CHAPTER 10

ELEMENTS OF POTENTIAL FLOW

10.1 INCOMPRESSIBLE FLOW

Most of the problems we are interested in involve low speed flow about wings and bodies. The equations governing incompressible flow are

Continuity

$$\nabla \bullet \bar{U} = 0 \tag{10.1}$$

Momentum

$$\frac{\partial \bar{U}}{\partial t} + \nabla \bullet \left(\bar{U}\bar{U} + \frac{P}{\rho}\bar{\bar{I}} \right) = \nu \nabla^2 \bar{U} \tag{10.2}$$

The convective term can be rearranged using $\nabla \bullet \bar{U} = 0$ and the identity

$$\bar{U} \bullet \nabla \bar{U} = \left(\nabla \times \bar{U} \right) \times \bar{U} + \nabla \left(\frac{\bar{U} \bullet \bar{U}}{2} \right) \tag{10.3}$$

The viscous term in (10.2) can be rearranged using the identity

$$\nabla \times \left(\nabla \times \bar{U} \right) = \nabla \left(\nabla \bullet \bar{U} \right) - \nabla^2 \bar{U} \tag{10.4}$$

Using these results and $\nabla \bullet \bar{U} = 0$ the momentum equation can be written in terms of the vorticity.

$$\bar{\Omega} = \nabla \times \bar{U} \tag{10.5}$$

in the form

$$\frac{\partial \bar{U}}{\partial t} + \bar{\Omega} \times \bar{U} + \nabla \left(\frac{P}{\rho} + \frac{\bar{U} \bullet \bar{U}}{2} \right) + \nu \nabla \times \bar{\Omega} = 0 \tag{10.6}$$

If the flow is irrotational, $\bar{\Omega} = 0$, and the velocity can be expressed in terms of a velocity potential.

$$\bar{U} = \nabla\Phi \tag{10.7}$$

The continuity equation becomes Laplace's equation

$$\nabla \bullet \bar{U} = \nabla \bullet \nabla\Phi = \nabla^2\Phi = 0 \tag{10.8}$$

and the momentum equation is fully integrable.

$$\nabla\left(\frac{\partial\Phi}{\partial t} + \frac{P}{\rho} + \frac{\bar{U}\bullet\bar{U}}{2}\right) = 0 \tag{10.9}$$

The quantity in parentheses is at most a function of time

$$\frac{\partial\Phi}{\partial t} + \frac{P}{\rho} + \frac{\bar{U}\bullet\bar{U}}{2} = f(t) \tag{10.10}$$

The expression (10.10) is called the Bernoulli integral and can be used to determine the pressure throughout the flow once the velocity potential is known from a solution of Laplace's equation (10.7).

Generally the flow is specified within a volume V surrounded by surface A (Figure 10.1). A solid body defined by the function $G(\bar{x},t) = 0$ may be imbedded inside V.

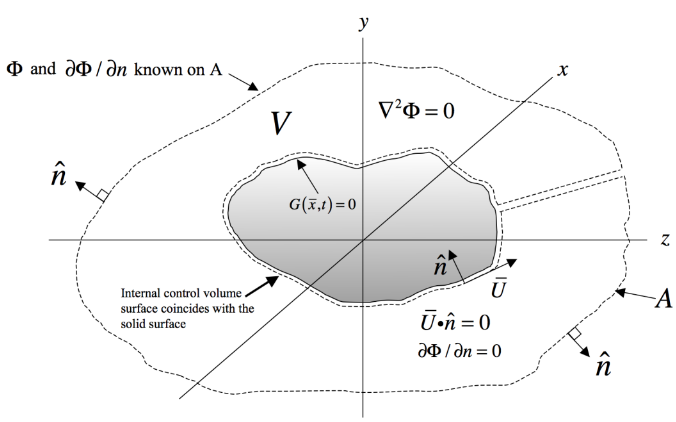


Figure 10.1 Flow volume V with surface A and imbedded solid body $G(\bar{x},t) = 0$.

