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AA284A
Advanced Rocket Propulsion

Session 2
1/9/20 - Review of Rocket Performance Parameters, Thrust, Impulse, Efficiency

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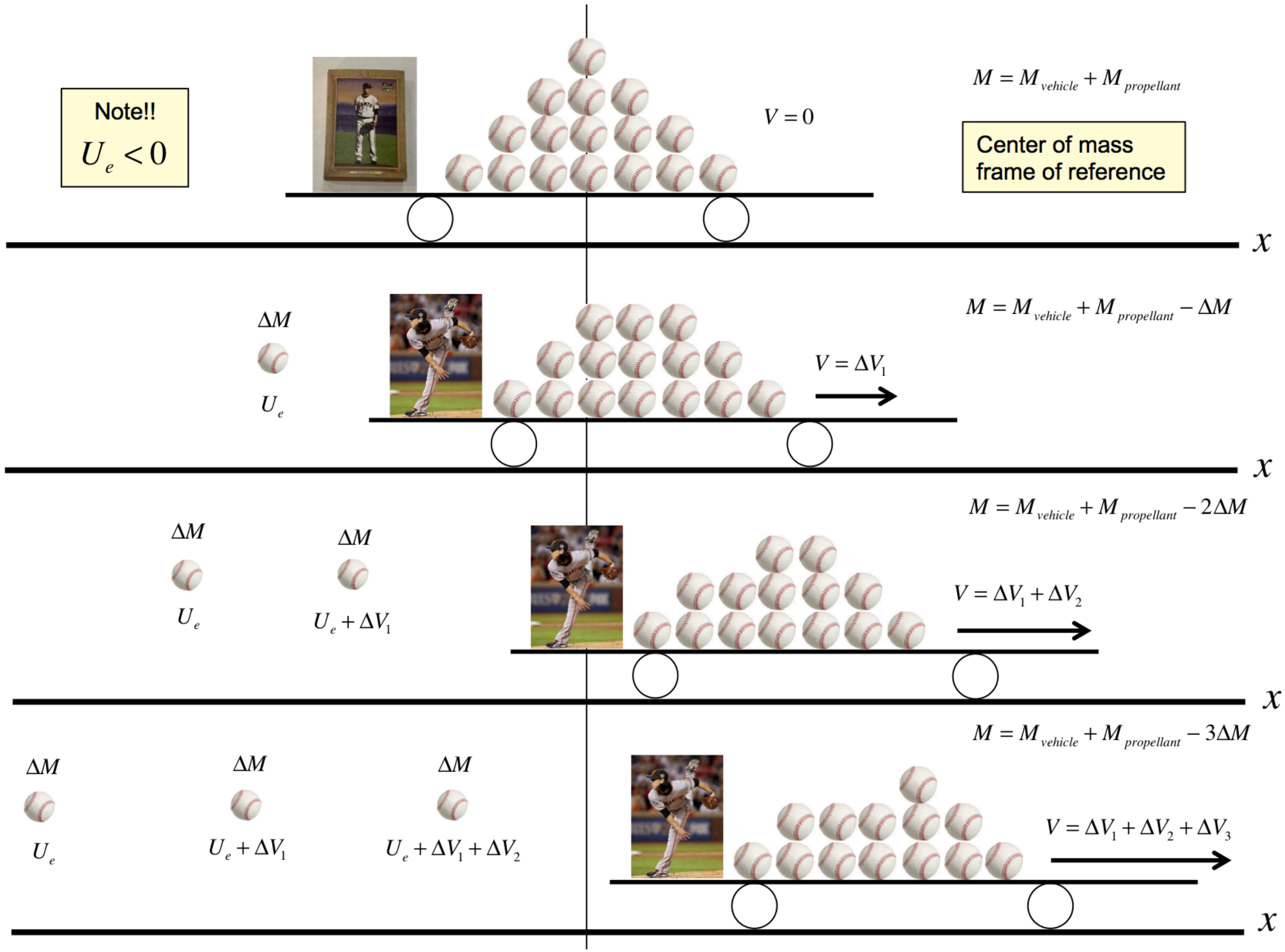
Thrust

Conservation of momentum − Tim Lincecum throwing baseballs from a frictionless cart



Bookkeeping momentum in the center-of-mass frame of reference

	mass and velocity of vehicle	*total momentum in the center-of-mass frame*
momentum of exhaust mass	$M = M_{vehicle} + M_{propellant} \quad V = 0$	$0 = \left(M_{vehicle} + M_{propellant}\right) \times 0$
$\Delta M\, U_e$	$M = M_{vehicle} + M_{propellant} - \Delta M$ $V = \Delta V_1$	$0 = \left(M_{vehicle} + M_{propellant} - \Delta M\right)\Delta V_1 + \Delta M U_e$
$\Delta M\, U_e$ $\Delta M\left(U_e + \Delta V_1\right)$	$M = M_{vehicle} + M_{propellant} - 2\Delta M$ $V = \Delta V_1 + \Delta V_2$	$0 = \left(M_{vehicle} + M_{propellant} - 2\Delta M\right)\left(\Delta V_1 + \Delta V_2\right) + 2\Delta M U_e + \Delta M\, \Delta V_1$ *Subtract* $0 = \left(M_{vehicle} + M_{propellant} - \Delta M\right)\Delta V_1 + \Delta M U_e$ *Equals* $0 = \left(M_{vehicle} + M_{propellant} - 2\Delta M\right)\Delta V_2 + \Delta M U_e$
$\Delta M\, U_e$ $\Delta M\left(U_e + \Delta V_1\right)$ $\Delta M\left(U_e + \Delta V_1 + \Delta V_2\right)$	$M = M_{vehicle} + M_{propellant} - 3\Delta M$ $V = \Delta V_1 + \Delta V_2 + \Delta V_3$	$0 = \left(M_{vehicle} + M_{propellant} - 3\Delta M\right)\left(\Delta V_1 + \Delta V_2 + \Delta V_3\right) + 3\Delta M U_e + 2\Delta M\, \Delta V_1 + \Delta M\, \Delta V_2$ *Subtract* $0 = \left(M_{vehicle} + M_{propellant} - 2\Delta M\right)\left(\Delta V_1 + \Delta V_2\right) + 2\Delta M U_e + \Delta M\, \Delta V_1$ *Equals* $0 = \left(M_{vehicle} + M_{propellant} - 3\Delta M\right)\Delta V_3 + \Delta M U_e$

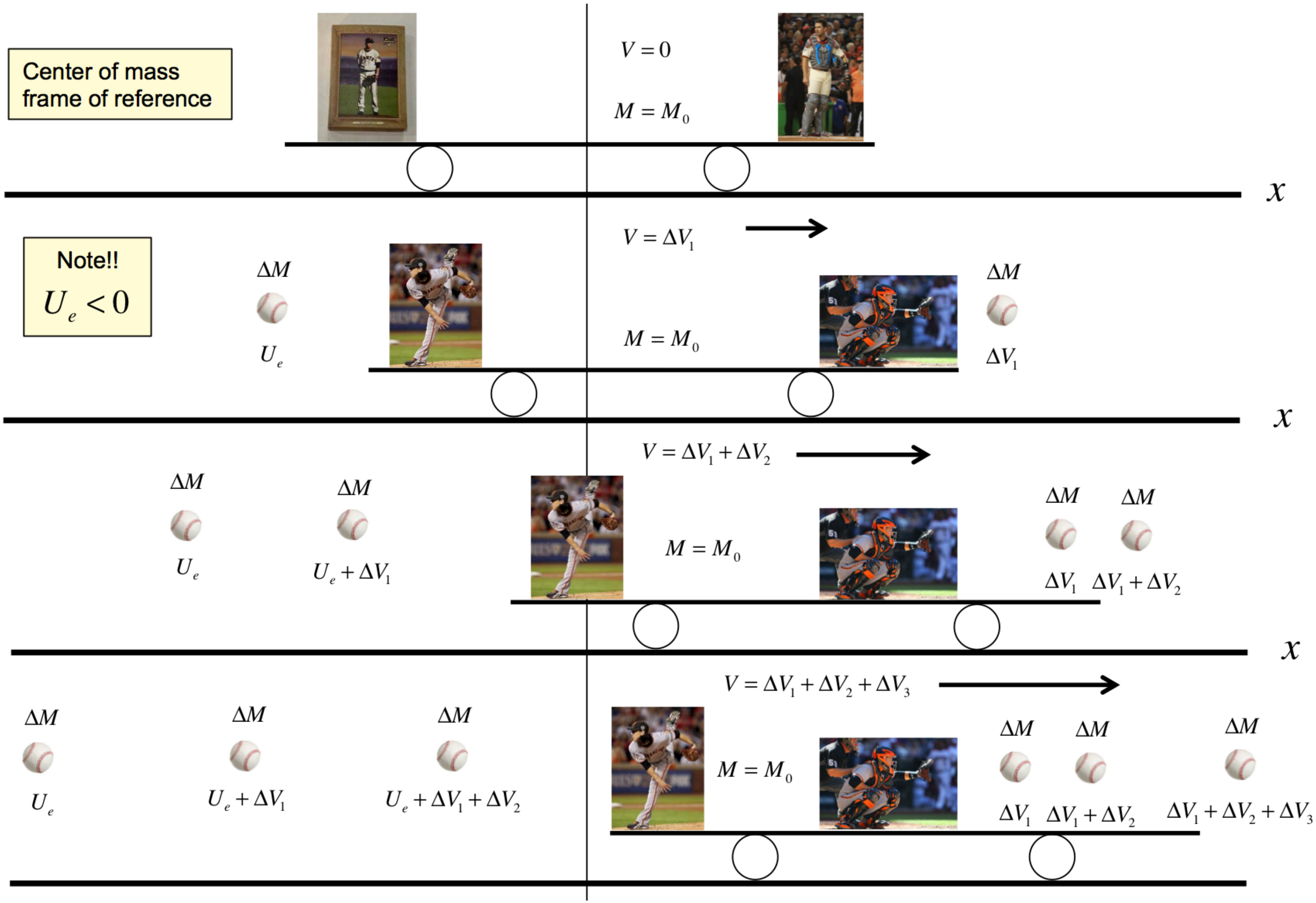
Conservation of momentum

$$M\,\Delta V = -\Delta M U_e$$

Thrust

$$T = M\frac{\Delta V}{\Delta t} = U_e\frac{\Delta M}{\Delta t}$$

What if there is incoming momentum − Posey catching incoming baseballs (air) then handing the baseballs to Tim who pitches them to the left.



Bookkeeping momentum in the center-of-mass frame of reference

	mass and velocity of vehicle	*total momentum in the center-of-mass frame*
momentum of exhaust mass	$M = M_0$ $\quad V = 0$	$0 = M_0 \times 0$
$\Delta M U_e$	$M = M_0$ $V = \Delta V_1$	$M_0 \Delta V_1 = -\Delta M \Delta V_1 - \Delta M U_e$
$\Delta M U_e$ $\Delta M (U_e + \Delta V_1)$	$M = M_0$ $V = \Delta V_1 + \Delta V_2$	$M_0(\Delta V_1 + \Delta V_2) = -\Delta M(\Delta V_1 + \Delta V_2) - \Delta M U_e - \Delta M(U_e + \Delta V_1)$ *Subtract* $M_0 \Delta V_1 = -\Delta M \Delta V_1 - \Delta M U_e$ *Equals* $M_0 \Delta V_2 = -\Delta M U_e - \Delta M(\Delta V_1 + \Delta V_2)$
$\Delta M U_e$ $\Delta M (U_e + \Delta V_1)$ $\Delta M (U_e + \Delta V_1 + \Delta V_2)$	$M = M_0$ $V = \Delta V_1 + \Delta V_2 + \Delta V_3$	$M_0(\Delta V_1 + \Delta V_2 + \Delta V_3) = -\Delta M(\Delta V_1 + \Delta V_2 + \Delta V_3) - \Delta M U_e - \Delta M(U_e + \Delta V_1) - \Delta M(U_e + \Delta V_1 + \Delta V_2)$ *Subtract* $M_0(\Delta V_1 + \Delta V_2) = -\Delta M(\Delta V_1 + \Delta V_2) - \Delta M U_e - \Delta M(U_e + \Delta V_1)$ *Equals* $M_0 \Delta V_3 = \Delta M U_e - \Delta M(\Delta V_1 + \Delta V_2 + \Delta V_3)$

Conservation of momentum

$$M_0 \Delta V = -\Delta M U_e - \Delta M V$$

Thrust

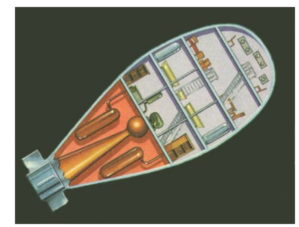
$$T = M_0 \frac{\Delta V}{\Delta t} = \frac{\Delta M}{\Delta t}\left(\left|U_e\right| - V\right)$$

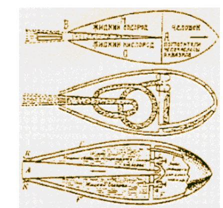
What is the final speed of the cart?

