}}{c}\text{ hyp. log. }\frac{am}{am-ct}+1};$$

or, putting $p = \frac{2ag\text{RV}}{c}$, the equation will be

$$v = \frac{\text{V}p\text{ . hyp. log. }\frac{am}{am-ct}}{p\text{ hyp. log. }\frac{am}{am-ct}+1}.$$

Now writing k for hyp. log. $am + \frac{1}{p}$, we shall have for the fluxion of the space $(\dot{s}) = v\dot{t}$, after reduction,

$$\text{V}\dot{t} - \frac{\text{V}\dot{t}}{p\left\{k - \text{hyp. log. }(am-ct)\right\}}.$$

Let $am-ct = z$; then $\dot{t} = \frac{-\dot{z}}{c}$ and $\dot{s} = -\frac{\text{V}\dot{z}}{c} + \frac{\dot{z}\text{V}}{pc\,(k - \text{hyp. log. }z)}$ = (by expanding the fraction $\frac{\dot{z}}{k-\text{hyp. log. }z}$) $-\frac{\text{V}\dot{z}}{c} + \frac{\text{V}}{cp}\left\{\frac{\dot{z}}{k} + \frac{\dot{z}}{k^2}\right.$ hyp. log.

$\dot{z} + \frac{\dot{z}}{k^3}$ hyp. log.^{2} $z + \frac{\dot{z}}{k^4}$ hyp. log.^{3} $z + \frac{\dot{z}}{k^5}$ hyp. log.^{4} z + &c. $\Big\}$;

the fluent of which is

$$s = \frac{-vz}{c} + \frac{v}{cp}\Big\{\frac{z}{k} + \frac{1}{k^2}(z \text{ hyp. log. } z - z) + \frac{1}{k^3}\Big(z \text{ hyp. log.}^2 z - 2A\Big) + \frac{1}{k^4}\Big(z \text{ hyp. log.}^3 z - 3B\Big) + \&c.\Big\};$$

where A, B, C, &c. denote the foregoing terms with their proper signs. Or from further reduction, $s = -\frac{vz}{c} + \frac{vz}{ckp}\Big\{1 + \frac{1}{k}(\text{hyp. log. } z - 1) + \frac{1}{k^2}\Big(\text{hyp. log.}^2 z - 2A\Big) + \frac{1}{k^3}\Big(\text{hyp. log.}^3 z - 3B\Big) + \&c.\Big\}$ A, B, &c. denoting in like manner the foregoing terms with their proper signs; and so forward.

Now when $s = 0$, $t = 0$, and $z = am$; therefore the correct equation or fluent will be

$$s = \frac{vam}{c} - \frac{vam}{pck}\Big\{1 + \frac{1}{k}(\text{hyp. log. } am - 1) + \frac{1}{k^2}\Big(\text{hyp. log.}^2 am - 2A\Big) + \frac{1}{k^3}\Big(\text{hyp. log.}^3 am - 3B\Big) + \&c.\Big\} - \frac{v(am - ct)}{c} + \frac{v(am-ct)}{ckp}\Big\{1 + \frac{1}{k}(\text{hyp. log. }(am - ct) - 1) + \frac{1}{k^2}\Big(\Big\{\text{hyp. log. }(am - ct)\Big\}^2 - 2A\Big) + \frac{1}{k^3}\Big(\Big\{\text{hyp. log. }(am-ct)\Big\}^3 - 3B\Big) + \&c.\Big\} =$$

$$\frac{vt(k-1)}{k} - \frac{vam}{pck^2}\Big\{(\text{hyp. log. } am - 1) + \frac{1}{k}\Big(\text{hyp. log.}^2 am - 2A\Big) + \frac{1}{k^2}\Big(\text{hyp. log.}^3 am - 3B\Big) + \&c.\Big)$$

$+ \frac{v\,(am-ct)}{ck^2p}\Big\{(\text{hyp. log. }(am-ct)-1)+\frac{1}{k}\Big(\big\{\text{hyp. log. }(am-ct)\big\}^2-2A\Big)+\frac{1}{k^2}\Big(\big\{\text{hyp. log. }(am-ct)\big\}^3-3B\Big)+\&c.\Big\}$; from whence, writing a for t, the deflection of the rocket at the end of its burning will be determined.

The fluent of $\frac{v\dot{z}}{pc\,(k-\text{hyp. log. }z)}$ might have been otherwise derived by dividing $\dot{z}$ by k *minus* the series expressing the hyp. log. of z, and then taking the fluent of each term separately. Thus the hyp. log. $z=(z-1)-\frac{1}{2}(z-1)^2+\frac{1}{3}(z-1)^3-\frac{1}{4}(z-1)^4+\&c.$; therefore by division we have,

$\frac{v\dot{z}}{pc\,(k-\text{hyp. log. }z)}=\frac{v}{pc}\Big\{\frac{\dot{z}}{k}+\frac{1}{k^2}(z-1)\dot{z}-\frac{k-2}{2k^3}(z-1)^2\dot{z}+\frac{k^2+3}{3k^4}(z-1)^3\dot{z}-\&c.\Big\}$; the fluent of which is $\frac{v}{pc}\Big\{\frac{1}{k}z+\frac{1}{2k^2}(z-1)^2-\frac{k-2}{6k^3}(z-1)^3+\frac{k^2+3}{12k^4}(z-1)^4-\&c.\Big\}$. So that s, or the fluent of $\frac{-v\dot{z}}{c}+\frac{v\dot{z}}{pc\,(k-\text{hyp. log. }z)}$, is $\frac{-vz}{c}+\frac{v}{pck}\Big\{z+\frac{1}{2k}(z-1)^2-\frac{k-2}{6k^2}(z-1)^3+\frac{k^2+3}{12k^3}(z-1)^4-\&c.\Big\}$;

which corrected, is

$\frac{vam}{c}-\frac{v}{pck}\Big\{am+\frac{1}{2k}(am-1)^2-\frac{k-2}{6k^2}(am-1)^3+\frac{k^2+3}{12k^3}(am-1)^4-\&c.\Big\}-\frac{v\,(am-ct)}{c}+\frac{v}{pck}$

$$\left\{ (am - ct) + \frac{1}{2k}(am - ct - 1)^2 - \frac{k-2}{6k^2}(am - ct - 1)^3 + \frac{k^2+3}{12k^3}(am - ct - 1)^4 - \&c. \right\} = \text{v}t - \frac{\text{v}}{pck}\left(\left\{ am + \frac{1}{2k}(am - 1)^2 - \frac{k-2}{6k^2}(am - 1)^3 + \frac{k^2+3}{12k^3}(am - 1)^4 - \&c. \right\} - \left\{ (am - ct) + \frac{1}{2k}(am - ct - 1)^2 - \frac{k-2}{6k^2} \times (am - ct - 1)^3 + \frac{k^2+3}{12k^3}(am - ct - 1)^4 - \&c. \right\}\right).$$

Where t being made $= a$, will give the deviation of the rocket from the line of projection at the end of its burning as before.

43. To find how much the rocket will be farther deflected during the remainder of the given time.

Let v now denote the velocity with which the wind strikes the body at the end of its burning; and v any accession of velocity of the rocket in its sideral motion after that period in the time t. Then $\frac{\text{R}(\text{v} - v)^2}{w}$ will be the accelerative force of the rocket; the weight of the whole mass being now a constant quantity. Hence,

$$\dot{v}(=2gf\dot{t}) = \frac{2g\text{R}\dot{t}(\text{v} - v)^2}{w}\text{; or, } \frac{\dot{v}}{(\text{v} - v)^2} = \frac{2g\text{R}\dot{t}}{w}$$

whereof the correct fluent, putting $q = \frac{2g\text{R}}{w}$, will be

$$\frac{v}{\text{v}^2 - \text{v}v} = qt;$$

whence by reduction, we shall have

$$v = \frac{\text{v}^2 qt}{\text{v}qt + 1};$$

where it is evident that v can never be equal to v, except in the case where t is infinite. Again,

$$s = v\dot{t} = \frac{\text{v}^2 qt\dot{t}}{\text{v}qt + 1} = \text{v}\dot{t} - \frac{\text{v}\dot{t}}{\text{v}qt + 1},$$

and

$$s = \text{v}t - \frac{1}{q} \text{ hyp. log. } (\text{v}qt + 1),$$

wanting no correction, since when $s = 0$, $t = 0$, and the whole vanishes. Therefore the additional deflection of the body from its original line of projection during the remainder of the given time is expressed by

$$\text{v}t - \frac{1}{q} \text{ hyp. log. } (\text{v}qt + 1).$$

44. For an example. Let us suppose that the wind is blowing the common gale of 15 miles an hour; or with the velocity of 22 feet per second; and that the time of motion of the rocket as given by the proposition is 63″; also let the values of the other letters included in the problem be as follow: namely,

w = 18 lbs. = 288 ozs.	h = 3 ft.
c = 10 lbs. = 160 ozs.	V = 22 ft. (as just mentioned.)
m = 28 lbs. = 448 ozs.	$n = 1\frac{2}{9}$
a = 3 sec. $r = \frac{1}{2}$ ft.	g = 16 feet.

Then p (first part of the investigation) $= \frac{2ag\text{RV}}{c} =$ $\frac{2ag\text{RV}}{c} = \frac{2ag\text{V}}{c} \times \frac{nrh}{3g} = \frac{2anrh\text{V}}{3c} = \frac{121}{240}$; and k $= \text{hyp. log. } am + \frac{1}{p} = 9{\cdot}186876$; which values, with the rest, being substituted in the first 20 terms of the first series expressive of the deflection of the rocket at the end of the time a, will be found $= 7{\cdot}10096$ feet. Now

$$v = \frac{\text{v}p \,.\, \text{hyp. log. } \frac{am}{am - ct}}{p \,.\, \text{hyp. log. } \frac{am}{am - ct} + 1};$$

Therefore making $t = a$, and reducing the expression, $v = 4$ feet; and hence the value of v in the second part of the process will be $= 18$ ft. Also $q = \frac{2g\text{R}}{w} = \frac{2g}{w} \times \frac{nrh}{3g} = \frac{2nrh}{3w} = \frac{11}{2592}$; and $t = 60''$. Whence,

$$s = \text{v}t - \frac{1}{q} \text{ hyp. log. } (\text{v}qt + 1) =$$

$$1080 - \frac{2592}{11} \text{ hyp. log. } \frac{67}{12} = 674{\cdot}75589 \text{ feet.}$$

And consequently the whole deflection of the rocket is 681·45685 feet.

When the velocity of the wind is not so considerable, the deflection will be accurately enough had from the latter formula only; for the deviation in such cases at the end of the rocket's burning will be very trifling, whether we consider the mass to vary (as it really does) during that time, or the constant weight of the rocket when its body is consumed. And the difference of the acquired velocities in the two cases will be too small to cause any sensible alteration in the final results.

45. For another example. Suppose the wind to blow at the very gentle rate of two feet per second, and the time of motion of the rocket as given by the proposition $50''$: also

$w = 14$ lbs. $= 224$ ozs.	$h = \frac{1}{2}$ feet.
$c = 8$ lbs. $= 128$ ozs.	$n = \frac{11}{9}$
$r = \frac{1}{3}$ foot.	$g = 16$ feet.

$$\text{Then } s = \text{v}t - \frac{1}{q} \text{ hyp. log. } (\text{v}qt + 1) =$$

$$100 - \frac{18144}{55} \text{ hyp. log. } \frac{5911}{4536} = 12{\cdot}655 \text{ feet.}$$

If the velocity of the wind be that of 11 feet per second; the deflection of the rocket will be 79 yards very nearly. But in neither of these examples is the weight of any appendage to the rocket taken into the account, which would alter the results very materially; making them much smaller than they are here found.

46. It may not be amiss now to enquire, how far a shell would be driven by the wind from the vertical line of motion during the whole time of its ascent and descent, which we will suppose to be 63″, as in the first of the foregoing examples. Let the shell be that, the external diameter of which is 13 inches, the weight whereof when loaded is 2 cwt. or 3584 ounces. Then $\frac{pnv^2r^2}{8g}$ (where $p = 3{\cdot}1416$) being the expression for the force of the fluid (*Art.* 30.) on the whole hemisphere of the body, we shall have R in this case $= \frac{pnr^2}{8g}$; and $q = \frac{2g\text{R}}{w} = \frac{2g}{w} \times \frac{pnr^2}{8g} = \frac{pnr^2}{4w} = \frac{243{\cdot}343}{3096576}$. Whence

$$s = \text{v}t - \frac{1}{q} \text{ hyp. log. } (\text{v}qt + 1) =$$

$$1386 - \frac{3096576}{243{\cdot}343} \text{ hyp. log. } \frac{27252{\cdot}7741}{24576} = 1386 -$$

$1315{\cdot}19 = 70{\cdot}41$ feet.

Therefore, notwithstanding the immense weight of the projectile, the wind acting upon it with a velocity of 22 feet per second, for 1 min. and 3 sec., will cause it to fall 70·41 feet from the point whence it was projected, an astonishing deviation for so ponderous a mass.

If the wind struck the body throughout its flight with the same velocity as at first, the deflection of the shell would be 75·480294 feet; or $23\frac{1}{5}$ yards nearly.

Ex. 2.—Let the same shell be thrown obliquely in a given direction, and suppose the time of flight 40″; also the wind to blow directly across the line of fire with the same velocity as before; then will the extreme error of the projectile be found = 31 feet.

If the direction of the wind makes any given angle with that of projection, the result as above determined must be lessened in the ratio of radius to the sine of that angle, to get the true distance of the body from the plane of projection at the end of its flight.

Another example of a cannon ball. Suppose a twelve-pounder, and the time of its motion at a certain elevation, 32″; moreover let the wind be supposed to blow perpendicularly to the vertical plane of projection with a velocity of $29\frac{2}{3}$ feet per second, or at the rate of 20 miles an hour, then we shall have for the maximum error in this case 67·8 feet nearly.

These examples are sufficient to demonstrate the effects of a disturbed atmosphere upon military projectiles, in driving them from their original courses, as well as to caution the practitioner, when in service, of the necessity of attending to this circumstance in cases of detached objects, where these are to be destroyed, and the air happens to be violently agitated; for without some alteration being made in the direction of the engine, the projectile may, in many instances, fall 30 or 40, or even 50 yards from the object, and consequently produce no sort of injury to it whatever. But when the wind is moderate, and does not blow so directly across the projectile, the directing the piece in the plane of the object, will be attended with more certainty perhaps, than when it is pointed somewhat different, from the smallness of alteration that will be required, which, if not strictly maintained, would incur greater error than if it were totally neglected.

PROP. XIV.