The solid body has the outward normal

$$\hat{n}_{Body} = \frac{\nabla G}{|\nabla G|} \tag{10.11}$$

Laplace's equation is second order in the spatial derivatives and two conditions must be known to construct a solution. Generally the value of Φ is known on the boundary (Derichlet condition) as well as the derivative of Φ normal to the boundary $\partial\Phi / \partial n$

(Neumann condition). If the fluid is inviscid the velocity at the surface of the body is not zero and cannot penetrate the body. The velocity vector at the solid surface must be tangent to the surface and the velocity component normal to the surface must be zero. This is expressed by

$$\bar{U}\bullet\nabla G = 0 \qquad (10.12)$$

In other words, the flow satisfies the Neumann condition $\partial\Phi/\partial n = 0$ on the portion of A coincident with the surface of the body. The value of the potential and normal derivative are usually known on the rest of the boundary. Virtually all of the flows we will deal with are external flows such as that depicted schematically in Figure 10.1 and will involve potentials that decrease with distance from the body such that $\Phi = \partial\Phi/\partial n = 0$ on the outer boundary.

10.2 POTENTIALS

If Φ_1 and Φ_2 are solutions of Laplace's equation then so is

$$\Phi_3 = \Phi_1 + \Phi_2$$

The linearity of Laplace's equation allows solutions to be constructed from the superposition of simpler, elementary, solutions. This is the key feature of the equation that makes it a powerful tool for analyzing fluid flows. In this approach the requirement that the flow be divergence free and curl free everywhere is relaxed to permit isolated regions to exist within the flow where mass and vorticity can be created.

One can view an unsteady, incompressible flow as a field constructed from a scalar distribution of mass sources, $Q(\bar{x},t)$ and a vector distribution of vorticity sources, $\bar{\Omega}(\bar{x},t)$. In this approach the velocity field is generated from the linear superposition of two fields.

$$\bar{U} = \bar{U}_{sources} + \bar{U}_{vortices} \qquad (10.13)$$

The velocity field generated by the mass sources is irrotational and that generated by the vorticity sources is divergence free. The continuity equation for such a flow now has a source term.

$$\nabla\bullet\bar{U} = \nabla\bullet\bar{U}_{sources} = Q(\bar{x},t) \qquad (10.14)$$

The curl of the velocity is

$$\nabla\times\bar{U} = \nabla\times\bar{U}_{vortices} = \bar{\Omega}(\bar{x},t) \qquad (10.15)$$

The velocity field is constructed from the superposition of the velocities generated by a scalar potential Φ generated by the mass sources and a vector potential $\bar{A}$ generated by the vorticity sources.

$$\bar{U} = \nabla\Phi + \nabla \times \bar{A} \qquad (10.16)$$

The potentials satisfy a system of Poisson equations, a single equation for the scalar potential

$$\nabla \bullet \nabla\Phi = \nabla^2\Phi = Q(\bar{x},t) \qquad (10.17)$$

and three equations for the Cartesian components of the vector potential.

$$\nabla^2 \bar{A} = -\bar{\Omega}(\bar{x},t) \qquad (10.18)$$

where the identity $\nabla \times \nabla \times \bar{A} = \nabla(\nabla \bullet \bar{A}) - \nabla^2 \bar{A}$ has been used with the choice of a Coulomb gauge on the vector potential $\nabla \bullet \bar{A} = 0$. This choice has no effect on the velocity field generated by the vector potential since to any vector potential we can add the gradient of a scalar. Let $\bar{A} = \bar{A}' + \nabla s$. Clearly $\nabla \times \bar{A} = \nabla \times \bar{A}'$. Choose s so that $\nabla \bullet \bar{A} = \nabla \bullet \bar{A}' + \nabla^2 s = 0$.

This approach allows one to construct fairly complex flow fields that can be rotational while retaining the simplicity of working in terms of potentials governed by linear equations and the associated law of superposition. The flow is determined once the distribution of mass sources and vorticity sources are specified. Notice that this theory of potential flow is exactly analogous to the theory of potentials in electricity and magnetism. The mass sources coincide with the distribution of electric charges and the vorticity coincides with the electric currents.

10.3 USEFUL SPECIAL FUNCTIONS

A function that is highly useful in the development of potential theory is the smooth version of the Heavyside-theta function

$$h(x;\varepsilon) = \frac{1}{2}\left(1 + Erf\left(\frac{x}{2\varepsilon}\right)\right) \qquad (10.19)$$

where ε is a small parameter that determines the steepness of the smooth transition from 0 to 1. The Heavyside function is dimensionless. The small parameter has the same units as the argument x, $\hat{\varepsilon} = 1/\hat{x}$ and the Heaviside-theta function is dimensionless.