Basic equation of motion

$$M\,\Delta V = -\Delta M U_e$$

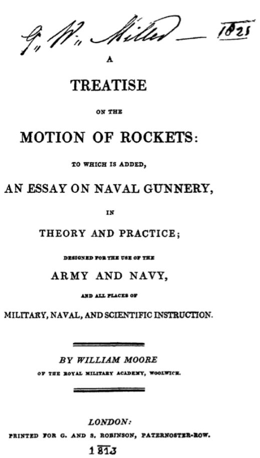
Convert to a differential. Take the limit $\Delta M \to 0$




Tsiolkovsky Rocket Designs

Note the change in sign of dM. The differential change in mass is now the change in mass of the vehicle.

$$dV = U_e \frac{dM}{M}$$


G. W. Miller — 1825

A
TREATISE
ON THE
MOTION OF ROCKETS:
TO WHICH IS ADDED,
AN ESSAY ON NAVAL GUNNERY,
IN
THEORY AND PRACTICE;
DESIGNED FOR THE USE OF THE
ARMY AND NAVY,
AND ALL PLACES OF
MILITARY, NAVAL, AND SCIENTIFIC INSTRUCTION.

BY WILLIAM MOORE
OF THE ROYAL MILITARY ACADEMY, WOOLWICH.

LONDON:
PRINTED FOR G. AND S. ROBINSON, PATERNOSTER-ROW.
1813

Fig. 1. The title paper of Moore's Treatise

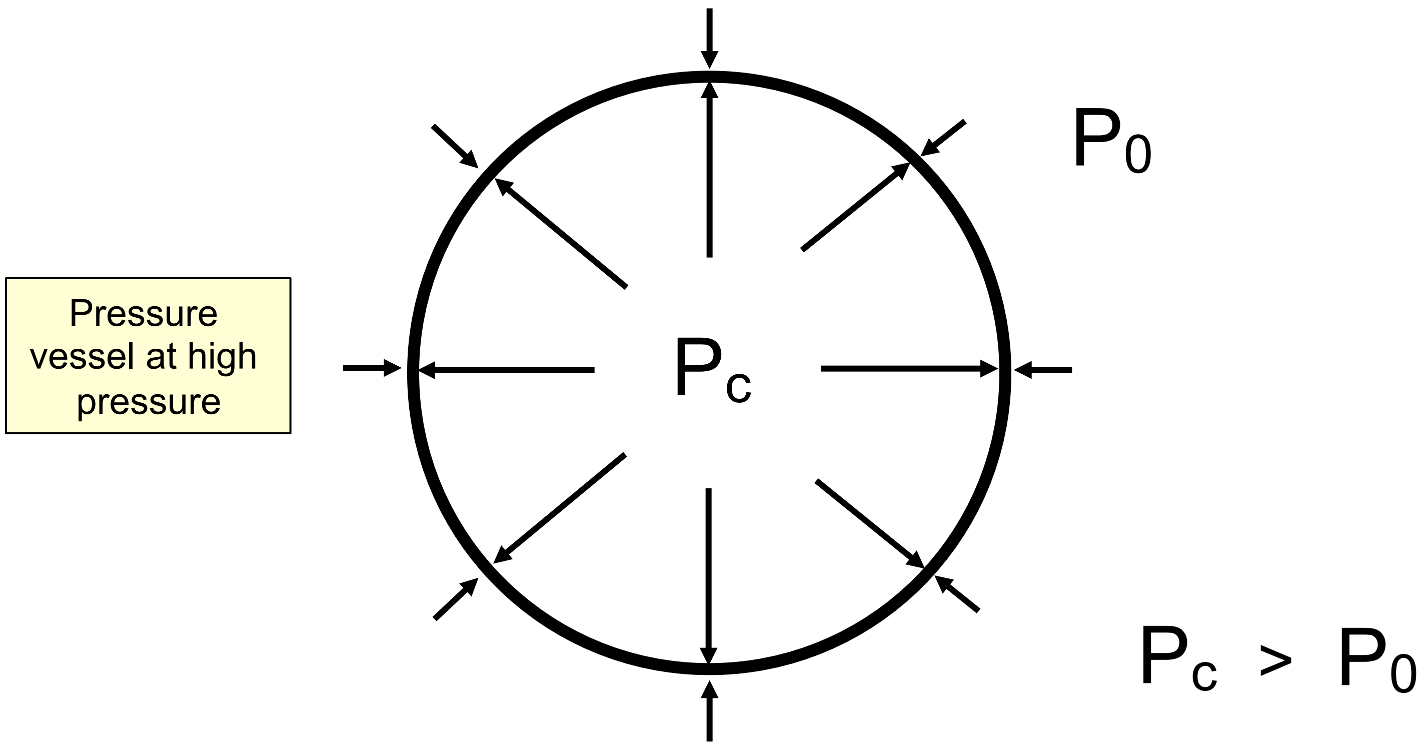
Integrate to produce the classical Moore-Tsiolkovsky rocket equation

$$V_{final} = U_e \ln\left(\frac{M_{final}}{M_{initial}}\right) = -U_e \ln\left(\frac{M_{delivered} + M_{propellant}}{M_{delivered}}\right)$$

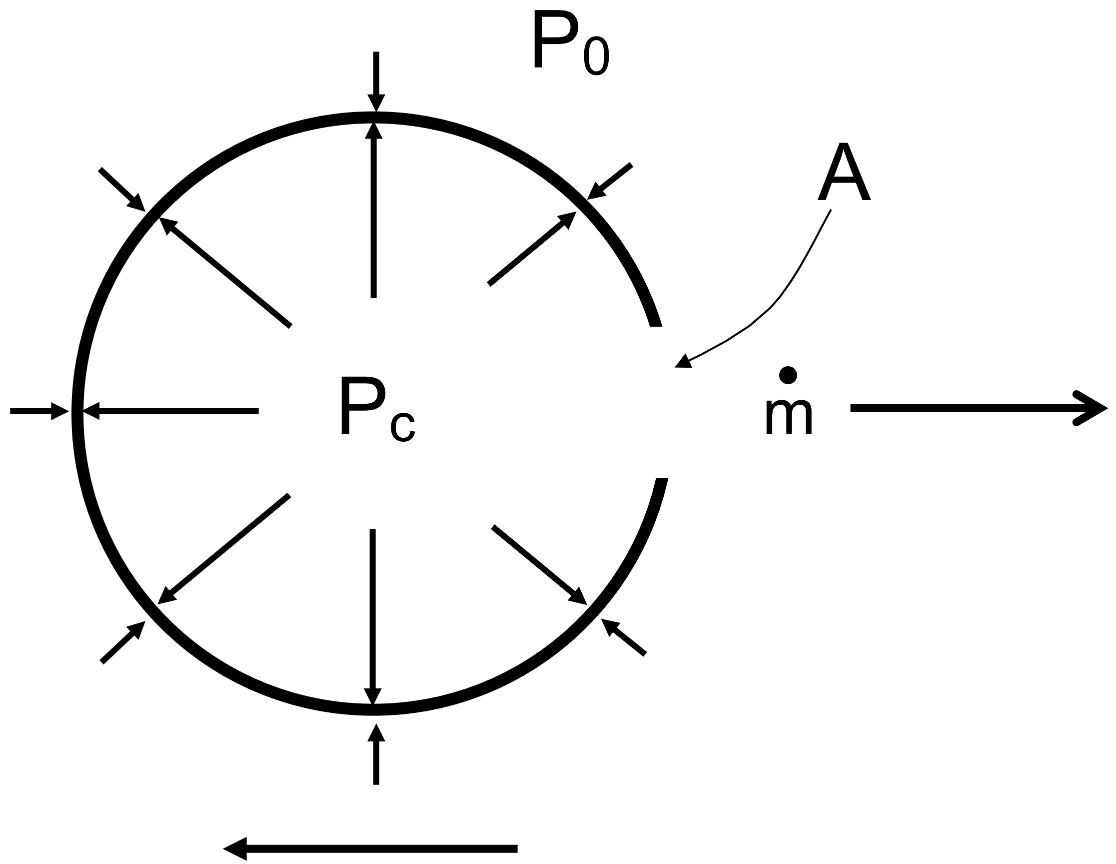
The motion of the cart is precisely analogous to the motion of a rocket in free space