47. *Given the time of flight of a rocket, and the angle and direction in which it is thrown, also the direction and velocity of the wind; to determine at what distance from the plane of projection, the rocket will fall; it being supposed not to revolve, but always to retain the position in which it first moved off; or to be parallel in its sideral motion to the line of projection.*

The method of solution to this problem is precisely similar to that of the foregoing. The angle of incidence of the wind against the rocket (considered as a mere cylinder) is given by the proposition: therefore, if this be denoted by f, we shall get for the force of the wind, moving with the velocity of 1 foot per second,

$$\frac{nrhf^3}{2g}\left\{1-\frac{3f^2-1}{6}+\frac{3(f^2-1)^2}{40}+\frac{(f^2+5)(f^2-1)^2}{112}+\&c.\right\}+\frac{np'r^2}{4g}(1-f^2)^{\frac{3}{2}},$$

(where $p' = 3{\cdot}1416$); which is the value of what R represents in the last problem. Hence $p = \frac{2ag\text{RV}}{c}$, and $q = \frac{2g\text{R}}{w}$ will be known; and also $k = \text{hyp. log. } am + \frac{1}{p}$; which being severally substituted in the general expression for the whole deflection of the rocket in the direction of the wind, (determined in the foregoing proposition), namely,

$$\frac{vt(k-1)}{k}-\frac{vam}{pck^2}\left\{(\text{hyp. log. } am-1)+\frac{1}{k}\left(\text{hyp. log.}^2\right.\right.$$

$$am - 2A \Big) + \frac{1}{k^2} \Big(\text{hyp. log.}^3\, am - 3B \Big) + \&c. \Big\} +$$

$$\frac{v\,(am - ct)}{ck^2 p} \Big\{ (\text{hyp. log.}\,(am - ct) - 1) + \frac{1}{k} \Big(\Big\{ \text{hyp.} $$

$$\text{log.}\,(am - ct) \Big\}^2 - 2A \Big) + \frac{1}{k^2} \Big(\Big\{ \text{hyp. log.}\,(am - ct) \Big\}^3$$

$$- 3B \Big) + \&c. \Big\} + vt - \frac{1}{q}\, \text{hyp. log.}\,(vqt + 1),$$

the deflection as required by the proposition may hence be determined: the angle which the line of direction of the wind makes with that of projection being given, and the several letters denoting the same quantities in both investigations.

SCHOLIUM.

48. The solution to this problem, under the various considerations that it involves, even regarding the rocket a mere cylinder, without any appendage whatever, will, perhaps, long remain a desideratum in the true theory of rockets. The force of the wind upon the body at any given instant, as depending upon its position at that instant, is a circumstance which a correct solution must necessarily embrace; and this is of itself no easy thing to determine, including in it the computation of two separate rotations; namely, the one resulting from the action of the wind; and the other as produced by the resistance of the air to the rocket in its descent to the earth by gravity. That there will be these two rotatory motions is evident. For with regard to the first; though the rocket in its sideral motion can never meet with any resistance from the medium, yet the inertia of the varying mass will, in conjunction with the force of the wind (the centre of which force never lying in the same right-

line with the centre of gravity of the varying mass), produce rotation in the body; making that end of it move to leeward which is less heavy than the other. This rotation of the body, like that of the other, will be effected about an imaginary axis, always passing through the centre of gravity of the whole mass; the place of which axis will, therefore, be variable, as long as the rocket continues to burn; receding from the centre of the axis of the rocket towards the head, till a certain quantity of the composition is consumed, when it will return again towards that centre, and at last come into it*. And the angular velocity at any given instant, will be the same about the centre of gravity of the body at that instant,

* To find the greatest distance of the varying centre of gravity of the mass from the centre of the axis of the rocket. Put $a = \frac{1}{2}$ the length of the cylinder or axis, and $x = \frac{1}{2}$ the length of the unconsumed cylinder of composition: then $a - x$ will be the distance of the centres of gravity of the case and of the consumed column of composition. Let $w =$ weight of the whole of the composition; and d that of the case of the rocket; and we shall have $2a : w :: 2x : \frac{wx}{a}$ for the weight of the unfired cylinder of composition: whence $d + \frac{wx}{a}$, will be the weight of the entire mass. And by the nature of the common centre of gravity $d + \frac{wx}{a} : \frac{wx}{a} :: a - x : \frac{w(ax - x^2)}{ad + wx}$ for the distance of that centre from the centre of the axis of the rocket, which when a maximum, its fluxion will be $= 0$; therefore the fluxion of $\frac{w(ax - x^2)}{ad + wx}$, or of $\frac{ax - x^2}{ad + wx}$, being taken and put $= 0$; we shall get finally, $x = \frac{a}{w}\left(dw + d^2\right)^{\frac{1}{2}} - \frac{ad}{w}$; whence the question itself becomes determined.

as about the corresponding centre of spontaneous rotation.

As to the second rotation, it is obvious, that if any body, the ends of which are unequally heavy, move in a resisting medium towards a centre of force, that the heavier end, having greater power to overcome any resistance, will preponderate, and consequently will cause the body to revolve; and the revolution will continue until the body comes into a vertical position, when if no other force acted upon it, it would proceed forward in that position.

The first of these rotations will evidently be the cause of a sensible deflection of the rocket from the plane of projection, when the force of the wind is considerable, and the action of the same against the surface of the rocket not very oblique: nor will this deviation seem strange, when we consider the great velocity that the body acquires during the time it is on fire, and the consequent extensive range afterwards; that if the quantity of rotation be but small at the end of its burning, the ultimate error must be important.

Let us suppose, that at the complete exhaustion of the composition, the rocket should have revolved through an angle of 8°; or that its position at that instant, should make with the position in which it was projected, an angle of that magnitude: also, that it should have acquired a velocity that will carry it to the distance of 1000 yards on the horizontal plane, reckoning from the point where a perpendicular from the rocket falls upon that plane: then it will be found, that independent of the action of any other force, the greatest deflection of the rocket is 139 yards; which if diminished by the distance that it is carried through by the wind, the remainder would still be a difference too considerable to be disregarded in practice. It is on this account that the rocket is thrown in a side wind, in any particular warfare with these machines, somewhat to leeward of the object it is meant to destroy, for if this were not done, it is obvious, from what has been observed, that the weapon could have no effect whatever upon the object, from the distance it would fall from it, and even under the above circumstances, if the wind blew very strongly across the body of the machine, its effect, like all other projectiles, would be sometimes uncertain.

The rotation of a rocket, from windward to leeward, as produced by the action of the wind against it, being inevitable, unless the rocket's motion be directly with, or contrary to the motion of the wind, the rocket-engineer will do well, when in actual service, to bear in remembrance this particular, and to choose such a spot, if possible, from whence he can throw the rockets either directly with, or directly against the wind, at the object to be destroyed; when its effects cannot but be certain, if the object be within its sphere of conflagration. But although circumstances should not be favourable to the choice of such a position when the exigencies of the moment require the throwing of rockets, the certainty of their effects, even upon a single object, will be greatly secured by attending to the foregoing observations: but from no other knowledge than that derived from practice, can any system of warfare with rockets, be so much advanced and brought to perfection.

Having thus far proceeded in the theory of rockets moving in an abandoned state, in different mediums, and pointed out some of the difficulties that must be encountered to the farther extension of it, as well as to its perfection; I shall, after giving a few examples for practice in this section, proceed to determine the circumstances attendant on the motion of wheels, when influenced by the impelling force of rockets attached to their circumferences; the wheels being suspended on fixed horizontal axes.

EXAMPLES FOR PRACTICE.

EXAMPLE I.

The weight of the case and head of a cylindrical rocket is 14 lbs.; the radius of the base, and length of the case 5 and 33 inches; and the radius of the base, and height of the conical head 5 and 12 inches respectively: to find to what height the rocket will rise in the atmosphere in a vertical ascent.

EXAMPLE II.

Let a rocket of the above dimensions, &c. move off in a direction inclined to the horizon in an angle of 30°; to find the height of the rocket from the earth at the end of its burning; granting it not to revolve, but to retain throughout the position in which it was projected.

EXAMPLE III.

The weight of the case and head of a rocket is given equal to 16 lbs.; the radius of its base and also that of the head (which is conical) $5\frac{1}{4}$ inches; the length of the cylindric case 3 feet; and the altitude of the head 9 inches. If when the rocket is thrown perpendicularly to the horizon it attains the height of $1\frac{1}{2}$ mile from the earth, what will be the time of its motion?

EXAMPLE IV.

How high would a 24-pounder cast iron ball rise in the atmosphere, if projected perpendicularly to the horizon with a velocity of 1200 feet per second?

EXAMPLE V.

Let a 10-inch shell, the weight of which unloaded is 89lbs., be projected vertically in the air with a velocity of 1700 feet per second; to determine where it will fall, the velocity of the wind being 19 feet per second.

EXAMPLE VI.

Suppose a solid cylinder of brass of 3 inches radius, and 2 feet in altitude, and having a hemispheric end of the same diameter as the base of the cylinder, to be projected vertically in the atmosphere with a velocity of 1500 feet per second: to determine the period of its return to the earth, it being supposed not to revolve, or to change the position in which it was projected; which it will not if the atmosphere continues calm.

EXAMPLE VII.

Given the same as in the last, and the time of the cylinder's return as thence determined; to find where it will fall; supposing the wind to have blown the smart gale of 40 miles an hour.

EXAMPLE VIII.

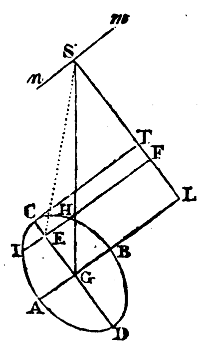
The time of flight of the rocket, *Ex.* 3., is given equal to 26″, and the angle at which it is thrown 43°; also the direction of projection north-east by north. The wind blows at the rate of 26 miles an hour directly from the south. What then is the maximum deflection of the rocket from the plane of projection?

SECTION IV.

ON THE APPLICATION OF THE FORCE OF ROCKETS TO THE MOTION OF WHEELS SUSPENDED ON FIXED HORIZONTAL AXES.

LEMMA I.

49. *Let* CD *be a circular plane, vibrating about an horizontally fixed axis* nsm, *parallel to the diameter* AB; *and inclined to* SG, *in any given angle* SGC: *to find the force of the plane* CD *to effect rotation about* nsm.



Draw GL perpendicular to the plane CD at G, and SL perpendicular to GL at L. Draw the diameter CD perpendicular to AB, and IH any chord parallel to the same, and join ES, GS; also let EF and CT be parallel to GL, and meeting SL in F and T.

Put a = ST,
b = GL, or EF, or CT,
r = CG, the rad. of the given circle,
x = CE,
p = 3·1416,

Then pr^2 is the area of CD. Now by the circle EH $= (2rx - x^2)^{\frac{1}{2}}$, also SE^2, or the square of the distance of IH from the axis of motion $= SF^2 + EF^2 = b^2 + (a + x)^2$. Therefore $\left\{ b^2 + (a + x)^2 \right\} (2rx - x^2)^{\frac{1}{2}}$, will be the force of all the particles in the semi-chord EH, and $\dot{x} \left\{ b^2 + \right.$

$(a+x)^2\}$ $(2rx - x^2)$ the fluxion of the force of ECH, which, putting f instead of $b^2 + a^2$, is equal to

$$f\dot{x}\,(2rx - x^2)^{\frac{1}{2}} + 2ax\dot{x}\,(2rx - x^2)^{\frac{1}{2}} + x^2\dot{x}(2rx - x^2)^{\frac{1}{2}},$$

and the force of ECH itself

$$f \text{ area ECH} + 2ar \text{ area ECH} - \frac{(2rx - x^2)^{\frac{3}{2}}}{3} + \frac{5r}{4}$$

$$\times \left(r \text{ area ECH} - \frac{(2rx - x^2)^{\frac{3}{2}}}{3}\right) - \frac{x\,(2rx - x^2)^{\frac{3}{2}}}{4} =$$

(when $x=0$) 0. Therefore making $x=2r$, we shall have

$$\frac{1}{2}\,pr^2\left(f + 2ar + \frac{5r^2}{4}\right)$$

for the force of the semicircle CED, and consequently

$$pr^2\left(f + 2ar + \frac{5r^2}{4}\right)$$

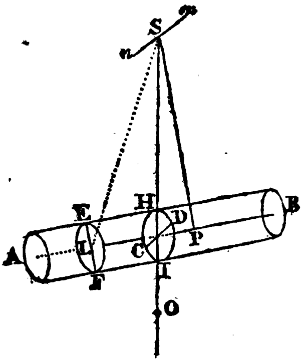
for the force of the whole circle as required: or restoring the value of f, and calling the distance SG $= \{b^2 + (a+r)^2\}^{\frac{1}{2}}$, g, the force of the whole circle will be truly expressed by

$$pr^2\,(g^2 + \frac{1}{4}\,r^2).$$

Whence it appears, that the problem is in no way affected by the inclination of the circle to SG; the result being independent of any quantity expressive of that inclination. Hence, in all positions of the given circular plane, if the axis nsm, be constantly parallel to the diameter AB, its force to produce rotation about nsm, will be the same. And hence the distance of the centre of oscillation of CD, equal to this force divided by g into pr^2, will not be changed from the circumstance of inclination of the plane.

LEMMA 2.

50. *Let the cylinder* AB *vibrate about an horizontally fixed axis* nsm, *parallel to the diameter* CD *of the circular section* CHDI, *and in any inclined position* SIF; *to find its centre of oscillation.*



Let a = SP,

b = AP,

d = AB, the length of the cylinder,

r = the radius of its base,

x = AL, any variable distance from A,

g = the distance of the centre of gravity of the solid from S,

p = 3·1416.

By the preceding lemma, the force of the section EF, to cause rotation about nsm is

$$pr^2\,(\text{SL}^2 + \frac{1}{4}r^2).$$

Whence $pr^2\dot{x}\left\{(b-x)^2 + a^2 + \frac{1}{4}r^2\right\}$, is the fluxion of the force of that part of the cylinder, the length of which is x; therefore the fluent

$$pr^2x\,(g^2 + \frac{1}{12}x^2 + \frac{1}{4}r^2),$$

wanting no correction, is the force itself of that part. Wherefore, when $x = d$, we shall have

$$dpr^2\,(g^2 + \frac{1}{12}d^2 + \frac{1}{4}r^2)$$

for the force of the whole cylinder. This divided by g into the solid gives,

$$g + \frac{d^2}{12g} + \frac{r^2}{4g}$$

for the distance SO of the centre of oscillation from S; which being also independent of any quantity expressing the inclination of the cylinder, shews, that whether the solid vibrates in a horizontal, vertical, or any oblique position, if the axis nsm, continues parallel to CD, the solution to the problem will be the same as above.

COR.—Because by mechanics, the distance of the centre of gyration of a body, from the axis of motion, is a mean proportional between the distances of the centres of gravity and of oscillation; we shall have for the distance of the centre of gyration of a cylinder vibrating horizontally or vertically, or in any inclined position, about an horizontal axis, as nsm, parallel to CD,

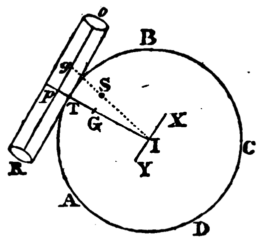
$$(g^2 + \frac{1}{12}d^2 + \frac{1}{4}r^2)^{\frac{1}{2}};$$

where g and d denote the same quantities as in the problem.

PROP. XV.

51. *Let* ABCD *be a solid cylindrical wheel, of any given substance, suspended on an horizontal axis* XY, *passing through*

the centre of gravity I; *and supposing a rocket* RO, *considered as a mere cylinder, and whose case is so light that its weight may be neglected, to be strongly attached, at its middle point, to the circumference at* T; *to determine the velocity of the wheel's motion at any given instant.*



Let φ = weight of the wheel,

r = IT its radius,

c = weight of the rocket composition,

a = time in which the same is consuming itself uniformly,

L = length of the rocket,

d = diameter of its base,

h = Ip = $r + \frac{1}{2}d$,

S = $sned^2$ (See *Art.* 17. *Prop.* 1st.) = the force of a laminum of the composition when inflamed,

v = velocity of the point p at the end of the time t,

l = IG $\left(= \frac{r}{2\frac{1}{2}}\right)$ the distance of the centre of gyration of the wheel from its centre of gravity.