The first derivative of $h(x;\varepsilon)$ is a smooth version of the Dirac delta function

$$\delta(x;\varepsilon) = \frac{e^{-\frac{x^2}{4\varepsilon^2}}}{2\varepsilon\sqrt{\pi}} \qquad (10.20)$$

The integral of $\delta(x;\varepsilon)$ is

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{4\varepsilon^2}}}{2\varepsilon\sqrt{\pi}}\,dx = 1 \tag{10.21}$$

Integrating the product of the Dirac delta function and some function $f(x)$ is

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-a)^2}{4\varepsilon^2}}}{2\varepsilon\sqrt{\pi}} f(x)\,dx = f(a) \tag{10.22}$$

In three dimensions

$$\delta(x;\varepsilon)\delta(y;\varepsilon)\delta(z;\varepsilon) = \frac{e^{-\frac{r^2}{4\varepsilon^2}}}{8\varepsilon^3\pi^{3/2}} \tag{10.23}$$

where $r^2 = x^2 + y^2 + z^2$. The units of the Dirac delta function are $\hat{\delta} = 1/\hat{x}$ and its derivative is

$$\delta'(x;\varepsilon) = \frac{-xe^{-\frac{x^2}{4\varepsilon^2}}}{4\varepsilon^3\sqrt{\pi}} = -\frac{x}{2\varepsilon^2}\delta(x;\varepsilon) \tag{10.24}$$

The derivative satisfies

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\frac{e^{-\frac{x^2}{4\varepsilon}}}{2\sqrt{\pi\varepsilon}} f(x)\right)dx = \int_{-\infty}^{\infty}\left(f(x)\frac{\partial}{\partial x}\left(\frac{e^{-\frac{x^2}{4\varepsilon}}}{2\sqrt{\pi\varepsilon}}\right) + \frac{\partial f}{\partial x}\left(\frac{e^{-\frac{x^2}{4\varepsilon}}}{2\sqrt{\pi\varepsilon}}\right)\right)dx = \left(\frac{e^{-\frac{x^2}{4\varepsilon}}}{2\sqrt{\pi\varepsilon}} f(x)\right)_{-\infty}^{\infty} = 0$$

(10.25)

In the limit $\varepsilon \to 0$

$$\int_{-\infty}^{\infty} f(x)\frac{\partial\delta(x)}{\partial x}dx = -\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}\delta(x)dx \tag{10.26}$$

The result (10.26) can be used to convert integrals involving derivatives of the Dirac delta function to the basic form (10.22).

10.4 Point source solution of the Poisson equation

The figure below shows a smooth spherically symmetric source of mass centered at the origin of a set of spherical polar coordinates.

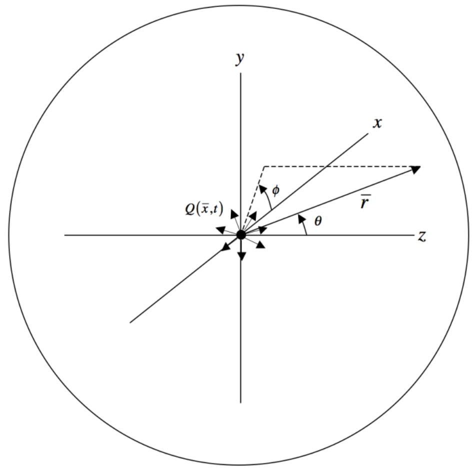


Figure 10.2 Mass source at the origin.

Let the smooth distribution of sources be

$$Q(\bar{x},t;\varepsilon)=Q(t)\delta(x;\varepsilon)\delta(y;\varepsilon)\delta(z;\varepsilon)=\frac{Q(t)}{8\varepsilon^3\pi^{3/2}\rho}e^{-\frac{r^2}{4\varepsilon^2}} \tag{10.27}$$

where the small parameter ε is real and positive and Q is the total strength of the source with units $Mass\,/\,Sec$. The governing equation for the scalar potential is the Poisson equation.

$$\nabla^2\Phi=\frac{Q(t)}{8\varepsilon^3\pi^{3/2}\rho}e^{-\frac{r^2}{4\varepsilon^2}} \tag{10.28}$$