The Basics of Rocket Thrust

Thrust is produced, not by expelling baseballs, but by expelling a gas



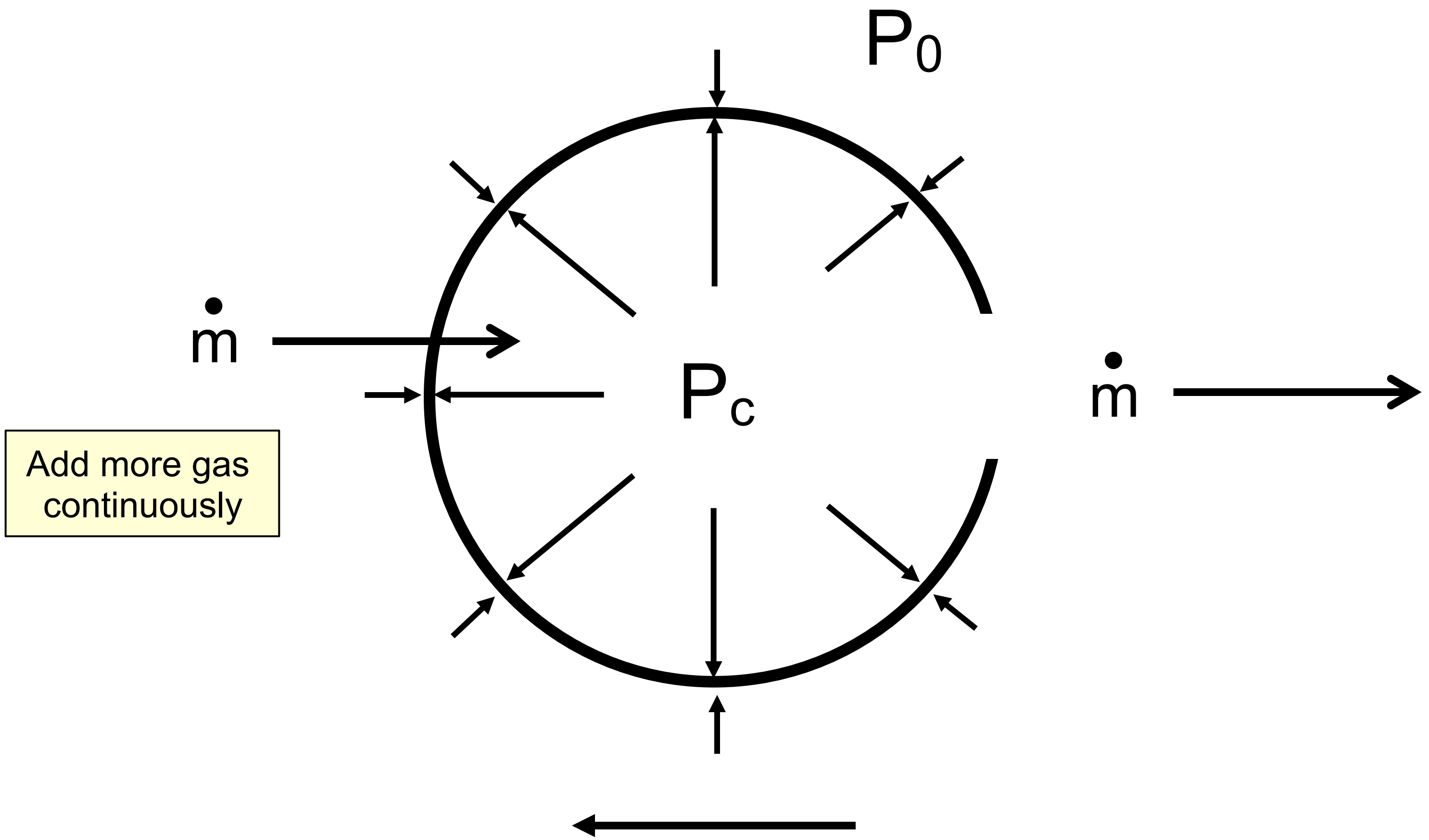
The Basics of Rocket Thrust



Open a hole

$$\text{Thrust} \sim (P_c - P_0) \times A$$

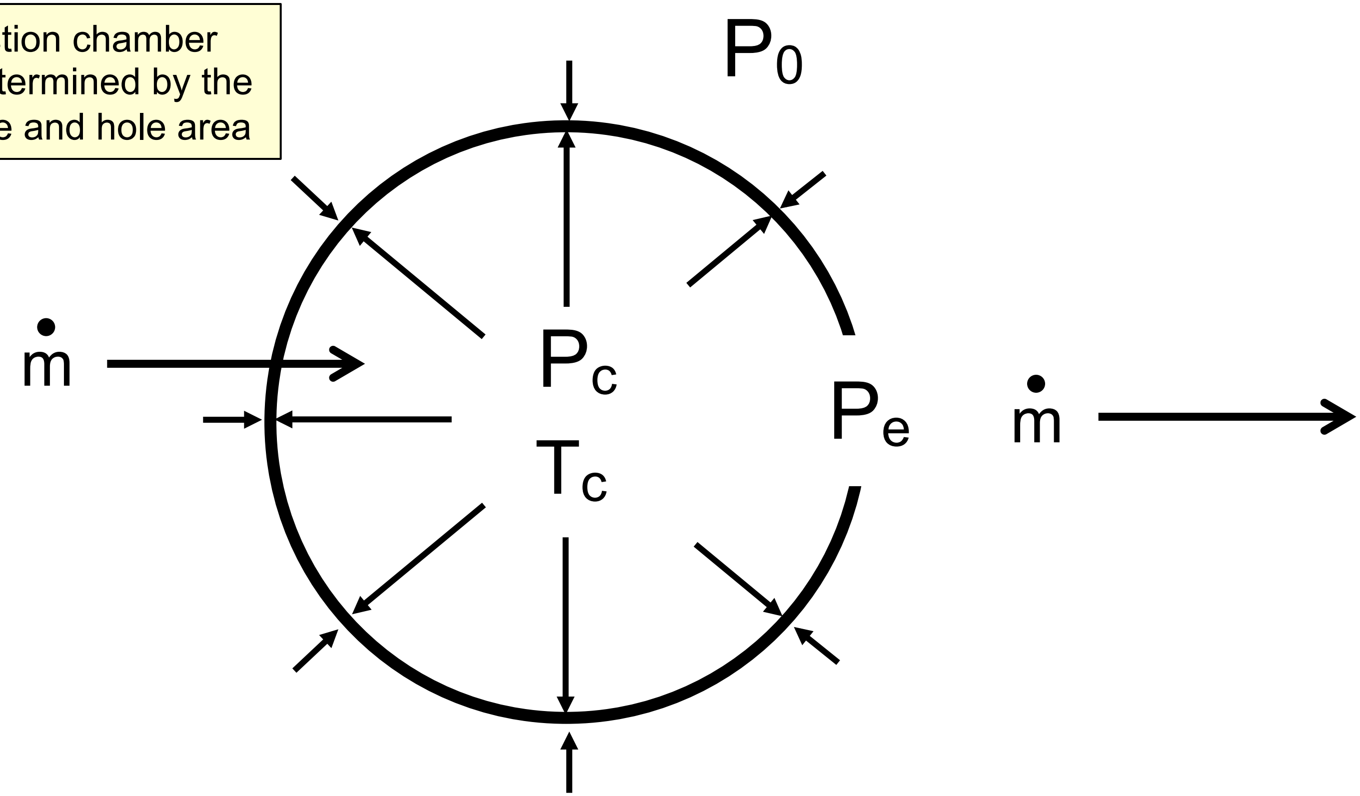
The Basics of Rocket Thrust



Thrust ~ $(P_c - P_0) \times A$

The Basics of Rocket Thrust

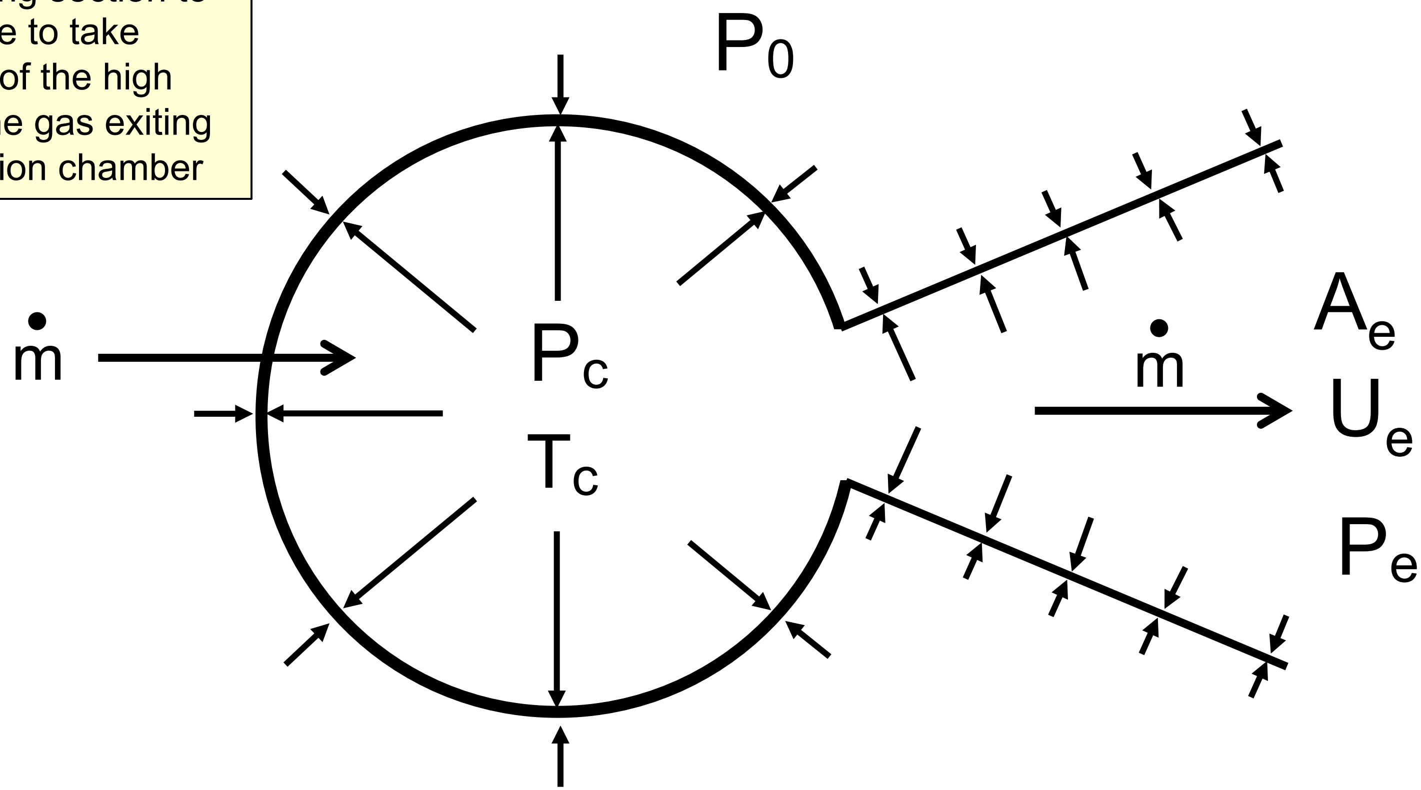
The combustion chamber pressure is determined by the mass flow rate and hole area



$$\text{Thrust} \sim P_c A + (P_e - P_0) A \sim \dot{m} \times \sqrt{\frac{R_u T_c}{M_w}} + (P_e - P_0) A$$

The Basics of Rocket Thrust

Add a diverging section to the nozzle to take advantage of the high pressure of the gas exiting the combustion chamber



$$\text{Thrust} = \dot{m} \times U_e + (P_e - P_0) \times A_e$$

7.1 Thrust – A more fluids-based approach

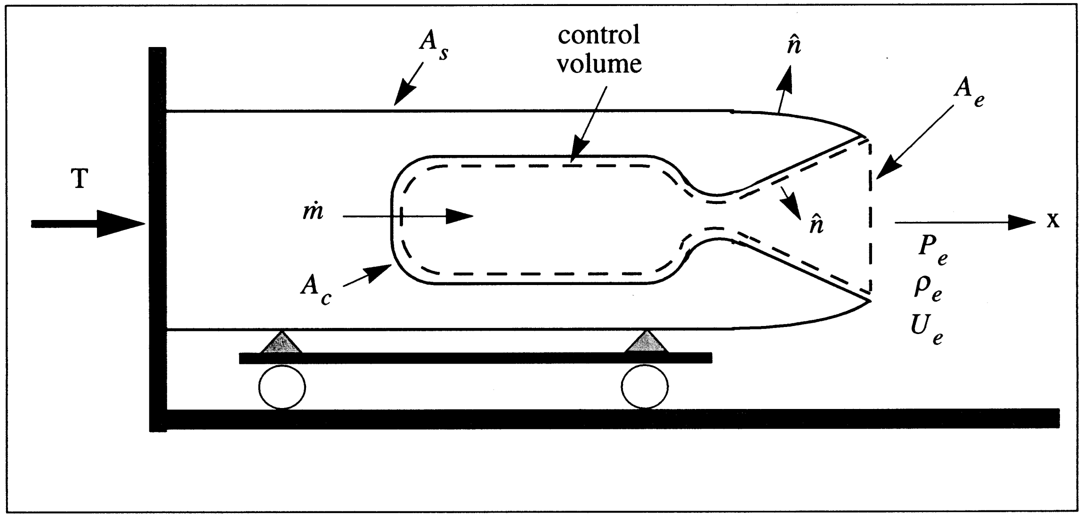


Figure 7.1 Rocket thrust schematic

$$
\begin{aligned}
A_s &= \text{outside surface of the vehicle exposed to } P_0 \\
A_c &= \text{inside surface of the combustion chamber} \\
A_e &= \text{nozzle exit area} \\
\hat{n} &= \text{outward unit normal} \\
P_e &= \text{area averaged exit gas pressure} \\
\rho_e &= \text{area averaged exit gas density} \\
U_e &= \text{area averaged x-component of velocity at the nozzle exit}
\end{aligned}
\qquad (7.1)
$$

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X-momentum of injected propellant

The total force on the rocket is zero

$$0 = T + \int_{A_s} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n}\, dA \Big|_x + \int_{A_c} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n}\, dA \Big|_x + \dot{m} U_{xm} \qquad (7.2)$$

If the rocket were inactive (turned off)

$$0 = \int_{A_s} P_0 \bar{\bar{I}} \bullet \hat{n}\, dA \Big|_x + \int_{A_c} P_0 \bar{\bar{I}} \bullet \hat{n}\, dA \Big|_x \qquad (7.3)$$

In this situation the control volume contains fluid all at rest.

$$0 = \int_{A_c} P_0 \bar{\bar{I}} \bullet \hat{n}\, dA \Big|_x + \int_{A_e} P_0 \bar{\bar{I}} \bullet \hat{n}\, dA \Big|_x \qquad (7.4)$$

The last equation can be written as

$$0 = \int_{A_c} P_0 \bar{\bar{I}} \bullet \hat{n}\, dA \bigg|_x + P_0 A_e . \tag{7.5}$$

Thus (7.3) can be written

$$0 = \int_{A_s} P_0 \bar{\bar{I}} \bullet \hat{n}\, dA \bigg|_x - P_0 A_e \tag{7.6}$$

The original force balance becomes

$$0 = T + P_0 A_e + \int_{A_c} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n}\, dA \bigg|_x + \dot{m} U_{xm} \tag{7.7}$$

The assumption in this last relation is

$$\left(\int_{A_s} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n} dA \Big|_x\right)_{after\ engine\ turn\ on} = \left(\int_{A_s} P_0\bar{\bar{I}} \bullet \hat{n} dA \Big|_x\right)_{before\ engine\ turn\ on} \tag{7.8}$$

Momentum balance

$$\frac{d}{dt}\int_V \rho\bar{U} dV = \int_V \frac{\partial \rho\bar{U}}{\partial t} dV = -\int_V \nabla \bullet (\rho\bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) dV \tag{7.9}$$

Convert the right hand side to a surface integral

$$0 = \int_V \nabla \bullet (\rho\bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) dV =$$
$$\int_{A_c} (\rho\bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n} dA + \int_{A_e} (\rho\bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n} dA \tag{7.10}$$

On the combustion chamber surface the velocity is zero by the no-slip condition. Use area averaging over the nozzle exit.

$$\left.\int_{A_c} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \hat{n}\, dA\right|_x + \dot{m}U_{xm} + \rho_e U_e^2 A_e + P_e A_e = 0 \qquad (7.11)$$

Now the force balance on the rocket is

$$0 = T + P_0 A_e - (\rho_e U_e^2 A_e + P_e A_e). \qquad (7.12)$$

Finally our rocket thrust formula is

$$T = \rho_e U_e^2 A_e + (P_e - P_0) A_e \qquad (7.13)$$

The propellant mass flow is

$$\dot{m} = \rho_e U_e A_e \tag{7.14}$$

and so the thrust formula is often written as

$$\boxed{T = \dot{m} U_e + (P_e - P_0) A_e} \tag{7.15}$$

Same as the thrust definition for air-breathing engines without the incoming air momentum term.