Then by the laws of revolving motion, $\frac{\varphi l^2}{h^2}$ is the mass which being condensed into p, and the matter of the whole wheel removed, will resist the motion of p, in the same manner, as the wheel itself does in its natural state. Now to determine, at any time, what similar mass must be substituted in p, for the matter in the rocket; the centres of gyration and of gravity of the latter must be first found; as the places of these points will not be fixed (*Art.* 48.), but will vary during the whole time of the rocket's combustion.

To find the places therefore of these two points at the end of the time t. Let g (in the axis RO) denote the centre of gravity, and join Ig. Now $c - \frac{ct}{a}$ will be the weight of the unconsumed cylinder of composition at the end of the time t, and $L - \frac{Lt}{a}$ its length; also $\frac{1}{2}L - \frac{Lt}{2a}$, the distance of the place of the centre of gravity of the said cylinder from either end of it; and $\frac{1}{2}L - \left(\frac{1}{2}L - \frac{Lt}{2a}\right) = \frac{Lt}{2a}$, the distance of the same point from p. Hence $Ig^2 = Ip^2 + pg^2 = h^2 + \frac{L^2t^2}{4a^2}$ is the square of the distance of g from the centre of motion I.

Now the square of the distance of the centre of gyration (SI^2) from the same point, by *Cor.* to last lemma, is

$$g^2 + \frac{1}{12}d^2 + \frac{1}{4}r^2$$

where d is the length of the cylinder, and $g = Ig$. Whence $IR^2 = \frac{L^2t^2 + 4a^2h^2}{4a^2} + \frac{L^2(a-t)^2}{12a^2} + \frac{r^2}{4} =$

$$\frac{3L^2t^2 + 12a^2h^2 + L^2(a-t)^2 + 3a^2r^2}{12a^2};$$

and therefore,

$$\frac{(3L^2t^2 + 12a^2h^2 + L^2(a-t)^2 + 3a^2r^2)\,c\,(a-t)}{12a^3h^2}$$

is the mass which being substituted in p, will afford the same resistance to the motion of that point as the mass of the rocket at the end of the time t. To this add the mass $\frac{\phi l^2}{h^2}$, and the sum

$$\frac{12a^3\phi l^2 + c\,(3L^2t^2 + 12a^2h^2 + L^2(a-t)^2 + 3a^2r^2)\,(a-t)}{12a^3h^2}$$

will be the whole inertia that resists the communication of motion to the point p*. Hence,

$$\frac{12a^3h^2s}{12a^3\phi l^2 + c\,(3L^2t^2 + 12a^2h^2 + L^2(a-t)^2 + 3a^2r^2)\,(a-t)}$$

is the actual force accelerating the point p at the end of the time t.

Now $\dot{v} = 2fg\dot{t}$; therefore

$$\dot{v} = \frac{24a^3gh^2s\dot{t}}{12a^3\phi l^2 + c\,(3L^2t^2 + 12a^2h^2 + L^2(a-t)^2 + 3a^2r^2)\,(a-t)}$$

Let $z = a - t$; then $t = a - z$, and $\dot{t} = -\dot{z}$. Therefore

$$\dot{v} = \frac{-24a^3gh^2s\dot{z}}{12a^3\phi l^2 + cz\left\{3L^2(a-z)^2 + 12a^2h^2 + L^2z^2 + 3a^2r^2\right\}}$$

or,

$$\dot{v} = -\frac{6a^3gh^2s}{cL^2} + \frac{\dot{z}}{\frac{3a^2\phi l^2}{cL^2} + \frac{3a^2(L^2+4h^2+r^2)}{4L^2}z - \frac{3}{2}az^2 + z^3}$$

To find the fluent of this equation. Let

$$\frac{K}{z-k} + \frac{U}{z-u} + \frac{W}{z-w} =$$

* should be × (into) × not + (plus.)

* I do not consider the weight or gravity of the rocket to have any effect upon the wheel's motion. For, supposing any number of complete revolutions, the retardation and acceleration from this circumstance, must so nearly counterbalance each other, that no sensible error can possibly arise from the neglect of it. And in any given part of a revolution, it can make but a very small impression.

$$\frac{1}{z^3 - \frac{3}{2}az^2 + \frac{3a^2(L^2+4h^2+r^2)}{4L^2}z + \frac{3a^2\varphi l^2}{cL^2}}$$

k, u, and w being the roots of the denominator assumed $=0$: then working according to the known rules for these cases, we shall get

$$K = \frac{1}{(k-u)(k-w)}$$

$$U = \frac{1}{(u-k)(u-w)}$$

$$W = \frac{1}{(w-k)(w-u)}$$

Hence K, U, and W being known, and the given fluxion justly characterised by the sum of the fractions $\frac{K\dot{z}}{z-k} + \frac{U\dot{z}}{z-u} + \frac{W\dot{z}}{z-w}$ into the given quantity $-\frac{6a^3gh^2s}{cL^2}$, its fluent, (calling $\frac{6a^3gh^2s}{cL^2}$, P,) will be

$-P\left\{K \text{ hyp. log. } (z-k) + U \text{ hyp. log. } (z-u) + W \times \text{hyp. log. } (z-w)\right\}$. Now $z = a-t$. And when $t=0$, $v=0$. Therefore the correct fluent or general expression for the actual velocity of the point p will be

$$P\left(K \text{ hyp. log. } \frac{a-k}{z-k} + U \text{ hyp. log. } \frac{a-u}{z-u} + W \text{ hyp. log. } \frac{a-w}{z-w}\right).$$

If two of the roots of the foresaid denominator be equal, as k and u, then assuming

$$\frac{1}{z^3 - \frac{3}{2}az^2 + \frac{3a^2(L^2+4h^2+r^2)}{4L^2}z + \frac{3a^2\varphi l^2}{cL^2}}$$

$$= \frac{Lz + M}{(z-k)^2} + \frac{N}{z-w}.$$

and reducing the fractions to a common denominator, and equating the numerators, we shall find

$$L = \frac{-1}{(k-w)^2},\ M = \frac{2k-w}{(k-w)^2},\ \text{and } N = \frac{1}{(k-w)^2}.$$

Hence the fluent of

$$\frac{\dot{z}}{z^3 + \frac{3}{2}az^2 + \frac{3a^2(L^2 + 4h^2 + r^2)}{4L^2}z + \frac{3a^2\varphi l^2}{cL^2}}$$

$= $ the fluent of $\frac{Lz\dot{z} + M\dot{z}}{(z-k)^2} + \frac{N\dot{z}}{z-w}$: where L, M, and N are known. And the fluent of this is

$$L \text{ hyp. log. } (z-k) - \frac{Lk + M}{z-k} + N \text{ hyp. log. } (z-w),$$

(as will be readily perceived by substituting a single variable letter for the compound quantity $z-k$), which multiplied into $-$ P, and corrected, gives

$$P \left\{ L \text{ hyp. log. } \frac{a-k}{z-k} - (Lk + M)\left(\frac{1}{a-k} - \frac{1}{z-k}\right) + N \text{ hyp. log. } \frac{a-w}{z-w} \right\},$$

for the general value of the actual velocity of the point p in this case.

But in the above solutions we take for granted that the roots of the denominator of the fraction are all of them possible, which may not be the case under numerous particular data of the problem. It will therefore be proper to integrate the fluxion upon the supposition that the cubic involves imaginary roots. Let these be k and u (for being a cubic equation it must have two impossible roots, if any,) and the real root w: then the two fluxional fractions $\frac{K\dot{z}}{z-k}$ and $\frac{U\dot{z}}{z-u}$, in which the imaginary roots enter, being incorporated together in order that the impossible parts may vanish, we shall have

$$-\mathrm{P}\left\{\frac{(\mathrm{K}+\mathrm{U})z\dot{z}-(\mathrm{K}u+\mathrm{U}k)\dot{z}}{z^2-(k+u)z+ku}+\frac{\mathrm{W}\dot{z}}{z-w}\right\}$$

for the transformed given fluxion; the fluent of which is resolved as follows.

Suppose $c=\mathrm{K}+\mathrm{U}$, $d=\mathrm{K}u+\mathrm{U}k$, $a=k+u$, and $b=ku$; then will

$$\frac{(k+\mathrm{U})z\dot{z}-(\mathrm{K}u+\mathrm{U}k)\dot{z}}{z^2-(k+u)z+ku}=\frac{cz\dot{z}-d\dot{z}}{z^2-az+b}.$$

Let, now, $x=z-\frac{a}{2}$: then $z=x+\frac{a}{2}$, $z^2=x^2+az,\frac{a^2}{4}$ and $z^2-az+b=x^2+b-\frac{a^2}{4}=$ (writing m^2 for $b-\frac{a^2}{4}$, which is a positive quantity by supposition) x^2+m^2. Therefore since $\dot{x}=\dot{z}$, the given fluxion $\frac{cz\dot{z}-d\dot{z}}{z^2-az+b}$ will be transformed into $\frac{cx\dot{x}+\left(\frac{ac}{2}-d\right)\dot{x}}{x^2+m^2}$,

the fluent of which is $\frac{1}{2}c$ hyp. log. $(x^2+m^2)+\frac{\frac{ac}{2}-d}{m^2}$ into a cir: arc of rad. m and tangent $x=$ (restoring the values of a, b, m, x, &c.)

$$\frac{1}{2}(\mathrm{K}+\mathrm{U})\,\text{hyp. log.}\left\{\left(z-\frac{k+u}{2}\right)^2+ku-\frac{(k+u)^2}{4}\right\}$$

$$+\frac{\frac{1}{2}(\mathrm{K}+\mathrm{U})(k+u)-(\mathrm{K}u+\mathrm{U}k)}{ku-\frac{(k+u)^2}{4}}$$

into a cir. arc. of rad. $\left\{ku-\frac{(k+u)^2}{4}\right\}^{\frac{1}{2}}$ and tan.

$\left(z - \frac{k+u}{2}\right)$. Consequently the whole fluent is $-$ P $\times \left\{ \frac{1}{2} (\text{K}+\text{U}) \text{ hyp. log.} \left\{ \left(z - \frac{k+u}{2} \right)^2 + ku - \frac{(k+u)^2}{4} \right\} + \frac{2(\text{K}-\text{U})(k-u)}{4ku - (k+u)^2} \text{ cir. arc of rad. } \frac{1}{2} \left\{ 4ku - (k+u)^2 \right\}^{\frac{1}{2}} \text{ and tan. } \left(z - \frac{k+u}{2} \right) + \text{W} \times \right.$ $\left. \text{hyp. log. } (z - w) \right\}$. This corrected, taking $z=a$ when $v=0$, gives, for the general value of the actual velocity of p,

$$\text{P} \left\{ \frac{1}{2} (\text{K}+\text{U}) \text{ hyp. log.} \frac{a^2 - (k+u)a + ku}{z^2 - (k+u)z + ku} + \frac{2(\text{K}+\text{U})(k+u)}{4ku - (k+u)^2} \right.$$ into the difference of two circular arcs $(\text{A}-b)$ whose common rad. is $\frac{1}{2} \left\{ 4ku - (k+u)^2 \right\}^{\frac{1}{2}}$, and tangents $a - \frac{k+u}{2}$ and $z - \frac{k+u}{2}$ respectively

$$\left. + \text{W hyp. log.} \frac{a-w}{z-w} \right\} = \text{P} \left\{ \frac{1}{2} (\text{K}+\text{U}) \text{ hyp. log.} \frac{a^2 - (k+u)a + ku}{z^2 - (k+u)z + ku} + \frac{2(\text{K}+\text{U})(k+u)}{4ku - (k+u)^2} \times \right.$$

circular arc of radius $\frac{1}{2} \left\{ 4ku - (k+u)^2 \right\}^{\frac{1}{2}}$ and tangent

$$\frac{\left\{ ku - \frac{(k+u)^2}{4} \right\} \times \left\{ \left(a - \frac{k+u}{2} \right) - \left(z - \frac{k+u}{2} \right) \right\}}{ku - \frac{(k+u)^2}{4} + \left(a - \frac{k+u}{2} \right) \left(z - \frac{k+u}{2} \right)} +$$

$$\left. \text{W hyp. log.} \frac{a-w}{z-w} \right\}.$$

Let us now restore the values of K, U, and W, and we shall have,

$$K+U=\frac{1}{(k-u)(k-w)}+\frac{1}{(u-k)(u-w)}=\frac{1}{(w-k)(u-w)}$$

$$K-U=\frac{1}{(k-u)(k-w)}-\frac{1}{(u-k)(u-w)}=$$

$$\frac{v(u+k)-2w}{(k-u)(k-w)(u-w)},$$

$$\text{and, } 2(K-U)(k-u)=\frac{2(k+u)-4w}{(k-w)(u-w)}.$$

Therefore by substitution and reduction, the above expression becomes

$$P\left\{\frac{1}{2(w-k)(u-w)}\text{ hyp. log. }\frac{a^2-(k+u)a+ku}{z^2-(k+u)z+ku}+\right.$$

$$\frac{2(k+u)-4w}{(ku-(k+u)w+w^2)(4ku-(k+u)^2)}\text{ into the circular}$$

arc of rad. $\frac{1}{2}\left\{4ku-(k+u)^2\right\}^{\frac{1}{2}}$ and tangent

$$\frac{\left(ku-\frac{(k+u)^2}{4}\right)(a-z)}{ku+az-\frac{k+u}{2}(a+z)}+\frac{1}{(w-k)(w-u)}\text{ hyp. log.}$$

$\left.\frac{a-w}{z-w}\right\}$. And in the extreme case where $t=a$, or $z=0$,

it is, $P\left\{\frac{1}{2(w-k)(u-w)}\text{ hyp. log. }\frac{a^2-(k+u)a+ku}{ku}\right.$

$$+\frac{2(k+u)-4w}{(ku-(k+u)w+w^2)(4ku-(k+u)^2)}\text{ into the arc}$$

whose rad. is $\frac{1}{2}\left\{4ku-(k+u)^2\right\}^{\frac{1}{2}}$ and tangent

$$\left.\frac{\left(ku-\frac{(k+u)^2}{4}\right)a}{ku-\frac{(k+u)a}{2}}+\frac{1}{(w-k)(w-u)}\text{ hyp. log. }\frac{w-a}{w}\right\},$$

where it is evident that the impossible quantities k and u

(partaking of the forms $\pm n+m\sqrt{-1}$ and $\pm n-m\sqrt{-1}$) are so involved as to make all the terms in which they are contained real.

To illustrate this by an example.

Let $T = \frac{1}{2}$ ft. the thickness of the wheel which we will suppose of sound dry oak,

$r = 2\frac{1}{2}$ ft. its radius,

$c = 160$ ozs.

$a = 4$ sec.

$L = 3$ ft.

$d = \frac{1}{2}$ ft.

$b = r + \frac{1}{2}d = 2\frac{3}{4}$ feet.

$l^2 = \left(\frac{r}{2\frac{1}{2}}\right)^2 = \frac{25}{8}$ ft.

Then $\phi = 9081{\cdot}2875$ ozs. and $P = \frac{6a^3gb^2s}{cL^2} = 1457178{\cdot}3$.