In spherical polar coordinates the equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)=\frac{Q(t)}{8\varepsilon^3\pi^{3/2}\rho}e^{-\frac{r^2}{4\varepsilon^2}} \tag{10.29}$$

with solution

$$\Phi(r)=C_1-\frac{C_2}{r}-\frac{Q(t)}{4\pi r\rho}Erf\left(\frac{r}{2\varepsilon}\right) \tag{10.30}$$

At $r\to\infty$, $\Phi\to 0$ and therefore $C_1=0$. The second constant of integration can be determined by integrating the Poisson equation over the control volume.

$$\int_V(\nabla\bullet\nabla\Phi)dV=\frac{Q(t)}{8\varepsilon^3\pi^{3/2}\rho}\int_V e^{-\frac{r^2}{4\varepsilon^2}}dV \tag{10.31}$$

Use Gauss's theorem on the left hand side.

$$\int_0^{\pi}\int_0^{2\pi}\frac{\partial\Phi}{\partial r}r^2 Sin(\theta)\,d\phi\, d\theta=\frac{Q(t)}{8\varepsilon^3\pi^{3/2}\rho}\int_o^{r}\int_0^{\pi}\int_0^{2\pi}e^{-\frac{r^2}{4\varepsilon^2}}r^2 Sin(\theta)\,d\phi\, d\theta\, dr \qquad (10.32)$$

When (10.30) is substituted into (10.31) the result is $C_2 = 0$. This analysis leads to a smooth version of the source solution of the Poisson equation.

$$\Phi(r,t)=-\frac{Q(t)}{4\pi\rho r}Erf\left(\frac{r}{2\varepsilon}\right) \qquad (10.33)$$

In the limit $\varepsilon \to 0$, the source distribution reduces to a point and $-(1/4\pi r)Erf\left(r/(2\varepsilon)\right)$ becomes the classical Green's function for the Laplace equation, $G(\overline{x},\overline{x}_s)=-1/\left(4\pi\left|\overline{x}-\overline{x}_s\right|\right)$ where $\overline{x}_s$ is the vector position of the source. The fundamental point source solution of the Laplace equation for a source located as $\overline{x}_s$ is

$$\Phi(\overline{x},\overline{x}_s,t)=-\frac{Q(t)}{4\pi\rho\left|\overline{x}-\overline{x}_s\right|} \qquad (10.34)$$

10.5 General solution of the Poisson Equation

Figure 10.3 shows a general smooth distribution of mass sources and vorticity sources in a finite region near the origin. The source strength outside the finite region is zero and is said to be compact.

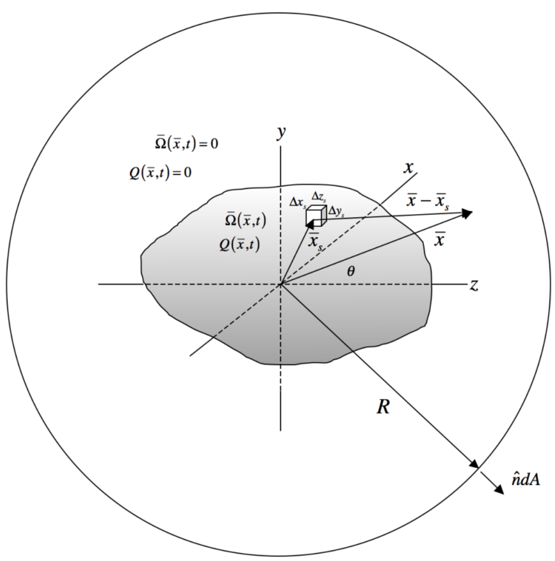


Figure 10.3 Smooth, finite distribution of mass and vorticity sources near the origin.

We can solve the Poisson equation by using the fundamental point solution (10.33) and superposing the scalar potentials produced by an infinite sum of differential sources. The incremental scalar potential, $\Delta\Phi$, produced by the mass source contained in the differential volume shown in the figure above is determined from the Poisson equation for this source. Namely

$$\nabla^2(\Delta\Phi) = \frac{Q(\overline{x}_s,t)\Delta x\Delta y\Delta z}{8\varepsilon^3\pi^{3/2}\rho} e^{-\frac{|\overline{x}-\overline{x}_s|^2}{4\varepsilon^2}} \tag{10.35}$$

where now Q is the source strength per unit volume with units $Mass/(Sec - Length^3)$. The solution of (10.35) with $\lim_{\varepsilon\to 0} Erf(|\overline{x}-\overline{x}_s|/2\varepsilon) = 1$ is

$$\Delta\Phi = -\frac{Q(\overline{x}_s,t)\Delta x_s\Delta y_s\Delta z_s}{4\pi\rho|\overline{x}-\overline{x}_s|} \tag{10.36}$$