7.2 Momentum conservation in the center-of-mass system

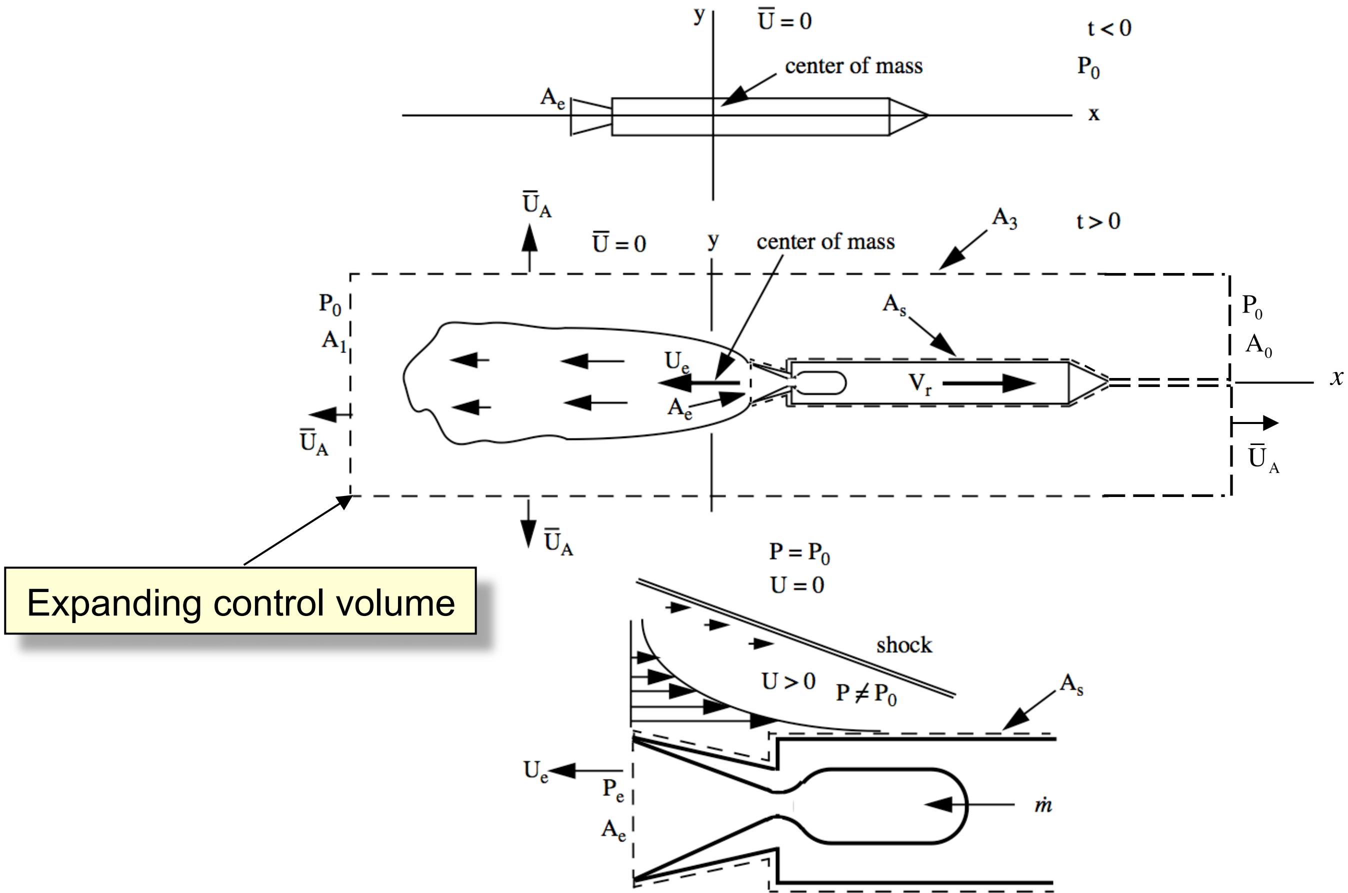

Figure 7.2 Center-of-mass description of the rocket and expelled propellant.

Conservation of momentum for the aggregate system comprising

1. Rocket vehicle plus onboard propellant
2. Expelled combustion gases
3. Air set into motion by the drag of the vehicle and through entrainment by the rocket plume.

$$\frac{D}{Dt}\left\{ \int_{V(t)} \rho \bar{U} dV \Big|_x + M_r(t) V_r(t) \right\} = 0 \qquad (7.16)$$

$M_r(t)$ is the vehicle mass and $V_r(t)$ is the vehicle velocity.

In the absence of gravity no external forces act on the system.

Momentum integral

$$\frac{D}{Dt}\int_{V(t)} \rho \bar{U} dV \Big|_x = -\int_{A(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_x \tag{7.17}$$

Over most of the control volume surface no additional momentum is being enclosed. The control volume is sufficiently large so that the fluid velocity on A_1, A_2 and A_3 is zero and the pressure is P_0. The pressure forces on A_1 and A_2 cancel and the pressure force on A_3 has no component in the *x* direction. Thus the momentum balance becomes

$$\frac{D}{Dt}\int_{V(t)} \rho \bar{U} dV \Big|_x = -\int_{A_s(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_x - \int_{A_e(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_x \tag{7.18}$$

Recall that the integral of the ambient pressure over the whole surface of the rocket is zero.

$$0 = \int_{A_c} P_0 \bar{\bar{I}} \bullet \hat{n} dA \Bigg|_x + \int_{A_e} P_0 \bar{\bar{I}} \bullet \hat{n} dA \Bigg|_x \tag{7.4}$$

Add the above equation to the previous result

$$\frac{D}{Dt} \int_{V(t)} \rho \bar{U} dV \Big|_x = - \int_{A_s(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + (P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_x -$$

$$\int_{A_e(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + (P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_x \tag{7.20}$$

On the no-slip surface of the rocket the fluid velocity is $\bar{U} = (V_r, 0, 0)$

On the rocket the control volume surface velocity is $\bar{U}_A = (V_r, 0, 0)$

$$\int_{A_s(t)} \rho \bar{U}(\bar{U} - \bar{U}_A) \bullet \bar{n} dV \Big|_x = 0 \tag{7.21}$$

Now

$$\frac{D}{Dt} \int_{V(t)} \rho \bar{U} dV \Big|_x = -\int_{A_s(t)} ((P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_x - \int_{A_e(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + (P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_x \tag{7.22}$$

The combination of viscous skin-friction drag, pressure drag and wave drag are all accounted for by the first integral.

$$D = -\int_{A_s(t)} ((P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_x \tag{7.23}$$

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Now we can evaluate the rate-of-change of the integrated gas momentum over the control volume.

$$\frac{D}{Dt}\int_{V(t)} \rho \bar{U} dV\Big|_x = D - \int_{A_e(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + (P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA\Big|_x \quad (7.24)$$

Fluid velocity at the nozzle exit in the center of mass frame

$$U = V_r + U_e \quad (7.25)$$

Area average over the nozzle exit

$$\int_{A_e(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + (P - P_0)\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA\Big|_x$$

$$= \rho_e A_e (U_e + V_r)(U_e + V_r - V_r) + (P_e - P_0) A_e \quad (7.26)$$

$$\frac{D}{Dt}\int_{V(t)} \rho \bar{U} dV\Big|_x = D - (\rho_e A_e (U_e + V_r)(U_e + V_r - V_r) + (P_e - P_0) A_e) \quad (7.27)$$

Momentum conservation of the aggregate system is

$$\frac{D}{Dt}(M_r(t)V_r(t)) + D - (\rho_e(U_e + V_r)U_e + (P_e - P_0))A_e = 0 \qquad (7.28)$$

Carry out the differentiation

$$M_r\frac{dV_r}{dt} + \cancel{V_r\frac{dM_r}{dt}} + D - \cancel{\rho_e U_e A_e V_r} - (\rho_e U_e^2 + (P_e - P_0))A_e = 0 \qquad (7.29)$$

Note

$$\frac{dM_r}{dt} = \rho_e U_e A_e \qquad (7.30)$$

Finally

$$\boxed{M_r\frac{dV_r}{dt} = (\rho_e U_e^2 A_e + (P_e - P_0)A_e) - D} \qquad (7.31)$$

Rocket mass × acceleration = Thrust - Drag

7.3 Effective exhaust velocity

Total impulse

$$I = \int_0^t T\,dt \tag{7.33}$$

Total propellant expended

$$M_p = \int_0^t \dot{m}\,dt \tag{7.34}$$

Effective exhaust velocity

$$C = \frac{dI}{dM_p} = \frac{T}{\dot{m}} = U_e + \frac{A_e}{\dot{m}}(P_e - P_0) \tag{7.35}$$

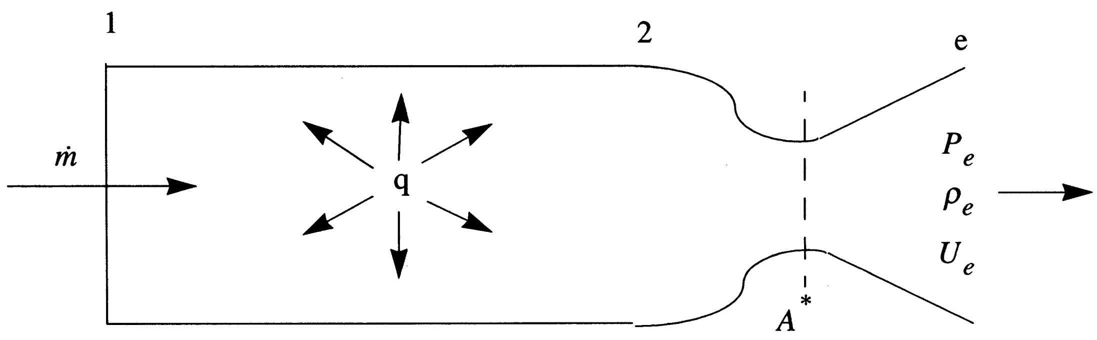
In terms of the exit Mach number

$$C = U_e\left(1 + \frac{P_e A_e}{\rho_e U_e^2 A_e}\left(1 - \frac{P_0}{P_e}\right)\right) \tag{7.36}$$

$$C = U_e\left(1 + \frac{1}{\gamma M_e^2}\left(1 - \frac{P_0}{P_e}\right)\right) \tag{7.37}$$

Note that for a large area ratio exhaust the pressure contribution is a diminishing proportion of the thrust.

Theoretical maximum exhaust velocity



$$h_{t2} = h_{t1} + q = h_e + \frac{1}{2}U_e^2 \tag{7.38}$$

$$C_{max} \cong \sqrt{2q} \tag{7.39}$$

$$C_{max} \cong \sqrt{2C_pT_{t2}} = \sqrt{\frac{2\gamma}{\gamma - 1}RT_{t2}}. \tag{7.40}$$

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Note that

$$R = \frac{R_u}{M_w} \tag{7.41}$$

The theoretical maximum exhaust velocity depends on the combustion chamber temperature and the molecular weight of the exhaust gases.

$$C_{max} \cong \sqrt{\frac{2\gamma}{\gamma - 1}\left(\frac{R_u}{M_w}\right)T_{t2}} \tag{7.42}$$

7.4 C* efficiency

Define C*

$$\dot{m} = \frac{P_{t2}A^*}{C^*} \qquad (7.43)$$

Recall our mass flow relation – at the choked throat

$$\dot{m} = P_{t2}A^*\left(\frac{\gamma}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\sqrt{\gamma\frac{R_u}{M_w}T_{t2}}}\right)$$

Under a constant heat capacity assumption C* would be

$$C^* = \frac{P_{t2}A^*}{\dot{m}} = \frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\sqrt{\gamma\frac{R_u}{M_w}T_{t2}}$$

C* is directly proportional to the speed of sound in the rocket chamber and provides a very useful measure of completeness of combustion. The definition (7.43) enables C* to be determined without making any assumptions about the gas passing through the nozzle. For a given mass flow and throat area, higher C* implies higher chamber pressure and higher thrust.

Rocket test schematic

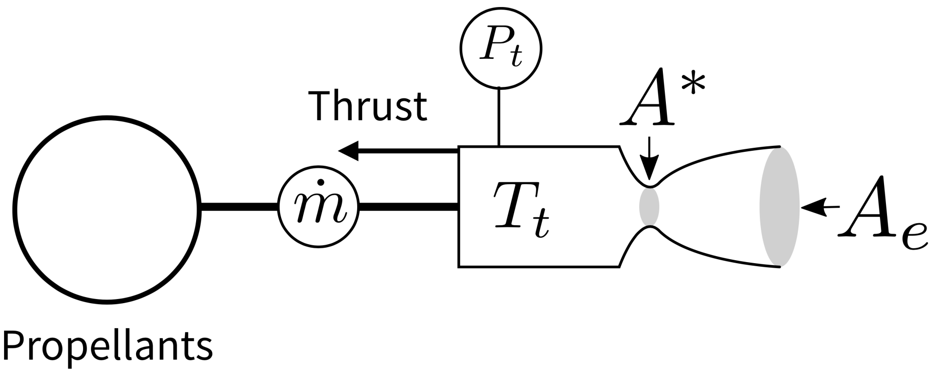


Figure – J Dyer

In a typical rocket test the mass flow, chamber pressure and throat area are relatively easy to measure with commonly available instrumentation allowing C* to be determined indirectly. The chamber temperature is very hard to measure. The C* efficiency is defined as

$$\eta_{C^*} \equiv \frac{\left(\dfrac{P_{t2}A^*}{\dot{m}}\right)_{measured}}{\left(\dfrac{P_{t2}A^*}{\dot{m}}\right)_{ideal}} \qquad (7.44)$$

For the same mass flow and area the C* efficiency is determined by the chamber pressure.

$$\eta_{C^*} \equiv \frac{P_{t2_{measured}}}{P_{t2_{ideal}}} \qquad (7.45)$$

The ideal chamber pressure is usually determined from an equilibrium combustion calculation.

7.5 Specific impulse

The specific impulse is defined as

$$\boxed{I_{sp} = \frac{T}{\dot{m} g_0} = \frac{C}{g_0}} \tag{7.46}$$

Note that for rockets the specific impulse is defined in terms of the total propellant mass flow whereas for air-breathing engines only the fuel mass flow is used. Note also that the specific impulse increases as the nozzle expansion ratio increases and as the ambient pressure decreases.