Now substituting the above values in the equation $\frac{3a^2\phi l^2}{cL^2} + \frac{3a^2(L^2+4b^2+r^2)}{4L^2}z - \frac{3}{2}az^2 + z^3 = 0$, it will become $z^3 - 6z^2 + \frac{182}{3}z + 946 = 0$; whereof one of the roots, by Cardan's rule, is $-6{\cdot}609$ nearly; and the other two are $6{\cdot}305 + \sqrt{-104}$ and $6{\cdot}305 - \sqrt{-104}$. Hence $P\left\{\frac{1}{2(w-k)(u-w)} \times \text{hyp. log.}\right.$

$$\frac{a^2-(k+u)a+ku}{ku} + \frac{2(k+u)-4w}{(ku-(k+u)w+w^2)(4ku-(k+u)^2)}$$

into a circular arc of rad. $\frac{1}{2}\left\{4ku-(k+u)^2\right\}^{\frac{1}{2}}$ and tangent

$$\frac{\left(ku-\frac{(k+u)^2}{4}\right)a}{ku-\frac{(k+u)a}{2}} + \frac{1}{(w-k)(w-u)}\text{ hyp. log.}\left.\frac{w-a}{w}\right\} =$$

$$1457178\cdot8\left\{-\frac{1}{541\cdot332}\text{ hyp. log. }\frac{109}{143\cdot69}-\frac{1\cdot24}{112597\cdot056}\right.$$

$$\left.\times\ 3\cdot3812+\frac{1}{270\cdot666}\text{ hyp. log. }\frac{10\cdot61}{6\cdot61}\right\}=1457178\cdot8$$

$(+\cdot00051043-\cdot00003723+\cdot00174833)=3237.1664$ feet; which is the actual velocity per second of the point p of the circumference of the wheel at the end of the rocket's burning; and consequently the angular velocity of the wheel itself, at that time, is 1294.8665 feet.

Hence, knowing the actual velocity of the point p, the number of revolutions per second that the wheel will for ever continue to make (no extraneous or other causes being here supposed to operate) may be determined: since it is only to divide the actual velocity of this point by the circumference of the wheel. In the present example therefore, where the circumference $=15\cdot708$ feet, the number will be 206.

PROP. 15.

52. *To find the number of revolutions the wheel makes during the time of the rocket's combustion.*

In the solution to this problem, I shall confine myself to the most difficult and laborious case, where the general value for the velocity found in the preceding proposition has been obtained on the supposition that the denominator of its fluxion contains two impossible, and one real root. Therefore $P\left\{\frac{1}{2(w-k)(u-w)}\right.$ hyp. log.

$$\frac{a^2-(k+u)a+ku}{z^2-(k+u)z+ku}+\frac{2(k+u)-4w}{(ku-(k+u)w+w^2)(4ku-(k+u)^2)}$$

$\times$ cir. arc of rad. $\frac{1}{2}\left\{4ku-(k+u)^2\right\}^{\frac{1}{2}}$ and tangent

$$\frac{\left(ku-\frac{(k+u)^2}{4}\right)(a-z)}{ku-\frac{(k+u)^2}{4}+\left(a-\frac{k+u}{2}\right)\left(z-\frac{k+u}{2}\right)}$$

$+\frac{1}{(w-k)(w-u)}$ hyp. log. $\frac{a-w}{z-w}\Big\}$ being the velocity, let us, in order to render the expression as simple as possible, put $A=\frac{1}{2(w-k)(u-w)}$, $B=a^2-(k+u)\times a+ku$, $D=\frac{2(k+u)-4w}{(ku-(k+u)w+w^2)(4ku-(k+u)^2)}$

$E=\frac{1}{2}\Big\{4ku-(k+u)^2\Big\}^{\frac{1}{2}}$, $n=k+u$, and $m=ku$; then it becomes $P\Big\{A$ hyp. log. $\frac{B}{z^2-nz+m}+D$ arc of rad.

E and tan. $\frac{m-\frac{n^2}{4}}{a-\frac{n}{2}}\times\frac{a-z}{\frac{m-\frac{an}{2}}{a-\frac{n}{2}}+z}$ + W hyp. log.

$\frac{a-w}{z-w}\Big\}$. Therefore since the fluxion of the space ($\dot{x}$) $=v\dot{t}=-v\dot{z}$; we get $\dot{x}=-P\Big\{A\dot{z}$ hyp. log. $\frac{B}{z^2-nz+m}$

$+D\dot{z}$ into arc of rad. E and tan. $\frac{m-\frac{n^2}{4}}{a-\frac{n}{2}}\times$

$(a-z)\div\left(\frac{m-\frac{an}{2}}{a-\frac{n}{2}}+z\right)+W\dot{z}\times$ hyp. log. of $\frac{a-w}{z-w}\Big\}$.

The fluent of the first term $\dot{z}$ hyp. log. $\frac{B}{z^2 - nz + m}$ ($\dot{F}$) omitting for the present the constant multiplier $-$PA, is $F = z$ hyp. log. $\frac{B}{z^2 - nz + m}$ $-$ fluent $z \times$ flux. of hyp. log. $\frac{B}{z^2 - nz + m} = z$ hyp. log. $\frac{B}{z^2 - nz + m}$ + flu. $\frac{2z^2\dot{z}}{z^2 - nz + m}$ $-$ flu. $\frac{nz\dot{z}}{z^2 - nz + m} = z$ hyp. log. $\frac{B}{z^2 - nz + m}$ $+ 2z +$ flu. $\frac{nz\dot{z}}{z^2 - nz + m}$ ($\dot{H}$) $-$ flu. $\frac{m\dot{z}}{z^2 - nz + m}$ ($\dot{G}$).

Let $x = z - \frac{1}{2}n$, then $\dot{x} = \dot{z}$, also $x^2 = z^2 - nz + \frac{1}{4}n^2$, and $z^2 - nz + m = x^2 - \frac{1}{4}n^2 + m =$ (writing e^2 for $-\frac{1}{4}n^2 + m$, which is a positive quantity) $x^2 + e^2$. Therefore $\dot{H} = \frac{nz\dot{z}}{z^2 - nz + m} = \frac{n(x\dot{x} + \frac{1}{2}n\dot{x})}{x^2 + e^2}$, and $H = \frac{n}{2}$ hyp. log. $(x^2 + e^2) + \frac{\frac{1}{2}n^2}{e^2}$ cir. arc of rad. e and tan. $x = \frac{n}{2}$ hyp. log. $(z^2 - nz + m) + \frac{\frac{1}{2}n^2}{-\frac{1}{4}n^2 + m}$ arc of rad. $(-\frac{1}{4}n^2 + m)^{\frac{1}{2}}$ and tan. $(z - \frac{1}{2}n)$. Also $\dot{G} = \frac{m\dot{z}}{z^2 - nz + m} = \frac{m\dot{x}}{x^2 + e^2}$, and $G = \frac{m}{-\frac{1}{4}n^2 + m}$ arc. of rad. $(-\frac{1}{4}n^2 + m)^{\frac{1}{2}}$ and tan $(z - \frac{1}{2}n)$.

So that $F \times -$PA or the fluent of the first term of the given fluxion, is $-$ PA $\Big\{ z$ hyp. log. $\frac{B}{z^2 - nz + m} + 2z + \frac{n}{2}$ hyp. log. $(z^2 - nz + m) + \frac{\frac{1}{2}n^2}{-\frac{1}{4}n^2 + m}$ arc of rad.

$(-\frac{1}{4}n^2 + m)^{\frac{1}{2}}$ and tan. $(z-\frac{1}{2}n) - \frac{m}{m-\frac{1}{4}n^2}$ arc of rad. $(-\frac{1}{4}n^2 + m)^{\frac{1}{2}}$ and tan. $(z-\frac{1}{2}n)\}$ = − PA $\{z$ hyp. log. B $+ \left(\frac{n}{2} - z\right)$ hyp. log. $(z^2-nz+m) + \frac{\frac{1}{2}n^2-m}{-\frac{1}{4}n^2+m}$ arc of rad. $(-\frac{1}{4}n^2+m)^{\frac{1}{2}}$ and tan. $(z-\frac{1}{2}n)\}$; which being corrected, will be PA $\{(a-z)$ hyp. log. B $+ \left(\frac{n}{2} - a\right)$ hyp. log. $(a^2-na+m) - \left(\frac{n}{2} - z\right)$ hyp. log. (z^2-nz+m) $+ \frac{\frac{1}{2}n^2-m}{-\frac{1}{4}n^2+m}$ arc of rad. $(-\frac{1}{4}n^2+m)^{\frac{1}{2}}$ and tangent $\frac{(-\frac{1}{4}n^2+m)\{(a-\frac{1}{2}n)-(z-\frac{1}{2}n)\}}{-\frac{1}{4}n^2+m+(a-\frac{1}{2}n)(z-\frac{1}{2}n)}\}$ = (when $z=0$ or $t=a$) PA $\{a$ hyp. log. B $+ \left(\frac{n}{2} - a\right)$ hyp. log. $(a^2 - na + m) - \frac{n}{2}$ hyp. log. $m + \frac{\frac{1}{2}n^2-m}{-\frac{1}{4}n^2+m}$ arc of rad. $(-\frac{1}{4}n^2 + m)^{\frac{1}{2}}$ and tan. $\frac{(-\frac{1}{4}n^2+m)a}{m-\frac{1}{2}an}\}$.

Next for the fluent of the second term of the given flux. $\dot{z}$ into arc of rad. E and tan. $\frac{m-\frac{n^2}{4}}{a-\frac{n}{2}} \times \frac{a-z}{\frac{m-\frac{an}{2}}{a-\frac{n}{2}}+z}$ ($\dot{F}$), (omitting for the present the multiplier − PD).

Writing q for $\frac{m-\frac{n^2}{4}}{a-\frac{n}{2}}$, and p for $\frac{m-\frac{an}{2}}{a-\frac{n}{2}}$, then $\dot{F} = \dot{z} \times$

arc of rad. E and tan. $\frac{q(a-z)}{p+z}$ − fluent of z into the flux. of the arc of rad. E and tan. $\frac{q(a-z)}{p+z}$ $(\dot{L})$. Now the fluxion of an arc, in terms of the tangent, is equal to the square of the radius into the fluxion of the tangent, divided by the sum of the squares of the radius and of the tangent: therefore $\dot{L} = \frac{-qE^2(p+a)z\dot{z}}{(p+z)^2} \div$

$$\frac{E^2(p+z)^2+q^2(a-z)^2}{(p+z)^2} = \frac{-qE^2(p+a)z\dot{z}}{E^2(p+z)^2+q^2(a-z)^2}$$

$$= \frac{-qE^2(p+a)z\dot{z}}{E^2p^2+a^2q^2+(E^2p-aq^2)2z+(E^2+q^2)z^2} =$$

$$\frac{-qE^2(p+a)}{E^2+q^2} \times \frac{z\dot{z}}{\frac{E^2p^2+a^2q^2}{E^2+q^2}+\frac{E^2p-aq^2}{E^2+q^2}2z+z^2}$$

Now if the roots of the denominator of this fluxion be impossible, then calling $\frac{-qE^2(p+a)}{E^2+q^2}$, V; $\frac{E^2p^2+a^2q^2}{E^2+q^2}$, R; and $\frac{E^2p-aq^2}{E^2+q^2}$, S; so that $\dot{L} = -V \times \frac{z\dot{z}}{R \pm 2Sz+z^2}$, we shall have when $2S$ is affirmative,

$L = -V \left\{ \frac{1}{2} \text{ hyp. log. } (z^2+2Sz+R) - \frac{S}{-S^2+R} \times \right.$ arc of rad. $(-S^2+R)^{\frac{1}{2}}$ and tan. $\left.(z+S)\right\}$; and when $2S$ is negative, $L = -V \left\{ \frac{1}{2} \text{ hyp. log. } (z^2-2Sz+R) + \right.$ $\frac{S}{-S^2+R}$ arc of rad. $(-S^2+R)^{\frac{1}{2}}$ and tan. $\left.(z-S)\right\}$.

Whence, − PD × F, or the whole fluent of the second term of the given fluxion uncorrected, will be

− PDz arc of rad. E and tan. $\frac{q(a-z)}{p+z}$ − PDV $\left\{ \frac{1}{2} \right.$ hyp.

log. $(z^2 \pm 2sz + R \mp \frac{s}{-s^2+R}$ arc of rad. $(-s^2+R)^{\frac{1}{2}}$ and tan. $(z \pm s)\}$. Now $z = a$, when the space and time are each $= 0$; therefore this fluent corrected is $-PDz$ arc of rad. E and tan. $\frac{q(a+z)}{p+z} + PDV \left\{ \frac{1}{2} \right.$ hyp. log. $\frac{a^2 \pm 2sa + R}{z^2 \pm 2sz + R} \mp \frac{s}{-s^2+R}$ arc of rad $(-s^2+R)^{\frac{1}{2}}$ and tan. $\left.\frac{(-s^2+R)\{(a \pm s) - (z \pm s)\}}{-s^2+R+(a \pm s) \times (z \pm s)}\right\} =$ (when $z = 0$ or $t = a$) $PDV \left\{ \frac{1}{2} \right.$ hyp. log. $\frac{a^2 \pm 2sa + R}{R} \mp \frac{s}{-s^2+R}$ arc of rad. $(-s^2+R)^{\frac{1}{2}}$ and tan. $\left.\frac{(-s^2+R)a}{R \pm sa}\right\}$.

This is the case when the roots of the quadratic denominator (assumed equal to 0) are impossible. But if the roots are real, and be denoted by i and j; then assuming $\frac{X}{z-i} + \frac{Z}{z-j} = \frac{1}{z^2 \pm 2sz + R}$, by reduction, &c. we get

$$X = \frac{1}{i-j} \text{ and } Z = \frac{1}{j-i}.$$

$$\text{Whence } \dot{L} = -V\left\{\frac{Xz\dot{z}}{z-i} + \frac{Zz\dot{z}}{z-j}\right\}, \text{ and}$$

$L = -VX\{z + i \text{ hyp. log. } (z-i)\} - VZ\ \{z+j \text{ hyp. log. } (z-j)\}$. Whence also, $-PD \times F$ will be $= -PDz \times$ arc of rad. E and tan. $\frac{q(a-z)}{p+z} - PDVX\{z + i$ hyp. log. $(z-i)\} - PDVZ\{z + j$ hyp. log. $(z-j)\}$;

which corrected, is

$$\text{PDV}\left\{x\left(a - z + i \text{ hyp. log. } \frac{a-i}{z-i}\right) + z\left(a - z + j \times \text{ hyp. log. } \frac{a-j}{z-j}\right)\right\} = (\text{when } z = 0)\ \text{PDV}\left\{x\left(a + i \text{ hyp. log. } \frac{i-a}{i}\right) + z\left(a + j \text{ hyp. log. } \frac{j-a}{j}\right)\right\}.$$

Lastly, to find the fluent of the remaining term $-\text{PW}\dot{z}$ hyp. log. $\frac{a-w}{z-w}$ ($\dot{\text{K}}$) of the original fluxion, it is $\text{K} = -\text{PW}\left\{z \text{ hyp. log. } \frac{a-w}{z-w}\right.$ — fluent z into the flux. of hyp. $\left.\text{log. } \frac{a-w}{z-w}\right\} = -\text{PW}\left\{z \text{ hyp. log. } \frac{a-w}{z-w} + z + w \times \text{ hyp. log. } (z - w)\right\}$; which corrected is $-\text{PW}\left\{a - z + w \text{ hyp. log. } \frac{a-w}{z-w} - z \text{ hyp. log. } \frac{a-w}{z-w}\right\} = \text{PW}\left\{a - z - (z-w) \text{ hyp. log. } \frac{a-w}{z-w}\right\} = (\text{when } z = 0)\ \text{PW} \times \left\{a + w \text{ hyp. log. } \frac{w-a}{w}\right\}$.