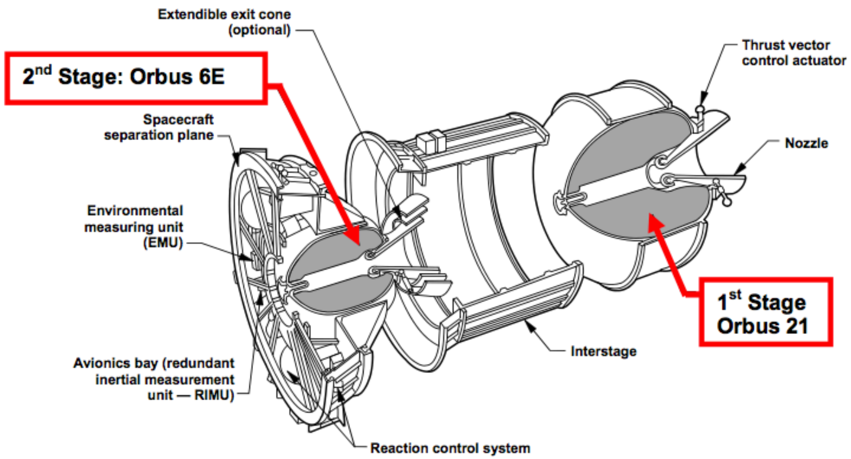
Typical solid rockets have ideal (fully expanded) specific impulses in the range of 230 to 290 sec.

Liquid rockets using hydrocarbon fuel burning with Oxygen have ideal specific impulses in the range of 360 to 370 sec. Hydrogen - Oxygen systems reach 450 sec.

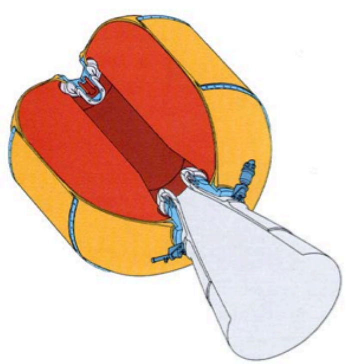
A meaningful statement about the specific impulse of a rocket motor should always specify the chamber pressure and area ratio of the motor and whether the information refers to ideal or delivered impulse.

Manufacturers will go to great lengths to achieve improvements of even a few seconds of specific impulse.

Boeing – CSD Inertial Upper Stage

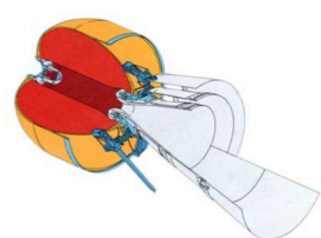


Air Force/NASA IUS, built by Boeing, a 2-Stage Space Vehicle using CSD's Orbus 21 and Orbus 6E Solid Propellant Rockets. It was Configured to Fly off both the Shuttle and Titan Launch Vehicles



Orbus 21: IUS 1st Stage

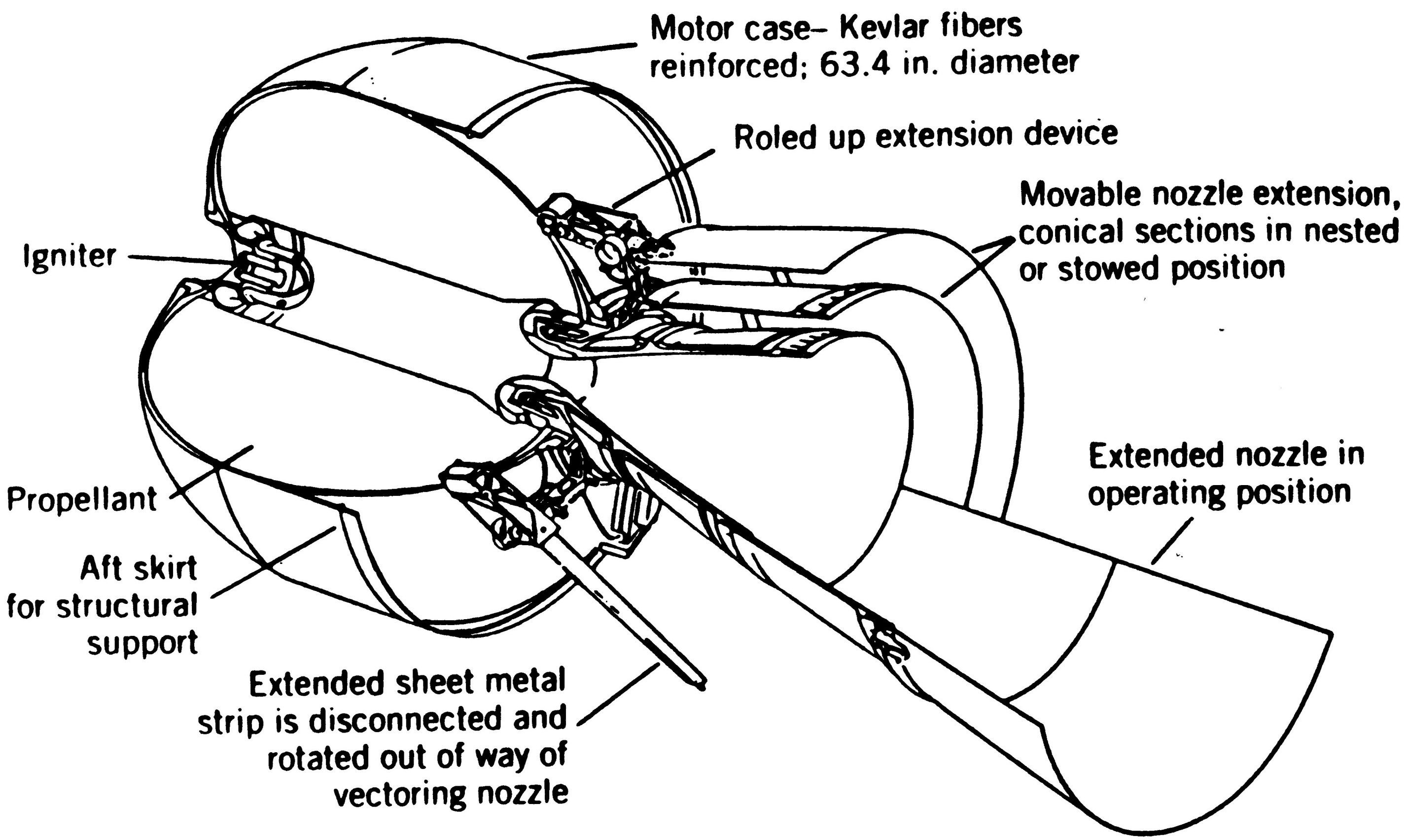
Diameter = 92-in
Wp = 21,400-lb



Orbus 6E: IUS 2nd Stage

Diameter = 63-in
Wp = 6,000-lb

Boeing inertial upper stage (IUS) with extensible vectored nozzle.
Nozzle area ratio could change from 49.3 to 181 increasing specific impulse by 14 seconds while allowing the motor to fit within a smaller length when the nozzle was stowed. The extensible nozzle added $1M to the cost of the motor.



7.4 Chamber pressure

Recall the all-important mass flow relation

$$\dot{m} = \rho U A = \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \frac{\gamma P_t A}{\sqrt{\gamma R T_t}} f(M) \tag{7.47}$$

where

$$f(M) = \frac{A^*}{A} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{M}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \tag{7.48}$$

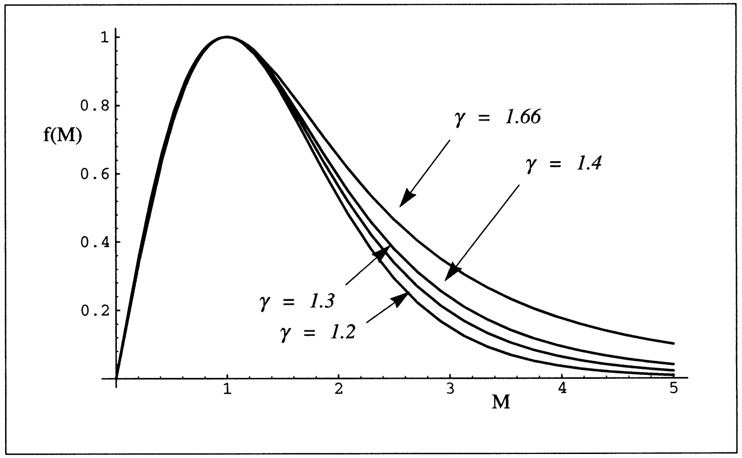


Figure 7.3 Area-Mach number relation

The stagnation temperature at station 2 is the sum of the incoming stagnation temperature and temperature increase due to heat release.

$$T_{t2} = T_{t1} + \frac{q}{C_p} \tag{7.49}$$

To a good approximation the stagnation temperature in the combustion chamber is independent of chamber stagnation pressure.

The chamber pressure is determined by the mass flow and chamber temperature. Virtually all rocket chambers run at a high enough pressure to insure that the nozzle is choked.

$$P_{t2} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{\sqrt{\gamma R T_{t2}}}{\gamma A^*} \dot{m} \tag{7.50}$$

7.5 Combustion chamber stagnation pressure drop

Recall the Rayleigh line relation

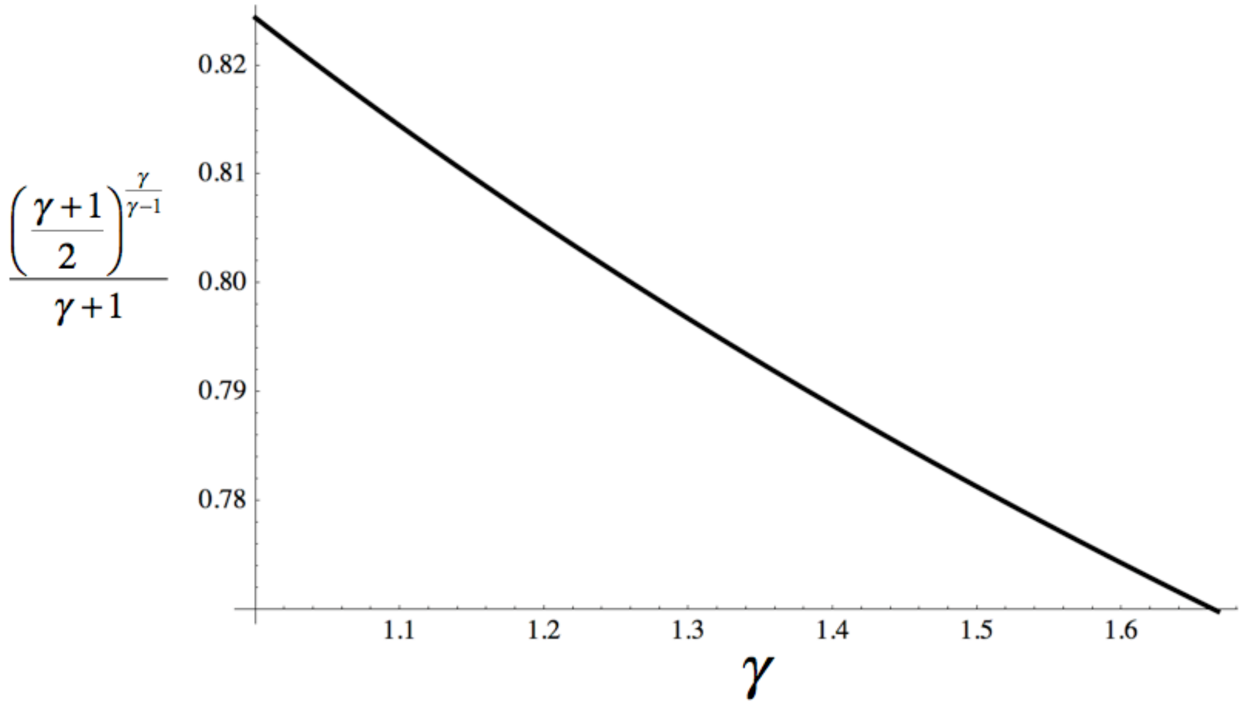
$$\frac{P_{t2}}{P_{t1}} = \left\{\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right\}\left\{\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right\}^{\frac{\gamma}{\gamma-1}} \tag{7.51}$$

The static pressure ratio is

$$\frac{P_2}{P_1} = \left\{\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right\} \tag{7.52}$$

The maximum stagnation pressure loss due to heat addition

$$\left(\frac{P_{t2}}{P_{t1}}\right)_{\substack{M_1=0\\M_2=1}} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}} = \frac{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}}{(1+\gamma)}$$



At station 1 the Mach number is small and so we can approximate conditions at station 2 just in terms of the Mach number at station 2.

$$\frac{P_{t2}}{P_{t1}} = \left\{\frac{1}{1+\gamma M_2^2}\right\}\left\{1+\frac{\gamma-1}{2}M_2^2\right\}^{\frac{\gamma}{\gamma-1}}$$

$$\frac{P_2}{P_1} = \left\{\frac{1}{1+\gamma M_2^2}\right\} \tag{7.53}$$

The Mach number at station 2 is determined by the nozzle area ratio.

$$\frac{A^*}{A_2} = \frac{M_2}{\left\{\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_2^2\right)\right\}^{\frac{\gamma+1}{2(\gamma-1)}}} \tag{7.54}$$

The internal area ratio of the nozzle determines the maximum stagnation pressure loss due to heat addition

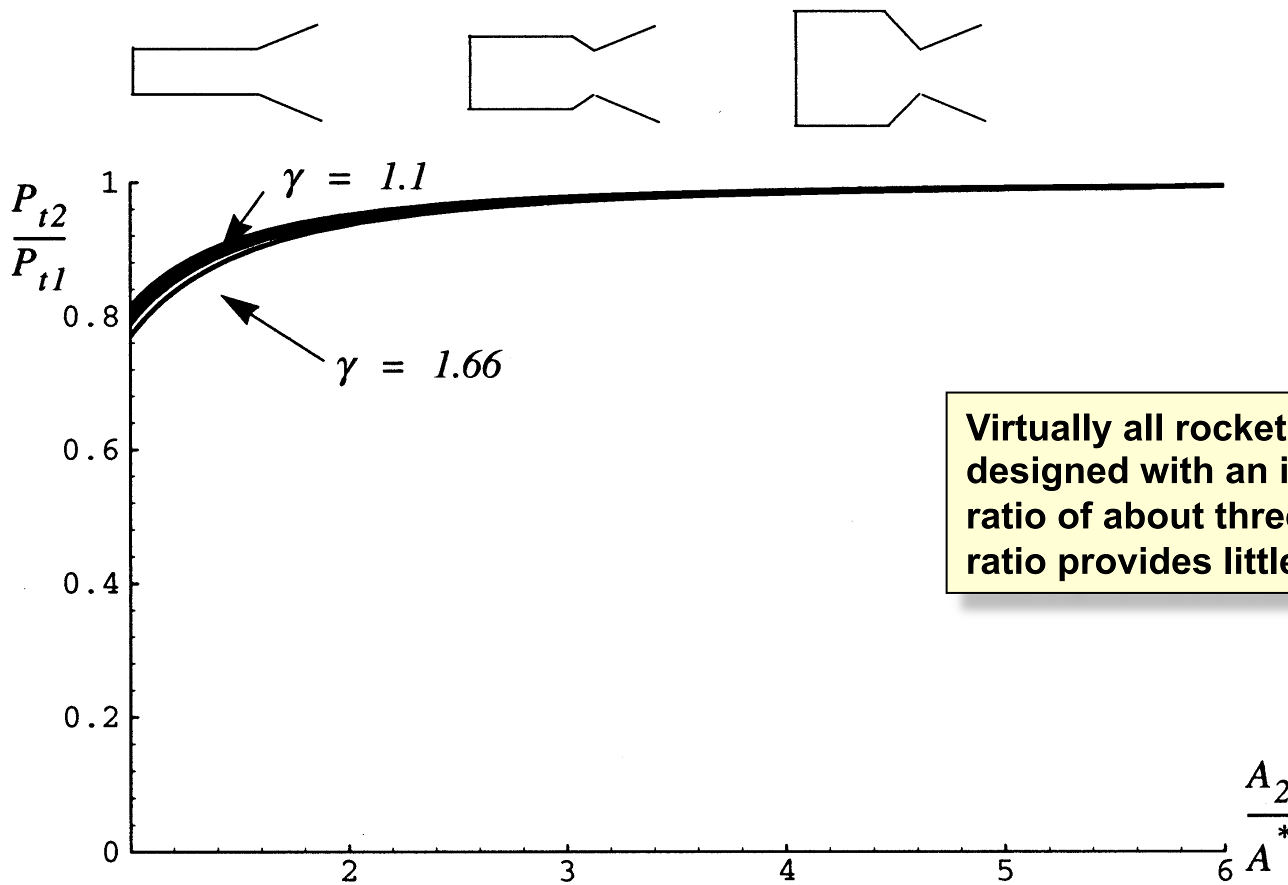


Figure 7.4 Combustion chamber stagnation pressure loss

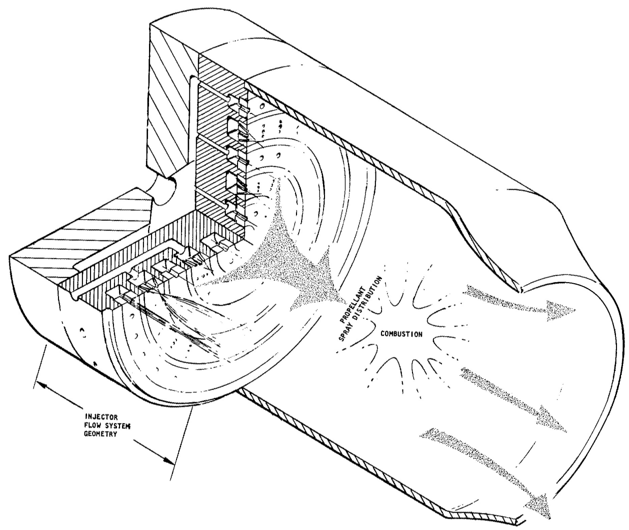
Virtually all rocket nozzles are designed with an internal area ratio of about three. A larger area ratio provides little improvement.

Description

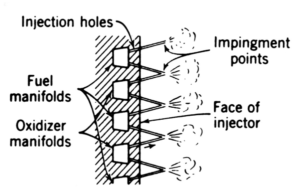
The Walter 109-509C two-chambered rocket engine was intended for use on Germany's Meserschmitt Me 263 and Junkers Ju 248 second generation rocket-powered interceptor aircraft, but hostilities in Europe ended before it was made operational. It burned a fuel mixture of methyl alcohol and hydrazine in hydrogen peroxide. The photograph shows a sectioned version of the smaller chamber. The fuel was pumped between the two layers of its double skin to keep it cool.

L*

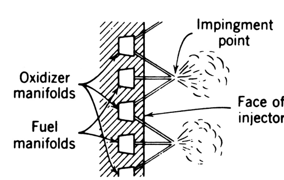
The length of the combustion chamber is determined by the distance needed for the propellants to atomize, mix and fully react in order to release all their chemical energy before exiting the combustion chamber.



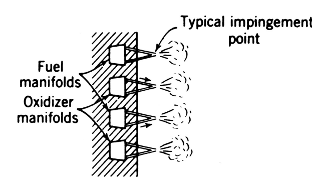
injector flow system



Doublet impinging stream pattern



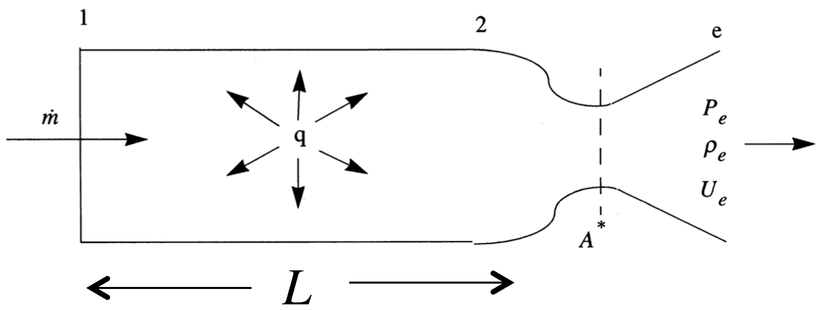
Triplet impinging stream pattern



Self-impinging stream pattern

Residence time t_r

Whether or not complete atomization, mixing and combustion is achieved is determined by the time that the propellants spend in the combustion chamber; the residence time.



$$t_r \equiv \frac{\text{mass of gas in the combustion chamber}}{\text{mass flow rate through the combustion chamber}} = \frac{M_g}{\dot{m}} = \frac{\rho_g V}{\rho^* U^* A^*} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{P_{t2} V}{\left(\frac{R_u}{M_w} T_{t2}\right)} \frac{\left(\gamma \frac{R_u}{M_w} T_{t2}\right)^{1/2}}{\gamma P_{t2} A^* f(M)}$$

$$t_r = \frac{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}{\left(\gamma \frac{R_u}{M_w} T_{t2}\right)^{1/2}} \frac{V}{A^*} = \frac{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}{\left(\gamma \frac{R_u}{M_w} T_{t2}\right)^{1/2}} L^*$$

$$L^* \equiv \frac{V}{A^*} = \frac{V}{A}\left(\frac{A}{A^*}\right) \cong 3\frac{V}{A} \cong 3L$$

Propellant Combination	L*, cm
Nitric acid/hydrazine-base fuel	76-89
Nitrogen tetroxide/hydrazine-base fuel	76-89
Hydrogen peroxide/RP-1 (including catalyst bed)	152-178
Liquid oxygen/RP-1	102-127
Liquid oxygen/ammonia	76-102
Liquid oxygen/liquid hydrogen (GH_2 injection)	56-71
Liquid oxygen/liquid hydrogen (LH_2 injection)	76-102
Liquid fluorine/liquid hydrogen (GH_2 injection)	56-66
Liquid fluorine/liquid hydrogen (LH_2 injection)	64-76
Liquid fluorine/hydrazine	61-71
Chlorine trifluoride/hydrazine-base fuel	51-89

7.8 The thrust coefficient

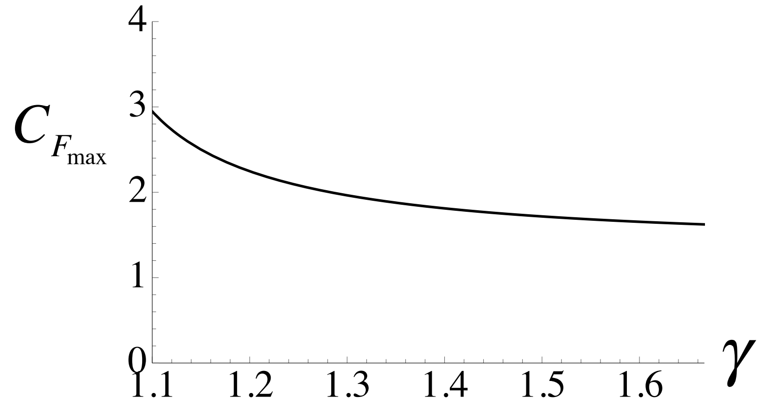
$$C_F = \frac{T}{P_{t2}A^*} = \frac{\dot{m}C}{P_{t2}A^*} = \frac{C}{C^*} = \frac{\dot{m}U_e + \left(P_e - P_0\right)A_e}{\dfrac{\dot{m}}{\gamma}\left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\sqrt{\gamma\dfrac{R_u}{M_w}T_{t2}}} \tag{7.65}$$

This can be expressed in terms of the exit Mach number and ambient pressure.

$$C_F = \frac{T}{P_{t2}A^*} = \frac{C}{C^*} = \frac{\left(\gamma M_e^2 + 1 - \dfrac{P_0}{P_e}\right)}{\left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} M_e\left(1 + \dfrac{\gamma-1}{2}M_e^2\right)^{1/2}} \tag{7.66}$$

For a rocket operating with a very large expansion ratio to vacuum the exit Mach number becomes large and the thrust coefficient has an upper limit.

$$C_{F_{max}} = \frac{\gamma}{\left(\frac{\gamma-1}{2}\right)^{1/2}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \tag{7.67}$$