Whence, the whole space passed over by the point p in the wheel, during the burning of the rocket, being now determined, the number of revolutions made during that time may be computed.

In the solutions of the foregoing propositions, we have supposed no other resistance to the wheel's motion than that which arises from the inertia of the mass about its axis. But if the wheel revolve in a medium (as in air for example), its motion will be further resisted from the action of the same against the rocket, and that very sensibly, when the velocity of revolution becomes great.

And there will be but this force of the air upon the rocket, opposed to the whole compound mass; unless it be said that some slight resistance is occasioned by the friction of the wheel against the fluid, which in air must be too inconsiderable to affect in any degree the result determined from the contrary supposition. That there will be considerable friction of the wheel upon its axis is evident, if the former be supposed possessing much weight, and ought to enter as an additional datum into the computation. Calling, therefore, the resistance to the rocket to any given angular velocity (1) of the wheel R, and v the corresponding velocity to time t, Rv^2 will be the resistance to that velocity, and F being taken for the quantity of friction on the axis, the fluxional expression for the velocity, namely, $\dot{v} = 2f\dot{g}t$ will become (*Vide Prop.* 14. *Art.* 51.)

$$\frac{-2g\dot{z}\,(12a^3b^2S - Rv^2 - F)}{12\varphi a^3l^2 + cz\,(3L^2(a-z)^2 + L^2z^2 + 12a^2b^2 + 3a^2r^2)}$$

or,

$$\dot{v} = \frac{-M\dot{z} + Ov^2\dot{z}}{P + Qz - Nz^2 + z^3}$$

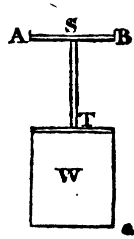
the fluent of which may be found by the method of infinite series, similarly to that at *Art.* 33. *Prop.* 9. and hence the space described be obtained.

Note.—When the rocket is fixed to the wheel in the manner prescribed by the proposition, the value of R will be had by a comparatively easy process, referring to what has been laid down in section 2. And when it is screwed upon the wheel, at the very extremity, so that no part of the surface of the cylinder meets the fluid, the resistance will be barely that upon the circular end, and consequently a problem of still easier solution.

SECTION V.

OF THE APPLICATION OF THE FORCE OF ROCKETS TO THE MOTION OF PENDULUMS.

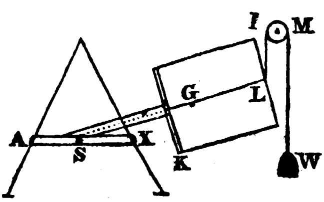
53. The pendulum, of which I here propose to consider the motion, is that denominated the ballistic; and as it will be required, in what follows on the subject, to know the centres of gravity and of oscillation of the machine; it will not be improper to give the methods by which the places of these points may be determined mechanically; and previously to which, a short description of the pendulum itself.



The ballistic pendulum is a massy block of wood W, hanging freely upon a strong horizontal fixed axis AB, at S, which axis is a part of the pendulum, to which the block W is connected by a strong inflexible wire or stem ST. It was invented by our late ingenious countryman Mr. Benjamin Robins, for the purpose of ascertaining the initial velocities of cannon balls, or the velocities with which they issue from the engines, and is, as Euler observes, one of the most useful discoveries ever made in artillery.

1. *To find its centre of oscillation.* It is well known that bodies vibrating in the arc of a cycloid, perform all their vibrations in the same time, from whatever point in the arc the vibration commences. But this is not the case when bodies vibrate in circular arcs, except those arcs be

very small. Therefore, to find the centre of oscillation, or which is the same thing, the length of a simple pendulum which shall vibrate isochronously with that of the ballistic, suspend it freely by a given point, and make it vibrate in a small arc not exceeding 4 or 5 degrees on each side of the vertical line of suspension, and by a good time-keeper, observe how many oscillations the pendulum makes in a given time (t), for instance 3 minutes, and call that number n; then by the theory of pendulums $n^2 : t^2 :: 39\frac{1}{8}$ inches (the length of a simple pendulum that vibrates seconds) : $\frac{t^2 \times 39\frac{1}{8}}{n^2}$, the length of the pendulum required; or the distance of the centre of oscillation from the point of suspension; where it is to be observed, that t must denote the number of seconds in the experiment.



2. *To find its centre of gravity.* Let a string or ribbon be fixed to the block at L, by means of which, raise the pendulum to a horizontal position; then let the string be put over a pulley M, so placed, that LM may be perpendicular to the horizon, or to the extremity IL, of the surface IK. The pendulum being horizontal, hang a weight W, at the end of the string LMW, just sufficient to keep it in that position. Then is SGL a lever of the second kind, the weight acting at G, the centre of gravity, is equal to that of the whole pendulum; and the weight or power W, acting in direction LM, preserves an equilibrium; therefore, calling the weight of the pendulum P, and the whole length

of it SL, g; we shall have P : W :: g : $\frac{Wg}{P}$, the distance of the centre of gravity from the point of suspension S.

Note.—It is plain, that P, the entire weight of the pendulum, is equal to the weight of the block and all its appendages, since in vibrating, the whole is in motion upon the pivots A and X.

PROP. 16.

54. *Let a rocket of given dimensions be strongly attached to the face of a given ballistic pendulum, so that the axis of the former, when produced, may intersect the axis of the latter perpendicularly: to determine the greatest arc through which the pendulum will be impelled.*

A little reflection on the nature of this problem, renders it obvious, that the pendulum will not have acquired its greatest ascent till the complete exhaustion of the composition of the rocket; for though the force of the mass, to prevent rotation about the axis of suspension at any intermediate time, may be an exact counterpoise to the force of the rocket, yet on account of the after combustion of the rocket, and consequent diminution of the weight of the remaining mass, the pendulum will ascend, and so continue, as long as the rocket remains on fire. To determine the problem, therefore, we have simply to find an expression for the gravitating force of the body under the circumstances here mentioned; which being made equal to the constant impelling force of the rocket, the equation thus resulting will afford us the means of determining the height required by the proposition.

Let the weight of the case of the rocket be inconsiderable with respect to the weight of the pendulum; and put

w = weight of the latter,

g = distance of its centre of gravity from the axis of suspension,

o = distance of the centre of oscillation,

$(go)^{\frac{1}{2}}$ = distance of the centre of gyration,

i = distance of the axis of the rocket,

r = radius of the rocket's base,

n = 230 ozs. the medium pressure of the atmosphere upon one square inch,

s = 1000,

p = 3·1416,

x = natural sine of the angle which the axis of the pendulum makes with the vertical line, when at its greatest altitude.

Then $snpr^2$ is the force of a surface of composition equal to the rocket's base, or the constant impelling force of the rocket. Now by the theory of rotatory motion, $\frac{gow}{i^2}$ is the mass which being condensed into that point of the axis of the pendulum the distance of which from the axis of suspension is represented by i, the motion, and every circumstance attending that motion of the pendulum, will be the same, as when it revolved in its natural form. Whence, $\frac{gowx}{i}$ will be the gravitating force of the pendulum when in the required position: therefore putting $\frac{gowx}{i} = snpr^2$, we shall have $x = \frac{snpr^2 i}{gow}$ for the natural sine of the angle sought.

For an example in numbers.

Let $w = 570$ lbs. or 9120 ozs.
$g = 78\frac{1}{2}$ in.
$o = 84\frac{7}{8}$ in.
$i = 60$ in.
$r = 1$ inch.
$n = 230$ ozs.
$n = 1000$.
$p = 3.1416$

Then $x = \frac{snpr^2i}{gow} = \frac{1000 \times 230 \times 3{\cdot}1416 \times 1 \times 60}{78\frac{1}{2} \times 84\frac{7}{8} \times 9120}$

$= {\cdot}7143045$, the natural sine answering to $45° \; 35'$.

If the arc through which the pendulum is impelled be given, the value of s, expressive of the force of the composition, in reference to the force of the atmosphere, denoted by 1, will be $\frac{gowx}{npr^2i}$.

Hence, a very easy and simple method of determining the strength of the composition of any species of rocket, or pyrotechnic arrow, by means of the pendulum: for in the experiment, it will be merely required to mark the precise height of the pendulum at the final instant of the burning of the rocket, and substitute the natural sine of the angle which it subtends, with the other known quantities contained in the foregoing expression for that strength. Thus, suppose the dimensions of the pendulum and of the rocket to be as in this proposition, and that the pendulum is urged through an arc of 30°, the natural sine of which is $\frac{1}{2}$; then will $s \left(= \frac{gowx}{npr^2i} \right)$ be found in this case equal to 700 very nearly, for the strength of the composition, which is therefore 700 times the elastic force of the atmosphere at a medium.

But in order to have the force of the composition as precise as possible, let us take into the account the weight of the case of the rocket; that is, instead of finding the centre of oscillation of the pendulum only, by the method laid down at *Art.* 53, find this point when the case of the rocket is fixed to the pendulum at the point where it is intended that the force of the latter should be applied. Also, for the centre of gravity of the compound pendulum, it will be had by a very easy process; for the centre of gravity of the pendulum without the case of the rocket annexed, is found by *Art.* 53; and the centre of gravity of the latter is known, being the middle point of its axis, the length of which is given; therefore, having also the distance between these two centres given, and the weights of the two bodies, their common centre of gravity will be had by saying, as the sum of the weights of the two bodies, is to the weight of either of them; so is the whole distance of their centres of gravity from each other, to the distance of their common centre of gravity from that of the centre of gravity of the other body; and this being known, the distance *(g)* of the same point from the axis of suspension may be determined.

As to those circumstances which may seem to cause some error in the result by diminishing the arc that the pendulum describes, such as the friction upon its axis, and the resistance of the air to the back of the pendulum, they are sufficiently balanced (so little as they exist) by the effect of the former upon the number of vibrations made by the pendulum in the experiment which determines its centre of oscillation. (See Dr.

Hutton's Tracts, 4to. ed. p. 120, &c.) Therefore, the force of the composition as above deduced, is perhaps as accurately defined by the fraction $\frac{gowx}{npr^2i}$, as the nature of the thing will possibly admit of.

Several other curious problems might now be proposed concerning the application of rockets to the motion of pendulums; but as they would be more speculative than practical, I shall pass them over, and conclude the section by a brief and popular account of the experiment for ascertaining the force of the composition.

The most striking object in the experiment being that of ascertaining the arc described by the pendulum; the means by which it is effected, cannot be too simple, and free from causes, that may tend to prevent its precise determination; considering how much the truth of the thing sought depends upon the accurate measurement of that arc. Now the best method with which I am acquainted is that given by Dr. Hutton (and invented by him), at p. 112, of the volume of Tracts before mentioned. It is as follows:—Let a sharp spear or stylette be conceived fixed in the centre of the bottom of the pendulum, and a block of wood to be placed immediately under the same having its upper surface formed into a circular arc, the centre of which is in the middle of the axis, and its radius equal to the length from the axis to the upper surface of the block;—then, in the middle of this arc, make a shallow grove* of 3 or 4 inches broad, running along the middle through the whole length of the arc, and fill it with a composition of soft-soap and wax of about the consistence of honey, or a little firmer, and having its upper surface smoothed off quite even with the general surface of the broad arc; then the

* groove,

whole being put into motion, the stylette proceeding from the bottom of the block, will move along the surface of the composition, and trace the precise vibration of the pendulum; the measure of which may be accurately determined by means of a scale of chords (previously constructed), answering to the radius, whose length is the distance between the axis of suspension and the upper surface of the block, by measuring first the chord of the arc marked out in the groove of composition, and then applying it to the said scale of chords. And thus having found the number of degrees in the arc of vibration, its natural sine (x), will be known. Whence, the values of the several letters contained in the expression for the force of the composition being now found, by substituting them in that expression, the force itself will be had in reference to the similar elastic force of the atmosphere denoted by unity.

ON

NAVAL GUNNERY.

55. Whatever is advanced towards the perfection of any system of warfare, whether for the use of the navy or for the army, must in the present day be considered as entitled to every attention. The following enquiries in naval gunnery are intended to obviate the evil arising from any undue allotment of charges for the artillery when in close action, for it has already been conjectured (See Preface) that the charges made use of are not always the most

eligible for producing the greatest destruction to the enemy's shipping; owing to their being too great; a circumstance that ought ever to be attended to in all cases of practice, as well military as naval.

The charges here given (which are computed for all the natures of ordnance generally used at sea) rest upon experiments, which, for accuracy, have never been excelled; and every circumstance that was likely to affect materially the quantity of them has been duly considered in the theory whence they are deduced. Many remarks might here be made in favour of their hoped-for utility; but as they will appear in the body of the work, it is unnecessary to repeat them in the introduction.

LEMMA 1.

56. *If two spheres of different diameters, and different specific gravities, impinge perpendicularly on two uniformly resisting fixed obstacles, and penetrate into them; the forces which retard the progress of the spheres, will be as the absolute resisting forces or strengths of the fibres of the substances directly, and the diameters and specific gravities of the spheres inversely.*

Let R and r denote the absolute resisting forces of the two substances; F and f the retardive forces; D, d, the diameters of the spheres; Q, q, their quantities of matter; and N and n their respective specific gravities. Then the whole resistance to the spheres being proportional to the quantities of motion destroyed in a given time, will be as the absolute resisting forces of the two substances and quantities of resisting surfaces jointly; or, as the resisting forces of the substances and squares of the diameters of the impinging spheres; because the surfaces of spheres are as the squares of their diameters; that is $\frac{M}{m} = \frac{R}{r} \times \frac{D^2}{d^2}$.

But in general, $\frac{M}{m} = \frac{F}{f} \times \frac{Q}{q}$. Therefore equating these two values of the whole resisting forces, we have $\frac{F}{f} \times \frac{Q}{q} = \frac{R}{q} \times \frac{D^2}{d^2}$, and $\frac{F}{f} = \frac{R}{r} \times \frac{D^2}{d^2} = \frac{q}{Q}$; and since the quantities of matter in spheres are in the conjoint ratio of their magnitudes and densities, or of the cubes of their diameters and densities; it is

$$\frac{F}{f} = \frac{R}{r} \times \frac{D^2}{d^2} \times \frac{d^3}{D^3} \times \frac{n}{N} = \frac{R}{r} \times \frac{d}{D} \times \frac{n}{N}.$$

That is, the forces retarding spheres penetrating uniformly resisting substances, are as the absolute strengths of the fibres of the substances directly, and the diameters and specific gravities of the spheres inversely.

Cor.—Because the whole resisting forces depend on the quantities of resisting surfaces, equal to the superficies of the spheres; it is evident that these forces will not be constant until after the spheres have penetrated to the depth of their radii. This circumstance however will not materially affect the conclusions we have derived from considering these forces as constant from the moment of impact, when the depths of penetration are considerable with respect to the radii of the spheres. And the times of penetration, the velocities, &c. when the depths are small, compared with the radii are considered in a subsequent part of the essay.

LEMMA 2.

The whole space or depths to which spheres impinging on differently resisting substances penetrate, are as the squares of the first velocities and the diameters and specific gravities of the spheres directly, and the absolute strengths of the resisting substances inversely: or, $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{D}{d} \times \frac{N}{n} \times \frac{r}{R}$.

For by mechanics, $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{f}{F}$: and by the preceding lemma $\frac{f}{F} = \frac{r}{R} \times \frac{D}{d} \times \frac{N}{n}$; therefore $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{D}{d} \times \frac{N}{n} \times \frac{r}{R}$.