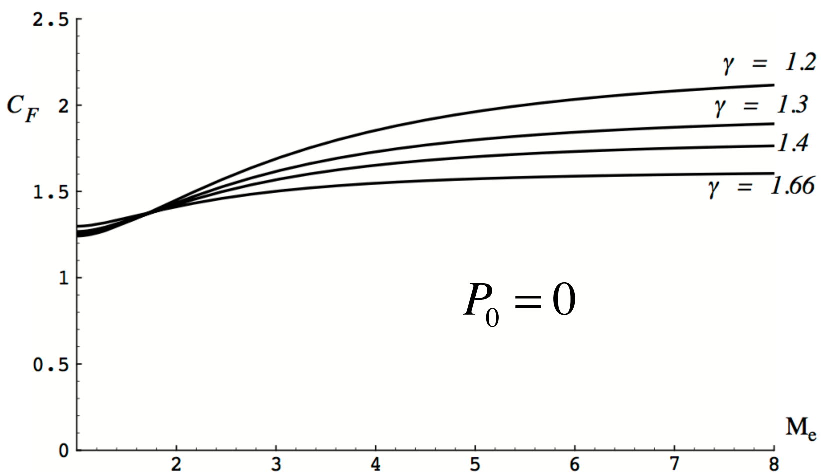


Figure 7.7: *Thrust coefficient versus Mach number.*

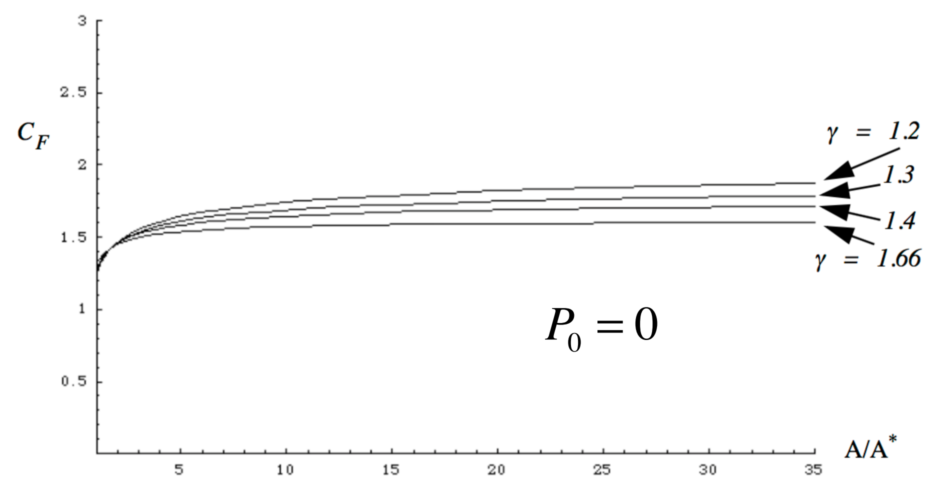
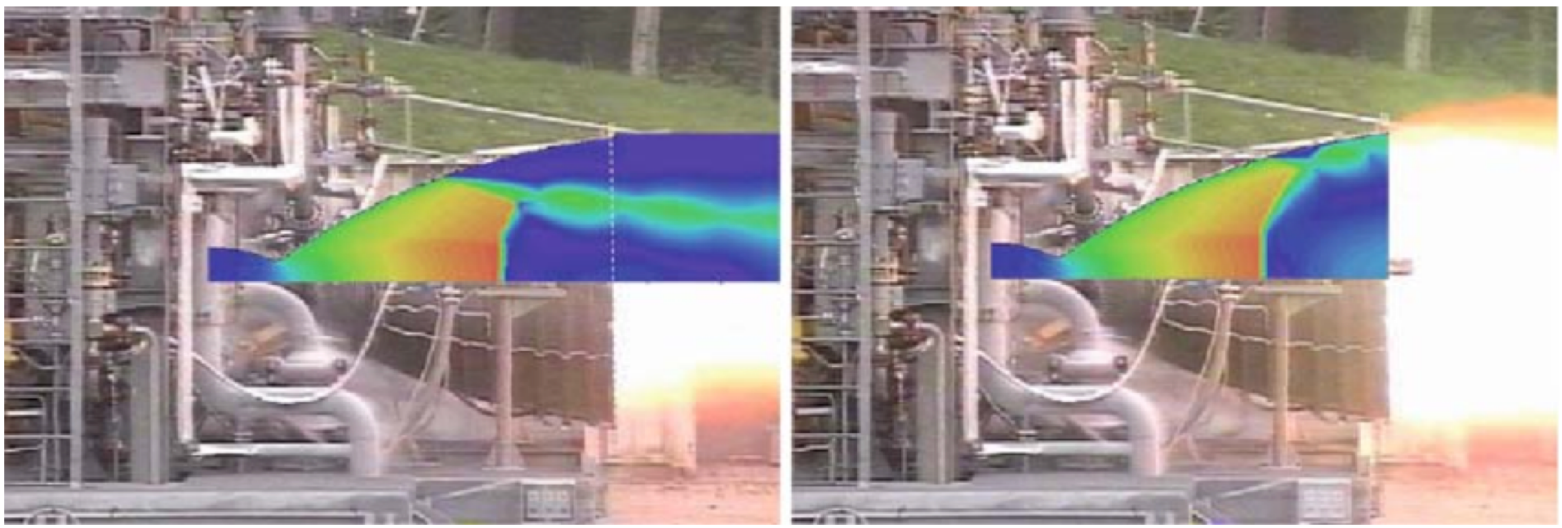


Figure 7.8: *Thrust coefficient versus area ratio.*

Nozzle performance

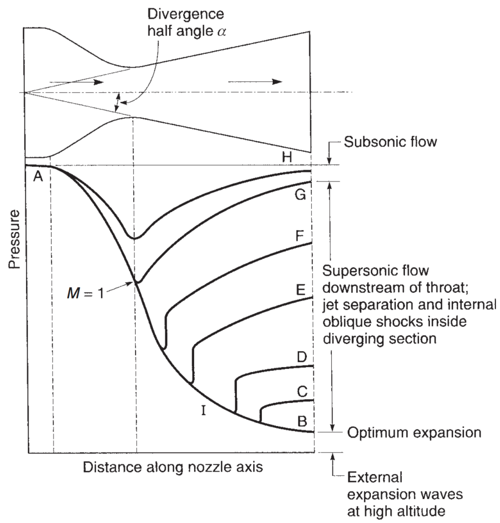
$$P_0 \neq 0$$

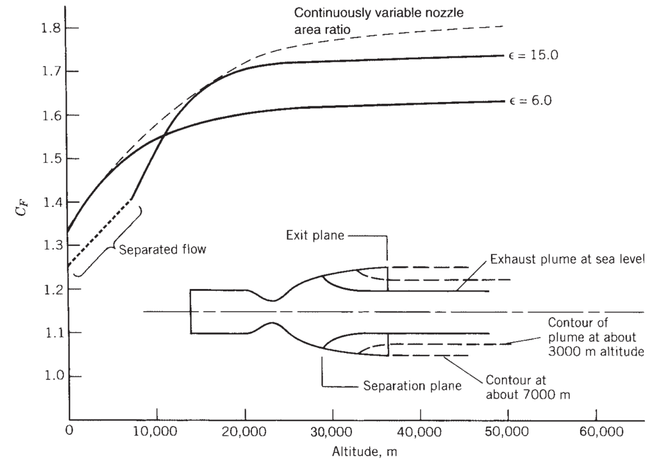
Due to the viscous boundary layer an over expanded nozzle will separate



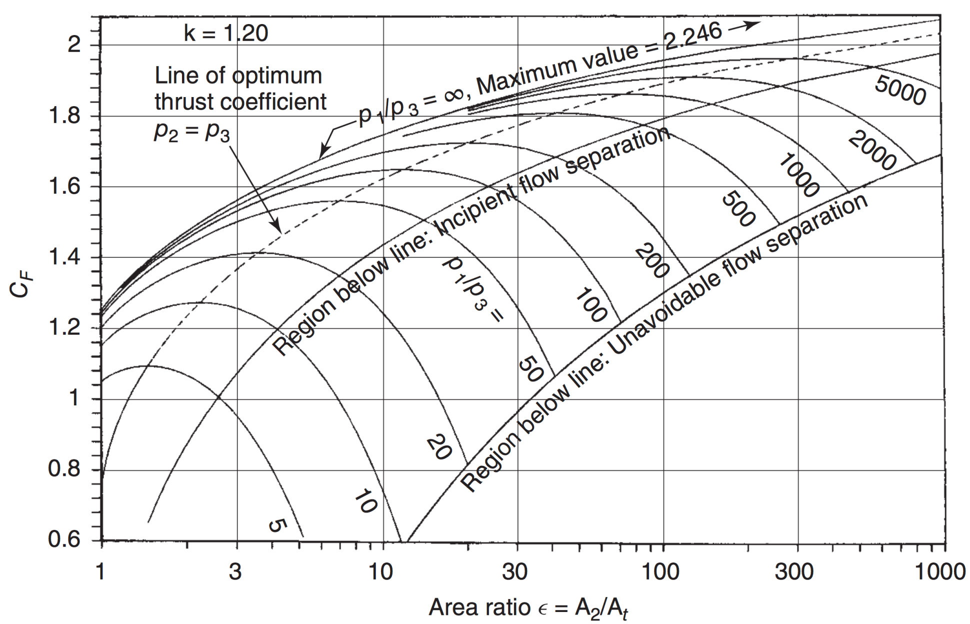
VULCAIN thrust chamber during start-up (at pressure ratio p c / p a of approximately 40), with free shock separation (FSS, left) and restricted shock separation (RSS, right) (Mach number distribution as inlay, increasing from blue to red)

Thrust coefficient with separation

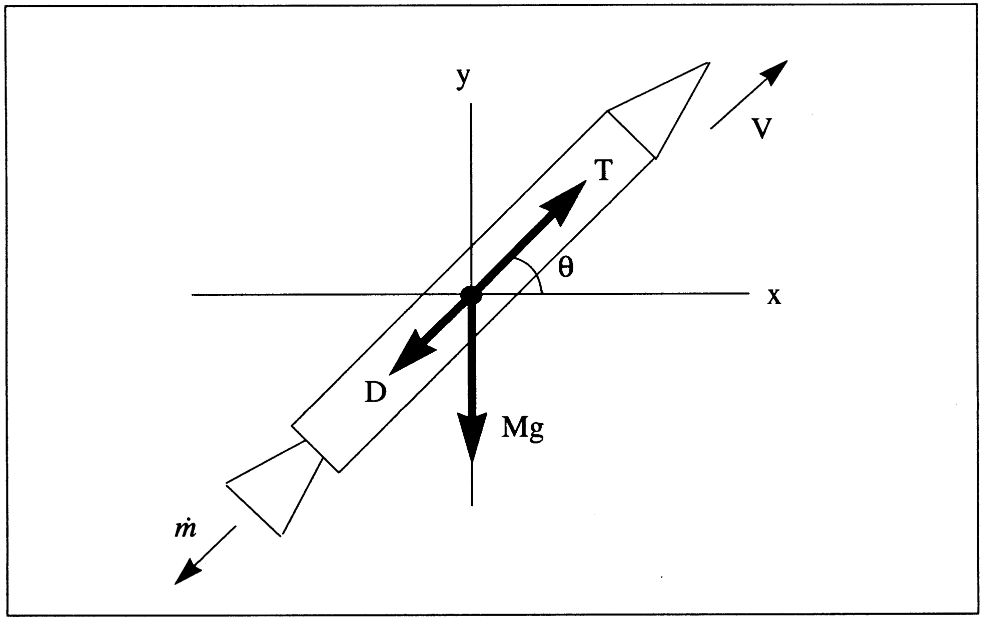




From Sutton: Rocket Propulsion Elements



7.6 The Moore - Tsiolkovsky rocket equation



T = *vehicle thrust*

D = *vehicle aerodynamic drag*

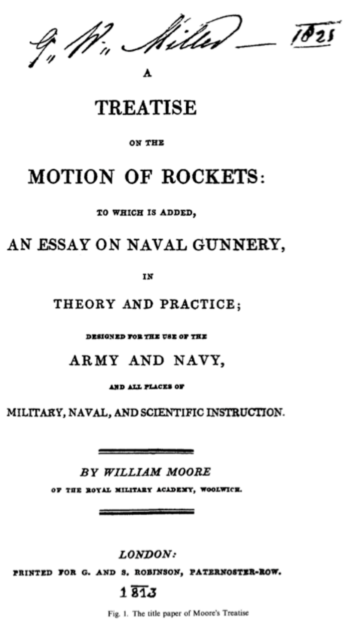
V = *vehicle velocity*

θ = *angle with respect to the horizontal*

$\dot{m}$ = *nozzle mass flow*

M = *vehicle mass*

g = *gravitational acceleration*



G. W. Miller — 1825

A

TREATISE

ON THE

MOTION OF ROCKETS:

TO WHICH IS ADDED,

AN ESSAY ON NAVAL GUNNERY,

IN

THEORY AND PRACTICE;

DESIGNED FOR THE USE OF THE

ARMY AND NAVY,

AND ALL PLACES OF

MILITARY, NAVAL, AND SCIENTIFIC INSTRUCTION.

BY WILLIAM MOORE

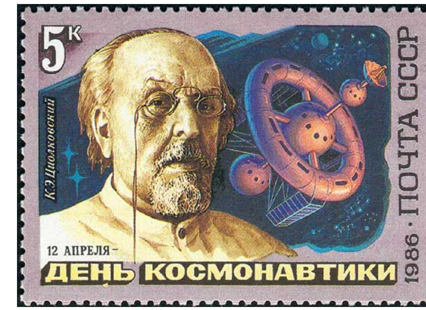
OF THE ROYAL MILITARY ACADEMY, WOOLWICH.

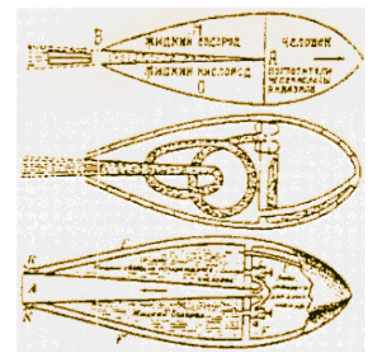
LONDON:

PRINTED FOR G. AND S. ROBINSON, PATERNOSTER-ROW.

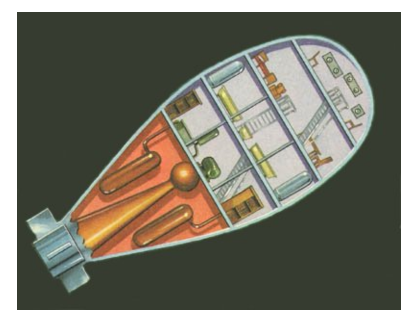
1813

Fig. 1. The title paper of Moore's Treatise





Tsiolkovsky Rocket Designs



Balance of forces along the direction of flight

$$M\frac{dV}{dt} = T - Mg\sin\theta - D \qquad (7.56)$$

Substitute the thrust equation expressed in terms of the effective exhaust velocity.

$$M\frac{dV}{dt} = -C\frac{dM}{dt} - Mg\sin\theta - D. \qquad (7.57)$$

Divide through by the current mass of the vehicle.

$$\frac{dV}{dt} = -C\frac{d(\ln M)}{dt} - g\sin\theta - \frac{D}{M} \qquad (7.58)$$

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Let

$$\begin{aligned} M_i &= \text{initial mass at } t = 0 \\ M_f &= \text{final mass at } t = t_b \\ t_b &= \text{time of burnout} \end{aligned} \tag{7.59}$$

The velocity change of the vehicle is.

$$\Delta V = V_b - V_0 = \Delta V|_{ideal} - \Delta V|_{gravitational} - \Delta V|_{drag} \tag{7.60}$$

where

$$\begin{aligned} \Delta V|_{gravitational} &= \int_0^{t_b} g \sin\theta dt \\ \Delta V|_{drag} &= \int_0^{t_b} \frac{D}{M} dt \end{aligned} \tag{7.61}$$

Ideal velocity increment

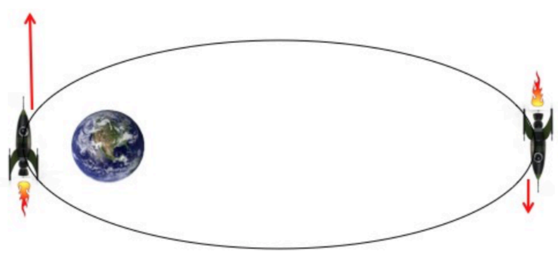
$$\boxed{\Delta V|_{ideal} = C\ln\left(\frac{M_i}{M_f}\right)} \qquad (7.62)$$

Thrust $T = M\frac{dV}{dt} = C\frac{dM}{dt}$

Power applied to the vehicle in the center-of-mass frame

$$P = TV = C\frac{dM}{dt}V$$

<u>The Oberth effect</u>
When adding energy to a spacecraft in a gravitational field the largest energy increase for a given mass of propellant burned occurs when the spacecraft is at its highest speed.