These being premised, I now proceed to the following important subject

ON

THE DESTRUCTION OF AN ENEMY'S FLEET AT SEA BY ARTILLERY.

PROP. I.

57. *To find a general formula which shall express the charge of gunpowder for any given piece of artillery, to produce the greatest destruction possible to an enemy's ship at sea; it being supposed of oak substance of given thickness, and at a distance not affecting in any sensible degree the initial velocity of the shot.*

By the last of the foregoing lemmata we have generally, $V = \left(\frac{SdnRv^2}{sDNr}\right)^{\frac{1}{2}}$. Also the charges of powder vary as the squares of the velocity and weight of the ball jointly. Hence, since it has been determined from experiment that a charge of half a pound, impelled a shot weighing one pound, with a velocity of 1600 feet per second, we shall, considering v the velocity of any ball impinging on the side of the vessel, have for the expression of the charge impelling it through the space s

$$\frac{\text{SR}dnv^2w}{2\text{DN}rs \times 1600^2}.$$

Now to apply this in the present instance, it is first necessary that a case be known concerning the penetration of a given shot into oak substance. Such a case we are furnished with at page 273 of Dr. Hutton's Robins's New Principles of Gunnery. It is there asserted, that an 18-pounder cast-iron ball penetrated a block of well seasoned oak (such as ships of war are generally built with) to the depth of $3\frac{1}{2}$ inches, when fired with a velocity of 400 feet per second. Making therefore this the standard of comparison for all cases where the object is of oak substance, we shall have for the charge generally,

$$\frac{400^2 \times \cdot 42}{2 \times 1600^2 \times \frac{7}{24}} \times \frac{\text{SR}nw}{\text{DN}r};$$

or, because the balls are of the same specific gravity, and the substance the same, or $\text{R} = r$, and $\text{N} = n$; it will be

$$\frac{400^2 \times \cdot 42}{2 \times 1600^2 \times \frac{7}{24}} \times \frac{\text{S}w}{\text{D}} = \cdot 045 \times \frac{\text{S}w}{\text{D}};$$

that is, the charge varies as the space to be penetrated and weight of ball directly, and diameter of the ball inversely.

But the charge, by the problem, being to produce the greatest effect possible in the destruction of the vessel; s, in the above formula must always be put equal to the given thickness of the side; since it is well ascertained, that, for a shot to produce the most damage to any splintering object, such as oak, it must lose all its motion just as it ceases to be resisted by the object, which happens when the ball has forced its first hemisphere out of the farther surface of it. And the quantity of motion destroyed during the penetration of the first hemisphere of the ball into, and the exit of the same out of the object, is precisely equal to what would be destroyed during the penetration of the ball through one of its radii if the quantity of resisting surface was equal to half its entire superficies. Hence the charge in question will be

$$\cdot 045 \times \frac{sw}{D},$$

s being the thickness of the side of the ship, *w* the weight of the ball, and D its diameter.

If it be desirable that the shot should pierce both sides of the vessel, and the greatest damage to the ship take place on the hithermost side; it will only be necessary to double the thickness of the side of the vessel, and take that charge in the following table corresponding with the result. It appears to me that this would be the most advantageous practice; for not only will there, in this case, be a chance of killing a greater number of men of the enemy, but of the ball's striking the masts of the ship; and every sailor who has experienced such an impact on a mast in the hull of the vessel, need not be informed of the resulting consequences.

REMARKS.

In this solution, no allowance is made for the splitting of the timber that may take place when the ball has nearly penetrated to the farther surface of the object, by which the shot would be there less resisted, and its force not wholly expended when it quitted the side of the vessel.—This circumstance would be a matter of some importance did not others of a contrary nature interpose to counterbalance its effects. Thus, the loss of motion which the ball suffers in passing through the intercepted space of air between the two vessels, has this tendency; for it must not be imagined that the firing commences, or can commence, when the ships are absolutely in contact with each other, this being impossible; nor can it be supposed that the shot will impinge in any instance precisely perpendicularly on the face of the ship, but will strike it somewhat a little obliquely, and thence cause a further compensation (from the greater space through which it will in such case have to penetrate) to the effects of splintering. These, and other considerations of less moment, but of an opposing nature to the one in question, will, it is hoped, be sufficient to justify the principles upon which the general expression for the charge has been computed, (and from which the following table of charges is derived), and render it of that signal practical advantage which it is desirable it should possess, but which no other criterion than that which long practice and experience afford, is able fully to confirm.

But it may now be urged that the foregoing solution does not apply to the case in hand, insomuch that the objects of penetration are at liberty to move, being afloat upon a very yielding fluid; whereas in the experiments upon which the theory hinges, the penetrated bodies were blocks of wood solidly fixed. The objection appertains to those cases where the weight of the shot bears a sensible proportion to that of the object; but in the instance of a ship of war, with all its immense weight of rigging, ordnance, and other appointments, it exists not to that degree as to make the difference in the depth of penetration an object of the smallest consideration in the allotment of the charge.

EXAMPLE.

An enemy's ship is in sight: required the charge for the 42-pounder guns to destroy her as quickly and completely as possible, when the ships have approached near to each other. The side of the enemy's vessel, a 74, being $1\frac{3}{4}$ foot thick of oak timber.

The diameter of a 42-pounder of cast iron being = ·557 ft. we get

$$\cdot045\times\frac{sw}{D}=\cdot045\times\frac{\frac{7}{4}\times42}{\cdot557}=5\cdot93806\text{ lbs. or, 5lb. 15ozs.}$$

for the weight of the charge sought.

ANOTHER EXAMPLE.

A piece of fortification is to be destroyed, consisting of a bank of firm dry earth 2 yards thick supported on each side by planks of oak $\frac{3}{4}$ foot thick; required the most efficacious charge for the battering 42-pounders.

A 24-pounder, fired with a velocity of 1300 feet per second, into a bank of the above soil, penetrates it to the exact depth of 15 feet. Wherefore, the quantity of charge that would just cause a 42-pounder to penetrate through the bank in question will be denoted by $\frac{sdv^2w}{2sD\times1600^2}$ (*Art.* 55.), which in numbers $=\frac{6\times\cdot46\times1300^2\times42}{2\times\cdot15\times557\times1600^2}$ $=4\cdot5796$ lbs.

And that which will just force it through the thickness of the planks ($\frac{3}{2}$ feet), by $5\cdot084186$ lbs. (See Table.) Whence, the charge required is $9\cdot663786$ lbs.

TABLE:

58. Containing the various charges for the 12, 18, 24, 32, 36, and 42-pounder guns for producing the greatest effect in the damage of the vessel in all cases of close action; the substance or object being of oak materials from the thickness of 1 foot to that of 6 feet, regularly ascending by 1 in the inches.

Nature of Ordnance.	Thickness of the side of the Vessel.			
	1 ft.	1 ft. 1 in.	1 ft. 2 in.	1 ft. 3 in.
Pounder.	lbs.	lbs.	lbs.	lbs.
12	1·471870	1·594526	1·717182	1·839838
18	1·928571	2·089285	2·249999	2·410713
24	2·336445	2·531149	2·725853	2·920557
32	2·830208	3·066059	3·301910	3·537761
36	3·061608	3·316742	3·571876	3·827010
42	3·391191	3·673790	3·956389	4·238988

	1 ft. 4 in.	1 ft. 5 in.	1 ft. 6 in.	1 ft. 7 in.
12	1·962494	2·085150	2·207806	2·330462
18	2·571427	2·732141	2·892855	3·053569
24	3·115261	3·309965	3·504669	3·699373
32	3·773612	4·009463	4·245314	4·481165
36	4·082144	4·337278	4·592412	4·847546
42	4·521587	4·804186	5·084186	5·369384

	1 ft. 8 in.	1 ft. 9 in.	1 ft. 10 in.	1 ft. 11 in.
12	2·453118	2·575774	2·698430	2·821086
18	3·214283	3·374997	3·535711	3·696425
24	3·894077	4·088781	4·283485	4·478189
32	4·717016	4·952867	5·188718	5·424569
36	5·102680	5·357814	8·612948	5·868082
42	5 651983	5·934582	6·217181	6·499780

Nature of Ordnance.	Thickness of the side of the Vessel.			
	2 ft. 0 in.	2 ft. 1 in.	2 ft. 2 in.	2 ft. 3 in.
Pounder.	lbs.	lbs.	lbs.	lbs.
12	2·943742	3·066398	3·189054	3·311710
18	3·857139	4·017853	4·178567	4·339281
24	4·672893	4·867597	5·062301	5·257005
32	5·660420	5·896271	6·132122	6·367973
36	6·123216	6·378350	6·633484	6·888618
42	6·782379	7·064978	7·347577	7·630176

	2 ft. 4 in.	2 ft. 5 in.	2 ft. 6 in.	2 ft. 7 in.
12	3·434366	3·557022	3·679678	3·802334
18	4·499995	4·660709	4·821423	4·982137
24	5·451709	5·646413	5·841117	6·035821
32	6·603824	6·839675	7·075526	7·311377
36	7·143752	7·398886	7·654[illegible]20	7·909154
42	7·912775	8·195374	8·477973	8·760572

	2 ft. 8 in.	2 ft. 9 in.	2 ft. 10 in.	2 ft. 11 in.
12	3·924990	4·047646	4·170302	4·292958
18	5·142851	5·303565	5·464279	5·624993
24	6·230525	6·425229	6·619933	6·814637
32	7·547228	7·783079	8·018930	8·254781
36	8·164288	8·419422	8·674556	8·929690
42	9·043171	9·325770	9·608369	9·890968

	3 ft. 0 in.	3 ft. 1 in.	3 ft. 2 in.	3 ft. 3 in.
12	4·415614	4·538270	4·660926	4·783582
18	8·785707	5·946421	6·107135	6·267849
24	7·009341	7·204045	7·398749	7·593453
32	8·490632	8·726483	8·962334	9·198185
36	9·184824	9·439958	9·695092	9·950226
42	10·173567	10·456166	10·738765	11·021364

Nature of Ordnance.	Thickness of the side of the Vessel.			
	3 ft. 4 in.	3 ft. 5 in.	3 ft. 6 in.	3 ft. 7 in.
Pounder.	lbs.	lbs.	lbs.	lbs.
12	4·906238	5·028894	5·151550	5·274206
18	6·428563	6·589277	6·749991	6·910705
24	7·788157	7·982861	8·177565	8·372269
32	9·434036	9 669887	9 905738	10·141589
36	10·205360	10·460494	10·715628	10·970762
42	11·303963	11·586562	12·869161	12·151760

	3 ft. 8 in.	3 ft. 9 in.	3 ft. 10 in.	3 ft. 11 in.
12	5·396862	5·519518	5·642174	5·764830
18	7·071419	7·232133	7·392847	7·553561
24	8·566973	8·761677	8·956381	9·151085
32	10 377440	10·613291	10·849142	11·084993
36	11·225896	11·481030	11·736164	11.991298
42	12·434359	12·716958	12·999557	13·282156

	4 ft. 0 in.	4 ft. 1 in.	4 ft. 2 in.	4 ft. 3 in.
12	5·887486	6.010142	6·132798	6·255454
18	7·714275	7.874989	8·035703	8·196417
24	9·345789	9.540493	9·735197	9·929901
32	11·320844	11.556695	11·792546	12·028397
36	12·246432	12.501566	12·756700	13·011834
42	13·564755	13.847354	14·129953	14·412552

	4 ft. 4 in.	4 ft. 5 in.	4 ft. 6 in.	4 ft. 7 in.
12	6·378110	6·500766	6·623422	6·746078
18	8·357131	8·517845	8·678559	8·839273
24	10·124605	10·319309	10·514013	10·708717
32	12·264248	12·500099	12·735950	12·971801
36	13·266968	13·522102	13·777236	14·032370
42	14·695151	14·977750	15·260349	15·542948

Nature of Ordnance.	Thickness of the side of the Vessel.			
	4 ft. 8 in.	4 ft. 9 in.	4 ft. 10 in.	4 ft. 11 in.
Pounder.	lbs.	lbs.	lbs.	lbs.
12	6·868734	6·991390	7·114046	7·236702
18	8·999987	9·160701	9·321415	9·482129
24	10·903421	11·098125	11·292829	11·487533
32	13·207652	13·443503	13·679354	13·915205
36	15·287504	14·542638	14·797772	15·052906
42	15·825547	16·108146	16·390745	16·673344

	5 ft. 0 in.	5 ft. 1 in.	5 ft. 2 in.	5 ft. 3 in.
12	7·359358	7·482014	7·604670	7·727366
18	9·642843	9·803557	9·964271	10·124985
24	11·682237	11·876941	12·071645	12·266349
32	14·151056	14·386907	14·622758	14·858609
36	15·308040	15·563174	15·818308	16·073442
42	16·955943	17·238542	17·521141	17·803740

	5 ft. 4 in.	5 ft. 5 in.	5 ft. 6 in.
12	7·849982	7·972638	8·095294
18	10·285699	10·446413	10·607127
24	12·461053	12·655757	12·850461
32	15·094460	15·33[illegible]311	15·566162
36	16·328576	16·583710	16·838844
42	18·086339	18·368938	18·651537

	5 ft. 7 in.	5 ft. 8 in.	5 ft. 9 in.
12	8·217950	8·340606	8·463262
18	10·767741	10·928555	11·089269
24	13·045165	13·239869	13·434573
32	15·802013	16·037864	16·273715
36	17·093978	17·349112	17·604246
42	18·934136	19·216735	19·499334

Nature of Ordnance.	Thickness of the side of the Vessel.		
	5 ft. 10 in.	5 ft. 11 in.	6 ft. 0 in.
Pounder.	lbs.	lbs.	lbs.
12	8·585918	8·708574	8·831230
18	11·249983	11·410697	11·571411
24	13·629277	13·823981	14·018685
32	16·509566	16·745417	16·981268
36	17·859380	18·114514	18·369648
42	19·781933	20·064532	20·347131

59. In this table, the first column contains the nature of the ordnance, and the numbers in the other columns are their respective charges of gunpowder in pounds, when the thickness of the object to be destroyed is as specified at the top of the columns. If the thickness be given in inches and parts of inches, take such parts of the difference between the charge for the given number of inches and that number increased by one, or the next greater, and add them to the charge first found for the given number of inches for the charge required.

The value of the decimal part of each will be had by multiplying it by 16, the number of ounces in a pound, and pointing off in the product from the right hand towards the left, as many places for decimals as are contained in the given decimal, and retaining the number on the left of the point for ounces, increasing it by $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, or 1, when the first figure of the decimal is 2, 3 or 4; 5 or 6; 7 or 8; and 9 respectively. This hint is merely given for those practitioners, into whose hands the table may fall, who are not very conversant with decimal arithmetic.

Ex.—Suppose we wanted to find the charge for the 24-pounder guns, for a thickness of 23¾ inches. By the table, the charge answering to 23 inches or 1 ft. 11 in. is 4·478580lbs; and for 24 inches 4·673300 lbs. the difference of which is ·194720 lb. this difference multiplied by 3 and divided by 4 gives ·14604 lb. for the quantity of charge for ¾ of an inch. Now let this be multiplied by 16, and the product is 2·33664 ozs. Whence, the first figure of the decimal being 3, a quarter of an ounce more must be added to the 2 ozs. cut off on the left; so that the charge required is 4lbs. 2¼ ozs. And thus for other like cases of thickness.