To minimize drag losses the rocket should be long and slender.

$$\Delta V_r\Big|_{drag} = \int_0^{t_b} \frac{D}{M_r}dt = \int_0^{t_b} \frac{1}{2}\frac{\rho V^2 C_D A}{M_{r\cancel{0}i}}\left(\frac{M_{ri}}{M_r}\right)dt = \frac{A}{M_{ri}}\int_0^{t_b} \frac{1}{2}(\rho V^2 C_D)\left(\frac{M_{ri}}{M_r}\right)dt \qquad (7.63)$$

$$M_{ri} \cong \rho_{vehicle} V_{vehicle}$$

$$\Delta V\Big|_{drag} \approx \frac{A}{V_{vehicle}} \approx \frac{1}{vehicle\ length} \qquad (7.64)$$

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Traditional Rocket Propulsive Efficiency

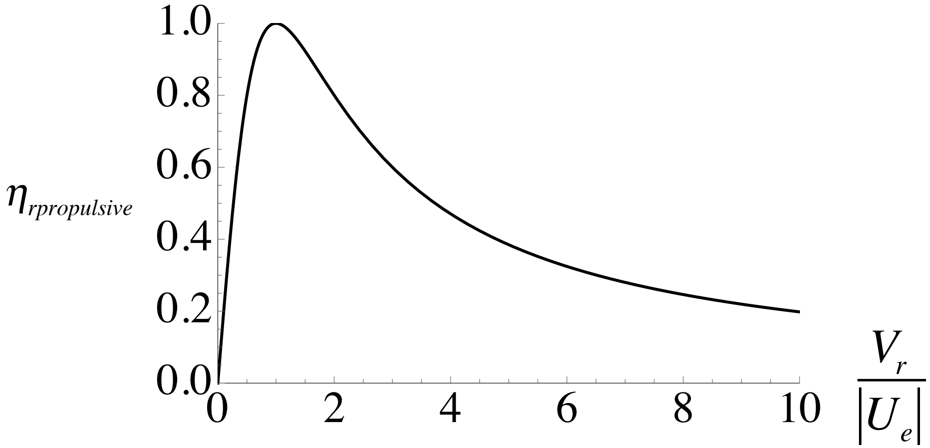
$$\frac{d}{dt}\left(\frac{1}{2}M_r V_r^2\right) = M_r V_r \frac{dV_r}{dt} + \frac{1}{2}V_r^2 \frac{dM_r}{dt}$$

$$\frac{dE_r}{dt} = M_r V_r \frac{dV_r}{dt} - \frac{1}{2}V_r^2 \frac{dm_p}{dt}$$

$$M_r \frac{dV_r}{dt} = \frac{dm_p}{dt}\left|U_e\right|$$

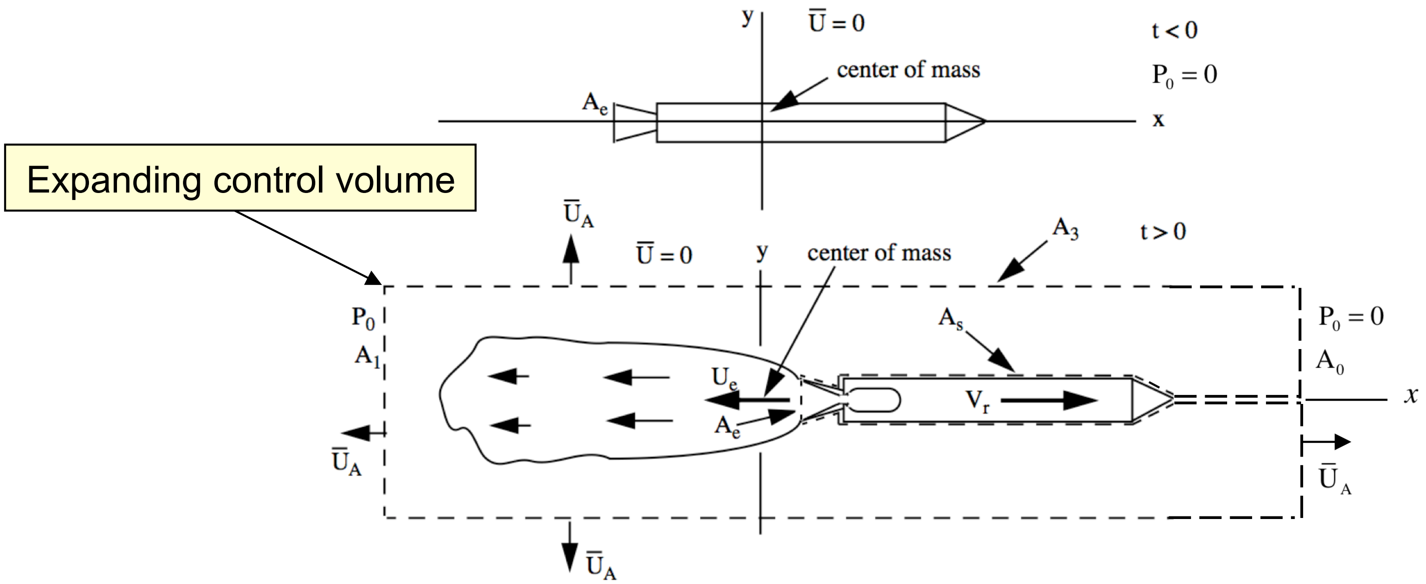
$$\frac{dE_r}{dt} = \left|U_e\right| V_r \frac{dm_p}{dt} - \frac{1}{2}V_r^2 \frac{dm_p}{dt}$$

$$\eta_{rpropulsive} = \frac{TV_r}{TV_r + \frac{1}{2}\frac{dm_p}{dt}\left(\left|U_e\right| - V_r\right)^2} = \frac{\left|U_e\right| V_r \frac{dm_p}{dt}}{\left|U_e\right| V_r \frac{dm_p}{dt} + \frac{1}{2}\frac{dm_p}{dt}\left(\left|U_e\right| - V_r\right)^2} = \frac{2\frac{V_r}{\left|U_e\right|}}{1 + \left(\frac{V_r}{\left|U_e\right|}\right)^2}$$



This provides a snapshot of the rocket efficiency at a moment during the flight but is not a useful measure of propellant utilization over the full burn.

Where does the energy go? Assume vacuum conditions, no gravity.



$M_d -$ Rocket delivered mass after all propellant is expended

$M_p -$ Initial propellant mass

$M_r -$ Rocket total mass at any instant

$m_p -$ Propellant mass expelled since the beginning of the burn

$V_r -$ Vehicle velocity during the burn

$V_{rf} -$ Vehicle velocity at the end of the burn

$E_{rf} -$ Vehicle kinetic energy at the end of the burn

$E_{pf} -$ Energy of the propellant in the rocket plume at the end of the burn

$U_e -$ Velocity of the propellant exiting the nozzle during the burn, assumed constant

$h_p -$ Propellant total enthalpy

$h_{te} = h_e + \frac{1}{2}U_e^2 -$ Enthalpy of the propellant exiting the nozzle during the burn

Energy that ends up in the plume

$$\frac{dE_p}{dt} = h_e \frac{dm_p}{dt} + \frac{1}{2}\left(-\left|U_e\right| + V_r\right)^2 \frac{dm_p}{dt}$$

$$dE_p = h_e dm_p + \frac{1}{2}\left|U_e\right|^2 \left(M_d + M_p\right)\left(-1 + Ln\left(\frac{1}{1 - \frac{m_p}{M_d + M_p}}\right)\right)^2 d\left(\frac{m_p}{M_d + M_p}\right)$$

$$x = \frac{m_p}{M_d + M_p}$$

$$E_{pf} = h_e \int_0^{M_p} dm_p + \frac{1}{2}\left|U_e\right|^2 \left(M_d + M_p\right)\int_0^{\frac{M_p}{M_d + M_p}} \left(1 + Ln\left(1 - x\right)\right)^2 dx$$

$$E_{pf} = h_e M_p + \frac{1}{2}\left|U_e\right|^2 \left(M_d + M_p\right)\left(x - \left(1 - x\right)\left(Ln\left(1 - x\right)\right)^2\right)_0^{\frac{M_p}{M_d + M_p}}$$

$$\boxed{E_{pf} = M_p\left(h_e + \frac{1}{2}\left|U_e\right|^2\right) - \frac{1}{2} M_p \left|U_e\right|^2 \left(\frac{M_d + M_p}{M_p} - 1\right)\left(Ln\left(1 - \frac{M_p}{M_d + M_p}\right)\right)^2}$$

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Define the rocket cycle efficiency as

$$\eta_{rc} = \left(\frac{h_e + \frac{1}{2} U_e^{\,2}}{h_{propellant}} \right)$$

This accounts for propellant energy losses between the storage tanks and the nozzle exit.

Define the rocket thermal efficiency as

$$\eta_{rth} = \left(\frac{\frac{1}{2} \left| U_e \right|^2}{h_e + \frac{1}{2} \left| U_e \right|^2} \right) = \left(\frac{\frac{\gamma - 1}{2} \gamma M_e^2}{1 + \frac{\gamma - 1}{2} \gamma M_e^2} \right)$$

This accounts for propellant energy not converted to nozzle exhaust kinetic energy.

The energy lost to the plume can be expressed as

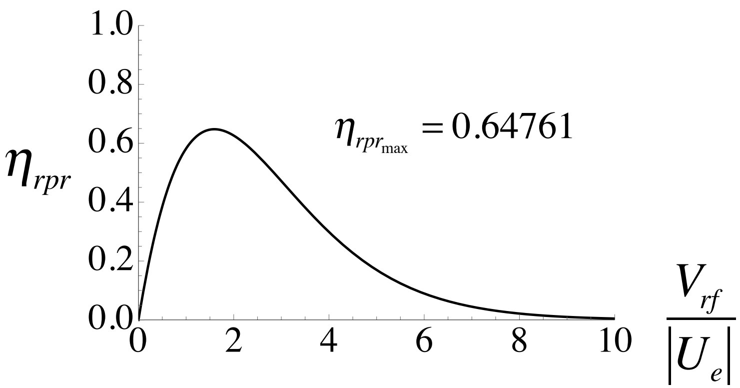
$$\frac{E_{pf}}{M_p \left(h_e + \frac{1}{2} \left| U_e \right|^2 \right)} = 1 - \eta_{rth} \left(\frac{M_d + M_p}{M_p} - 1 \right) \left(Ln \left(1 - \frac{M_p}{M_d + M_p} \right) \right)^2$$

Energy transferred to the rocket vehicle during the burn

$$V_{rf} = |U_e| Ln\left(\frac{1}{1-\dfrac{M_p}{M_d+M_p}}\right)$$

$$E_{rf} = \frac{1}{2}M_d V_{rf}^2$$

$$E_{rf} = \frac{1}{2}M_p|U_e|^2\left(\frac{M_d+M_p}{M_p}-1\right)\left(Ln\left(1-\frac{M_p}{M_d+M_p}\right)\right)^2$$



Rocket overall efficiency

$$\eta_{rov} = \frac{E_{rf}}{M_p\left(h_e + \frac{1}{2}|U_e|^2\right)}$$

$$\eta_{rov} = \eta_{th}\left(\frac{M_d+M_p}{M_p}-1\right)Ln\left(1-\frac{M_p}{M_d+M_p}\right)^2$$

Maximum fractional conversion of propellant energy to rocket kinetic energy occurs for

$$\frac{M_p}{M_d+M_p} = 0.7968$$

$$\frac{V_{rf}}{|U_e|} = 1.59362$$

Rocket propulsive efficiency

$$\eta_{rpr} = \frac{E_{rf}}{\frac{1}{2}M_p|U_e|^2} = \left(\frac{M_d+M_p}{M_p}-1\right)\left(Ln\left(1-\frac{M_p}{M_d+M_p}\right)\right)^2$$

$$\frac{M_p}{M_d+M_p} = 1-e^{-\frac{V_{rf}}{|U_e|}}$$

$$\eta_{rpr} = \left(\frac{1}{e^{\frac{V_{rf}}{|U_e|}}-1}\right)\left(\frac{V_{rf}}{|U_e|}\right)^2$$