60. The foregoing table of charges is not only useful for the navy (for which it is more expressly intended), but in many instances of operation for the artillerist on shore; as the bursting open gates of besieged towns with promptitude and effect; and breaking up all fortifications composed of wooden materials; especially those of a splintering nature, to which the charges apply most correctly. In the case of a naval action, where the object to be penetrated is of oak substance; the ball, by having a small motion when it quits the side of the ship, tears and splinters it excessively, breaking away large pieces before it, which are not so easily supplied in the reparation; whereas, on the other hand, if the shot had any considerable velocity when it quitted the side, the effect it produced would be merely a hole, which would be stopped instantly by the mechanic employed for that purpose, and indeed in a great degree by the wood itself from its own efforts of springiness. And therefore the sole mischief that the balls can do under such circumstances of extreme velocity is, the killing or wounding

those men who may chance to stand in the way of their motion.

If any object to be destroyed be so thick that it cannot be completely pierced by any common engine, or if it be of a very brittle nature, such as stone or brick; then that charge is to be used, which will give the greatest velocity to the shot to produce the greatest effect. But in many cases of bombardment this charge is by no means to be preferred; for although the effect produced each individual time be greater, yet in any considerable time the whole effect would be less than that from a smaller charge oftener fired, on account of the extreme heat it would give to the engine after a few discharges; and in consequence of which greater time would be required for cooling the gun and preparing it for farther service.

EXAMPLE.

61. Required the charge for a 24-pounder shot to force the gates of a city with the greatest ease possible, the substance of them being elm, 1 foot thick.

Here the object to be penetrated being elm, the small letters in the general formula for the charge, namely

$$\frac{sdv^2w}{2Ds \times 1600^2}$$

must be made to express the several numbers of some experiment made in the penetration of this substance. Now by a mean of many very accurate experiments made by Dr. Hutton at Woolwich, in the years 1783, 1784, and 1785, he found, that a cast iron ball of two inches diameter impinging perpendicularly on the face of a block of elm-wood, with a velocity of 1500 feet per second, penetrated 13 inches deep into its substance;

hence we shall have $d = \frac{1}{6}$ ft. $v = 1500$, and $s = \frac{13}{12}$ ft.; also by the question, $S = 1$ ft. $D = \cdot 46$, and $w = 24$ lbs. Therefore

$$\frac{Sdv^2w}{2Ds \times 1600^2} = \frac{1 \times \frac{1}{6} \times 1500^2 \times 24}{2 \times \cdot 46 \times \frac{13}{12} \times 1600^2} = \frac{45 \times 9}{104 \times 1 \cdot 11} =$$

3·50831 lbs. or 3 lbs. $8\frac{1}{8}$ ozs. for the weight of charge required in this case.

Retaining the experiment of Dr. Hutton as a standard for all cases where the object to be penetrated is of elm, we shall get by reduction

$$\frac{Sdv^2w}{2Ds \times 1600^2} = \cdot 0676 \times \frac{Sw}{D}$$

the charge for any piece of artillery, the diameter of the shot of which is D, and weight w; S being the thickness of the object as before.

It is not unworthy of remark, that the gates of a besieged town, or any like things, might be effectually broken open by the gun itself, charged only with powder, by placing it close to the gates, with its muzzle from them; the momentum of recoil being generally sufficient to force such objects completely. But this method for several reasons is not to be insisted upon.

From the circumstance, that no English admiral, or commander, seldom or ever commences firing till his ships are about to be grappled with those of the enemy, or until they have approached them so nearly as to effect in no sensible degree the first force of the shot; the above paper has, it is presumed, as much claim to utility as any that has ever yet been offered to the navy in the science of gunnery: and even if the vessels be not so closely engaged, but are fighting at the distance of about 30 or 40 feet from each other, no uncertainty of effect would result from the above charges, provided that the shot impinged perpendicularly on the side of the vessel; on account of the splitting of the timber in some degree, which would make ample compensation for the defect of velocity occasioned by the resistance of the medium.

It is impossible to deduce charges, that shall produce invariably the effect above stated, when fired at any considerable distance from the ship. The uncertainty of the impact being perpendicular, from the unsteadiness of the vessels, renders the thing at once nugatory, without any consideration of the real resistance of the medium to the ball, and the deflection of the latter from a right-lined direction. If the obliquity of the impact be given, or can be determined, then the problem being otherwise rightly solved, a charge can be found which shall produce the same effects as those above given; but if this be impossible (which it most decidedly is), then will the problem be at best but speculative upon certain hypotheses.

I shall, however, give an investigation of the problem on the principles of resistance generally allowed, and then conclude the subject by a few observations. But it will be proper first to peruse the following

LEMMA.

62. *To determine the velocity of a cannon-ball after passing through any space in air, into which it is projected with a given velocity.*

Put a = the projectile velocity,

s = any variable space described in the time t,

v = the velocity. Then,

Proposition 7, the retardive force of the ball at the end of the time t will be $\frac{3nv^2}{16gd\text{N}}$, where N and n denote the respective specific gravities of the ball and air, and d the ball's diameter. Therefore $-v\dot{v} = 2gf\dot{s} = \frac{3nv^2\dot{s}}{8Nd}$; and hence $-\frac{\dot{v}}{v} = \frac{3n\dot{s}}{8Nd} = \left(\text{putting } b \text{ for } \frac{3n}{8Nd}\right) b\dot{s}$: whereof the correct fluent is

$$\text{hyp. log. } \frac{a}{v} = bs.$$

Whence, if c be put $= 2{\cdot}71828$, the number, the hyp. log. of which is 1, we shall get

$$\frac{a}{v} = c^{bs}\text{, and } v = \frac{a}{c^{bs}} \text{ the velocity required.}$$

Hence the velocity lost in describing the space s, is

$$a - \frac{a}{c^{bs}} = \frac{a(c^{bs} - 1)}{c^{bs}}.$$

To find the time of describing the said space; we have $\dot{t} = \frac{\dot{s}}{v} = \frac{c^{bs}\dot{s}}{a}$. Put $z = c^{bs}$; then is $bs =$ hyp. log. z, and $b\dot{s} = \frac{\dot{z}}{z}$ or $\dot{s} = \frac{\dot{z}}{bz}$. Consequently $\dot{t} = \frac{c^{bs}\dot{s}}{a} = \frac{z\dot{s}}{a} = \frac{\dot{z}}{ab}$; and $t = \frac{z}{ab} = \frac{c^{bs}}{ab}$. Now when $t = 0$, $s = 0$, and $\frac{c^b}{ab} = \frac{1}{ab}$. Therefore the correct fluential equation is

$$t = \frac{c^{bs} - 1}{ab},$$

or restoring the value of b, it is

$$t = \frac{8Nd(c^{\frac{3ns}{8Nd}} - 1)}{3an}$$

63. Having determined an expression for the time in which a ball moves through any space in a resisting medium, it will not be unworthy now to enquire, whether there be a ball, which of all others, when projected with a given velocity, will describe a given space in the least time possible. To this end we have only to consider the diameter d as variable, and make the fluxion of the formula for the time $= 0$, and then solve the equation.

Let therefore $\frac{8Nd}{3an}\left(c^{\frac{3ns}{8Nd}} - 1\right)$ be put into fluxions, or because N, a, n, &c. are given quantities, $d\left(c^{\frac{3ns}{8Nd}} - 1\right)$ $= \left(\text{putting } q = \frac{3ns}{8N}\right) dc^{\frac{q}{d}} - d$; and we get $\dot{d}c^{\frac{q}{d}} - \frac{q\dot{d}c^{\frac{q}{d}}}{d} - \dot{d} = 0$; or $dc^{\frac{q}{d}} - qc^{\frac{q}{d}} - d = 0$. Whence it evidently appears, that there is a ball which will answer the conditions of the enquiry; and it is further obvious that the said ball will be different for different values of s, this quantity being included in the expression for q. The value of d will be readily found for any given space by the method of approximation.

Note.—In this proposition, it must be observed that the ball is supposed to move in a right-line, or very nearly so; or to be fired horizontally from the engine.

PROBLEM 2.

64. *To determine the same as in the last problem, when the engine is at any considerable distance from the object, and the resistance of the air taken into the account.*

Here, as in the former proposition, the velocity $V = \left(\frac{sdv^2}{Ds}\right)^{\frac{1}{2}}$ is to be esteemed the velocity of impact. Now on the principles of resistance before adverted to, which considers the fluid as infinitely compressed, and the particles thereof perfectly nonelastic, and affording no resistance to the body but what arises from their inertia; if a denote the first or initial velocity; x the distance of the gun from the object, $c = 2{\cdot}71828$ the number, the hyp. log. of which is 1, and $b = \frac{3n}{8ND}$, where N and n represent the respective specific gravities of the ball and medium, we shall, by the foregoing lemma, have

$$a = Vc^{bx}.$$

Hence by the law of variation of the charges, and proper substitution, the true expression for the charge in question will be

$$\frac{sdv^2wc^{\frac{3nx}{4ND}}}{2Ds\ 1600^2}$$

for a perpendicular impact, and

$$\frac{sdv^2wc^{\frac{3nx}{4ND}}}{2Ds\int 1600^2}$$

for an oblique one; $\int$ being the sine of the angle of incidence; the space (s) to be described in this case being the hypothenuse of a right-angled triangle, when the effect is the same.

EXAMPLE.

Resuming the first of the foregoing examples, what must be the charge of powder to cause the shot to produce the same effect in the vessel when fired at the distance of 300 feet from it?

Substituting for the several letters in the general expression for the charge

$$\frac{Sdv^2wc\,\frac{3nx}{4ND}}{2Ds\;1600^2}$$

their proper numerical values, namely,

$S = 1\frac{3}{4}$ ft.
$s = \frac{13}{12}$ ft.
$d = \frac{1}{6}$ ft.
$D = \cdot557$ ft.
$v = 1500$ ft.
$x = 300$ ft.
$w = 42$ lbs.
$N = 7\frac{1}{3}$.
$n = \cdot0012$.

we get $\dfrac{Sdv^2wc\,\frac{3nx}{4ND}}{2Ds\;1600^2} = 9\cdot530625$ lbs. or 9 lbs. $8\frac{1}{2}$ ozs. nearly for the weight of the charge sought; being 3 lbs. $9\frac{1}{2}$ ozs. more in this case than when the vessels are in close action.

Hence, not only is the destruction of the vessel more certain when the firing commences just as the ships touch each other, but a great saving of powder takes place besides, insomuch that not more than two-thirds of the quantity is expended, that would be required at the distance of 300 feet.

From this circumstance then, and the impossibility of solving the problem rightly, from the various causes already enumerated, the effects of which are not reducible to any regular laws; we conclude, that the foregoing table of charges for close fighting, is the only one that can be of the smallest service in practice; and that all attempts at others must be rendered completely futile from the nature and constitution of things.

PROBLEM 4.

65. *To determine the charge for any given piece of artillery, to cause its shot to penetrate a block of well seasoned oak, to any given depth not exceeding its radius.*

Before entering upon the solution of this problem, it is necessary that the strength of any given surface of fibres of oak, to resist a force acting perpendicularly against it be given. Let us, therefore, first determine this point, by referring to some known experiment concerning the penetration of a shot into a block of oak substance some considerable depth. For it must be observed, that the greater proportion the depth of penetration bears to the radius of the ball, the nearer we shall be to the truth of the thing in question, by supposing the resistance throughout uniform. Now the greatest penetration with which I am acquainted, is that of 34 inches, from an experiment made by Robins with an 18-pounder cast-iron ball, fired with a velocity of 1200 feet per second. The radius of the ball being $2\frac{1}{2}$ inches, we shall be extremely near the truth therefore, to consider the penetration under the supposition of the resistance being uniform from the moment of impact, 33 inches deep; since it is obvious, that the resistance cannot be uniform until the ball has penetrated to the full depth of its radius.

A body being vertically projected in vacuo with the velocity of the above impinging sphere (1200 feet per second), would, by the laws of ascending bodies near the earth's surface, rise to a height denoted by $\frac{v^2}{4g}$ (where $v = 1200$, and $g = 16$ feet) or 22500 feet; and the resisting forces being as the spaces described when the momenta are the same, we shall have the uniform resisting force to an 18-pounder penetrating oak to that of gravity, as 22500 to $\frac{33}{12}$, or as 8182 to 1 nearly.

Therefore the force that uniformly resists the ball is equal to $8182 \times 18 = 147276$ lbs.; and this is the strength of a laminum of oak fibres equal to half the surface of the shot (39·27 sq. in.), and consequently the force of 1 square inch of such fibres will be 3750·3438 lbs. Call this R.

Put r = the radius of the ball given in the proposition,
a = the hemispheric surface of the same,
w = the weight of the ball,
d = the depth to be penetrated,
x = any variable depth less than d.

Then the surfaces of spherical segments being as their heights, we have $r : a :: x : \frac{ax}{r}$ the surface of the segment penetrated; and $\frac{\text{R}ax}{r}$ is the resisting force to the ball at the depth x, and $\frac{\text{R}ax}{rw}$ the retardive force. Now by the theory of variable forces $-v\dot{v} = 2fg\dot{x}$ (the negative sign being taken because v is a decreasing quantity) $= \frac{2ag\text{R}x\dot{x}}{rw}$; the fluent of which is $-v^2 = \frac{2ag\text{R}x^2}{rw}$;

which corrected, for the case where $x = d$, is

$$v^2 = \frac{2agRd^2}{rw}.$$

Again, the charges vary as the square of the first velocity and weight of ball conjunctly. And it has been found, that a charge of half a pound, impelled a ball weighing 1 lb. with a velocity of 1600 feet per second. Therefore the general expression for the charge is

$$\frac{agRd^2}{r1600^2}.$$

For an example, suppose the ball a 32-pounder, the radius of which is ·254 feet, and that it is to penetrate the block to the exact depth of its radius; then the hemispheric surface of the shot being 58·45 square inches, and $r = d$; we shall have

$$\frac{agrR}{1600^2} = \cdot 347992 \text{ lbs. or } 5\cdot 56787 \text{ ozs.}$$

for the charge required.

EXAMPLES FOR PRACTICE.

EXAMPLE I.

What charge will be required for a 24-pounder cast-iron ball to cause it to penetrate to the depth of 1½ inch in a block of well seasoned oak?

EXAMPLE II.

For a 42-pounder shot, what charge is necessary to force it into a ship's side to the depth of its diameter?

EXAMPLE III.

The gate of a castle is closed against us by the enemy; it is of elm wood, and 1¼ foot thick; required the charge for the 18-pounder carronade to force it at once completely?

EXAMPLE IV.

The firing upon an enemy's frigate commences at the distance of 108 yards; the guns are 24-pounders; to find the charge that will cause the shot to do the most execution with regard to the destruction of the vessel?

EXAMPLE V.

What must be the radius of that cast-iron ball that shall penetrate to the depth of its radius in a block of oak when fired with a velocity of 800 feet per second?

EXAMPLE VI.

Required the diameter of that ball which just pierces a ship's side of oak $1\frac{3}{4}$ foot thick; its initial velocity being of 2000 feet per second?

EXAMPLE VII.

Required the most efficacious charge for the battering 68-pounders, to demolish the fortifications of a citadel, consisting of a bank of firm dry earth 8 feet thick, and supported on each side by elm planks (solidly fixed) of the thickness of 9 inches.

EXAMPLE VIII.

A piece of brick fortification is to be destroyed, the thickness of which is $4\frac{1}{4}$ feet: required the fittest charge for the 42-pounder guns; or that which will cause its shot to effect the most mischief possible in a given time.

A TABLE

OF HYPERBOLIC LOGARITHMS FOR ALL NUMBERS FROM ONE TO TWO THOUSAND.

0	Inf. Neg.	40	3·68887945	80	4·38202663
1	0·00000000	41	3·71357207	81	4·39444915
2	0·69314718	42	3·73766962	82	4·40671925
3	1·09861229	43	3·76120012	83	4·41884061
4	1·38629436	44	3·78418963	84	4·43081680
5	1·60943791	45	3·80666249	85	4·44265126
6	1·79175947	46	3·82864140	86	4·45434730
7	1·94591015	47	3·85014760	87	4·46590812
8	2·07944154	48	3·87120101	88	4·47[illegible]33681
9	2·19722458	49	3·89182030	89	4·48863637
10	2·30258509	50	3·91202301	90	4·49980967
11	2·39789527	51	3·93182563	91	4·51085951
12	2·48490665	52	3·95124372	92	4·52178858
13	2·56494936	53	3·97029191	93	4·53259949
14	2·63905733	54	3·98898405	94	4·54329478
15	2·70805020	55	4·00733319	95	4·55387689
16	2·77258872	56	4·02535169	96	4·56434819
17	2·83321334	57	4·04305127	97	4·57471098
18	2·89037176	58	4·06044301	98	4·58496748
19	2·94443898	59	4·07753744	99	4·59511985
20	2·99573227	60	4·09434456	100	4·60517019
21	3·04452244	61	4·11087386	101	4·61512052
22	3·09104245	62	4·12713439	102	4·62497281
23	3·13549422	63	4·14313473	103	4·63472899
24	3·17805383	64	4·15888308	104	4·64439090
25	3·21887582	65	4·17438727	105	4·65396035
26	3·25809654	66	4·18965474	106	4·66343909
27	3·29583687	67	4·20469262	107	4·67282883
28	3·33220451	68	4·21950771	108	4·68213123
29	3·36729583	69	4·23410650	109	4·69134788
30	3·40119738	70	4·24849524	110	4·70048037
31	3·43398720	71	4·26267988	111	4·70953020
32	3·46573590	72	4·27666612	112	4·71849887
33	3·49650756	73	4·29045944	113	4·72738782
34	3·52636052	74	4·30406509	114	4·73619845
35	3·55534806	75	4·31748811	115	4·74493213
36	3·58351894	76	4·33073334	116	4·75359019
37	3·61091791	77	4·34380542	117	4·76217393
38	3·63758616	78	4·35670883	118	4·77068462
39	3·66356165	79	4·36944785	119	4·77912349

120	4·78749174	165	5·10594547	210	5·34710753
121	4·79579055	166	5·11198779	211	5·35185813
122	4·80402104	167	5·11799381	212	5·35658627
123	4·81218436	168	5·12396398	213	5·36129217
124	4·82028157	169	5·12989871	214	5·36597602
125	4·82831374	170	5·13579844	215	5·37063803
126	4·83628191	171	5·14166356	216	5·37527841
127	4·84418709	172	5·14749448	217	5·37989735
128	4·85203026	173	5·15329159	218	5·38449506
129	4·85981240	174	5·15905530	219	5·38907173
130	4·86753445	175	5·16478597	220	5·39362755
131	4·87519732	176	5·17048400	221	5·39816270
132	4·88280192	177	5·17614973	222	5·40267738
133	4·89034913	178	5·18178355	223	5·40717177
134	4·89783980	179	5·18738581	224	5·41164605
135	4·90527478	180	5·19295685	225	5·41610040
136	4·91265489	181	5·19849703	226	5·42053500
137	4·91998093	182	5·20400669	227	5·42495002
138	4·92725369	183	5·20948615	228	5·42934563
139	4·93447393	184	5·21493576	229	5·43372200
140	4·94164242	185	5·22035583	230	5·43807931
141	4·94875989	186	5·22574667	231	5·44241771
142	4·95582706	187	5·23110862	232	5·44673737
143	4·96284463	188	5·23644196	233	5·45103845
144	4·96981330	189	5·24174702	234	5·45532112
145	4·97673374	190	5·24702407	235	5·45958551
146	4·98360662	191	5·25227343	236	5·46383181
147	4·99043259	192	5·25749537	237	5·46806014
148	4·99721227	193	5·26269019	238	5·47227067
149	5·00394631	194	5·26785816	239	5·47646355
150	5·01063529	195	5·27299956	240	5·48063892
151	5·01727984	196	5·27811466	241	5·48479693
152	5·02388052	197	5·28320373	242	5·48893773
153	5·03043792	198	5·28826703	243	5·49306144
154	5·03695260	199	5·29330482	244	5·49716823
155	5·04342512	200	5·29831737	245	5·50125821
156	5·04985601	201	5·30330491	246	5·50533154
157	5·05624581	202	5·30826770	247	5·50938834
158	5·06259503	203	5·31320598	248	5·51342875
159	5·06890420	204	5·31811999	249	5·51745290
160	5·07517382	205	5·32300998	250	5·52146092
161	5·08140436	206	5·32787617	251	5·52545294
162	5·08759634	207	5·33271879	252	5·52942909
163	5·09375020	208	5·33753808	253	5·53338949
164	5·09986643	209	5·34233425	254	5·53733427

255	5·54126355	300	5·70378247	345	5·84354442
256	5·54517744	301	5·70711026	346	5·84643878
257	5·54907608	302	5·71042702	347	5·84932478
258	5·55295958	303	5·71373281	348	5·85220248
259	5·55682806	304	5·71702770	349	5·85507192
260	5·56068163	305	5·72031178	350	5·85793315
261	5·56452041	306	5·72358510	351	5 86078622
262	5·56834450	307	5·72684775	352	5·86363118
263	5·57215403	308	5·73009978	353	5·86646806
264	5·57594910	309	5·73334128	354	5·86929691
265	5·57972983	310	5·73657230	355	5·87211779
266	5·58349631	311	5·73979291	356	5·87493073
267	5·58724866	312	5·74300319	357	5·87773578
268	5·59098698	313	5·74620319	358	5·88053299
269	5·59471138	314	5·74939299	359	5·88332239
270	5·59842196	315	5·75257264	360	5·88610403
271	5·60211882	316	5·75574221	361	5·88887796
272	5·60580207	317	5·75890177	362	5·89164421
273	5·60947180	318	5·76205138	363	5·89440283
274	5·61312811	319	5·76519110	364	5·89715387
275	5·61677110	320	5·76832100	365	5·89989735
276	5·62040087	321	5·77144112	366	5·90263333
277	5·62401751	322	5·77455155	367	5·90536185
278	5·62762111	323	5·77765232	368	5·90808294
279	5·63121178	324	5·78074352	369	5·91079664
280	5·63478960	325	5·78382518	370	5·91350301
281	5·63835467	326	5·78689738	371	5·91620206
282	5·64190707	327	5·78996017	372	5·91889385
283	5·64544690	328	5·79301361	373	5·92157842
284	5·64897424	329	5·79605775	374	5·92425580
285	5·65248918	330	5·79909265	375	5·92692603
286	5·65599181	331	5·80211838	376	5·92958914
287	5·65948222	332	5·80513497	377	5·93224519
288	5·66296048	333	5·80814249	378	5·93489420
289	5·66642669	334	5·81114099	379	5·93753621
290	5·66988092	335	5·81413053	380	5·94017125
291	5·67332327	336	5·81711116	381	5·94279938
292	5·67675380	337	5·82008293	382	5·94542061
293	5·68017261	338	5·82304590	383	5·94803499
294	5·68357977	339	5·82600011	384	5·95064255
295	5·68697536	340	5·82894562	385	5·95324333
296	5·69035945	341	5·83188248	386	5·95583737
297	5·69373214	342	5·83481074	387	5·95842469
298	5·69709349	343	5·83773045	388	5·96100534
299	5·70044357	344	5·84064166	389	5·96357934

390	5·96614674	435	6·07534603	480	6·17378610
391	5·96870756	436	6·07764224	481	6·17586727
392	5·97126184	437	6·07993320	482	6·17794411
393	5·97380961	438	6·08221891	483	6·18001665
394	5·97635091	439	6·08449941	484	6·18208491
395	5·97888576	440	6·08677473	485	6·18414889
396	5·98141421	441	6·08904488	486	6·18620862
397	5·98393628	442	6·09130988	487	6·18826412
398	5·98645201	443	6·09356977	488	6·19031541
399	5·98896142	444	6·09582456	489	6·19236249
400	5·99146455	445	6·09807428	490	6·19440539
401	5·99396143	446	6·10031895	491	6·19644413
402	5·99645209	447	6·10255859	492	6·19847872
403	5·99893656	448	6·10479323	493	6·20050917
404	6·00141488	449	6·10702289	494	6·20253552
405	6·00388707	450	6·10924758	495	6·20455776
406	6·00635316	451	6·11146734	496	6·20657593
407	6·00881319	452	6·11368218	497	6·20859003
408	6·01126717	453	6·11589213	498	6·21060008
409	6·01371516	454	6·11809720	499	6·21260610
410	6·01615716	455	6·12029742	500	6·21460810
411	6·01859321	456	6·12249281	501	6·21660610
412	6·02102335	457	6·12468339	502	6·21860012
413	6·02344759	458	6·12686918	503	6·22059017
414	6·02586597	459	6·12905021	504	6·22257627
415	6·02827852	460	6·13122649	505	6·22455843
416	6·03068526	461	6·13339804	506	6·22653667
417	6·03308622	462	6·13556489	507	6·22851100
418	6·03548143	463	6·13772705	508	6·23048145
419	6·03787092	464	6·13988455	509	6·23244802
420	6·04025471	465	6·14203741	510	6·23441073
421	6·04263283	466	6·14418563	511	6·23636959
422	6·04500531	467	6·14632926	512	6·23832463
423	6·04737218	468	6·14846830	513	6·24027585
424	6·04973346	469	6·15060277	514	6·24222327
425	6·05208917	470	6·15273269	515	6·24416690
426	6·05443935	471	6·15485809	516	6·24610677
427	6·05678401	472	6·15697899	517	6·24804287
428	6·05912320	473	6·15909539	518	6·24997524
429	6·06145692	474	6·16120732	519	6·25190388
430	6·06378521	475	6·16331480	520	6·25382881
431	6·06610809	476	6·16541785	521	6·25575004
432	6·06842559	477	6·16751649	522	6·25766759
433	6·07073773	478	6·16961073	523	6·25958146
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526	6·26530121	571	6·34738921	616	6·42324696
527	6·26720055	572	6·34913899	617	6·42486902
528	6·26909628	573	6·35088572	618	6·42648846
529	6·27098843	574	6·35262940	619	6·42810527
530	6·27287701	575	6·35437004	620	6·42971948
531	6·27476202	576	6·35610766	621	6·43133108
532	6·27664349	577	6·35784227	622	6·43294009
533	6·27852142	578	6·35957387	623	6·43454652
534	6·28039584	579	6·36130248	624	6·43615037
535	6·28226675	580	6·36302810	625	6·43775165
536	6·28413416	581	6·36475076	626	6·43935037
537	6·28599809	582	6·36647045	627	6·44094654
538	6·28785856	583	6·36818719	628	6·44254017
539	6·28971557	584	6·36990098	629	6·44413126
540	6·29156914	585	6·37161185	630	6·44571982
541	6·29341928	586	6·37331979	631	6·44730586
542	6·29526600	587	6·37502482	632	6·44888939
543	6·29710932	588	6·37672695	633	6·45047042
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546	6·30261898	591	6·38181602	636	6·45519856
547	6·30444880	592	6·38350663	637	6·45676966
548	6·30627529	593	6·38519440	638	6·45833828
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553	6·31535800	598	6·39359075	643	6·46614472
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556	6·32076829	601	6·39859493	646	6·47079950
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567	6·34035930	612	6·41673228	657	6·48768402
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663	6·49677499	708	6·56244409	753	6·62406523
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666	6·50128967	711	6·56667243	756	6·62804138
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669	6·50578406	714	6·57088296	759	6·63200178
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671	6·50876914	716	6·57368017	761	6·63463336
672	6·51025834	717	6·57507584	762	6·63594656
673	6·51174533	718	6·57646957	763	6·63725803
674	6·51323011	719	6·57786136	764	6·63856779
675	6·51471269	720	6·57925121	765	6·63987583
676	6·51619308	721	6·58063914	766	6·64118217
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679	6·52062113	724	6·58479139	769	6·64509097
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682	6·52502966	727	6·58892648	772	6·64898455
683	6·52649486	728	6·59030105	773	6·65027905
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686	6·53087763	731	6·59441346	776	6·65415252
687	6·53233429	732	6·59578051	777	6·65544035
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689	6·53524127	734	6·59850903	779	6·65801105
690	6·53669160	735	6·59987050	780	6·65929392
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692	6·53958596	737	6·60258789	782	6·66185474
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819	6·70808408	864	6·76157277	909	6·81234509
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822	6·71174040	867	6·76503898	912	6·81563999
823	6·71295620	868	6·76619171	913	6·81673588
824	6·71417053	869	6·76734313	914	6·81783057
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1224	7·10987946	1269	7·14598447	1314	7·18083120
1225	7·11069612	1270	7·14677218	1315	7·18159194
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1227	7·11232744	1272	7·14834574	1317	7·18311170
1228	7·11314211	1273	7·14913160	1318	7·18387072
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1237	7·12044437	1282	7·15617664	1327	7·19067603
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1340	7·20042489	1385	7·23345542	1430	7·26542972
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1356	7·21229447	1401	7·24494155	1446	7·27655640
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1744	7·46393760	1789	7·48941208	1834	7·51425465
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1771	7·47929964	1816	7·50439156	1861	7·52886926
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1776	7·48211892	1821	7·50714108	1866	7·53155238
1777	7·48268183	1822	7·50779008	1867	7·53208814
1778	7·48324442	1823	7·50823878	1868	7·53262362
1779	7·48380669	1824	7·50878717	1869	7·53315881
1780	7·48436864	1825	7·50933527	1870	7·53369371
1781	7·48493028	1826	7·50988306	1871	7·53422833
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1889	7·54380287	1934	7·56733568	1979	7·59034695
1890	7·54432211	1935	7·56786261	1980	7·59085212
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1892	7·54538975	1937	7·56789566	1982	7·59186171
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1894	7·54644627	1939	7·56992766	1984	7·59287029
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1896	7·54750168	1941	7·57095858	1986	7·59387784
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1901	7·55013534	1946	7·57353126	1991	7·59639230
1902	7·55066124	1947	7·57404501	1992	7·59689544
1903	7·55118687	1948	7·57455848	1993	7·59739632
1904	7·55171222	1949	7·57507170	1994	7·59789795
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1906	7·55276208	1951	7 57609734	1996	7·59890046
1907	7·55328661	1952	7·57660977	1997	7·59940133
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1913	7·55642797	1958	7·57967882	2003	7·60240134
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1917	7·55851674	1962	7·58171964	2007	7·60439635
1918	7·55903826	1963	7·58222919	2008	7·60489448
1919	7·55960950	1964	7·58273849	2009	7·60539236

THE END.

ERRATA.

Page 9, l. 2, for *are*, read *is*.

19, l. 6, for *v*, read *t*.

21, l. 4, after *therefore*, add, $\frac{W}{w} = \frac{D^3}{d^3} \times \frac{G}{g}$.

95, l. 16, for $+$, read $\times$.

115, l. 19, for *weights* of either, read *weight* of either.

116, l. 27, for *grove*, read *groove*.

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Whitefriars, London.