In the limit where the differential volume becomes infinitesimally small

$$d\Phi = -\frac{Q(\overline{x}_s,t)}{4\pi\rho|\overline{x}-\overline{x}_s|} dx_s dy_s dz_s \tag{10.37}$$

and the general solution to the scalar Poisson equation (10.17) is

$$\Phi(\overline{x},t) = -\frac{1}{4\pi\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{Q(\overline{x}_s,t)}{|\overline{x}-\overline{x}_s|} dx_s\, dy_s\, dz_s \tag{10.38}$$

The Poisson equation for the vector potential

$$\nabla^2\overline{A}(x,y,z,t) = -\overline{\Omega}(x,y,z,t) \tag{10.39}$$

is equivalent to three scalar equations relating the Cartesian components of the vector potential to the Cartesian components of the vorticity.

$$\nabla^2 A_x = -\Omega_x \qquad \nabla^2 A_y = -\Omega_y \qquad \nabla^2 A_z = -\Omega_z \tag{10.40}$$

The same procedure used to derive (10.38) can be applied to each of the Cartesian components of the vector potential. The fundamental solution of (10.39) at vector position $\overline{x}$ due to a point source of vorticity of strength $\Omega(\overline{x}_s,t)dx_s dy_s dz_s$ at source point $\overline{x}_s$ is

$$d\overline{A} = \frac{\overline{\Omega}(\overline{x}_s,t)dx_s dy_s dz_s}{4\pi|\overline{x}-\overline{x}_s|} \tag{10.41}$$

The general solution of (10.39) is

$$\bar{A}(\bar{x},t)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{\Omega}(\bar{x}_s,t)}{|\bar{x}-\bar{x}_s|}dx_s\,dy_s\,dz_s \tag{10.42}$$

Several examples are used to illustrate these ideas.

Example 1 – Velocity potential due to a finite line of sources along the z-axis.

The figure below shows a line of sources distributed along the z-axis between $a<z<b$.

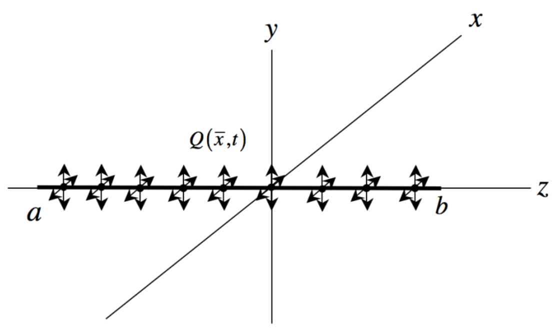


Figure 10.4 Finite line distribution of mass sources

The source distribution is

$$\frac{Q(\bar{x},t)}{\rho}=\dot{S}(t)\delta(x)\delta(y)u(b-z)u(z-a) \tag{10.43}$$

where the units of $\dot{S}$ are $Area/Time$. The strength $\dot{S}$ is the volume generated per unit length of the source distribution per unit time (the total volume generated per second is $(b-a)\dot{S}(t)$). The scalar potential is

$$\Phi=-\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\dot{S}(t)\delta(x_s)\delta(y_s)u(b-z_s)u(z_s-a)}{\left((x-x_s)^2+(y-y_s)^2+(z-z_z)^2\right)^{1/2}}dx_s\,dy_s\,dz_s=$$

$$-\frac{1}{4\pi}\int_a^b\frac{\dot{S}(t)}{\left(x^2+y^2+(z-z_z)^2\right)^{1/2}}dz_s= \tag{10.44}$$

$$\frac{\dot{S}(t)}{4\pi}Ln\left(2\left(z-z_s+\sqrt{x^2+y^2+(z-z_z)^2}\right)\right)\Big|_a^b$$

The potential (10.44) of a finite line of sources is

$$\Phi(x,y,z,t;a,b)=\frac{\dot{S}(t)}{4\pi}Ln\left(\frac{z-b+\sqrt{x^2+y^2+(z-b)^2}}{z-a+\sqrt{x^2+y^2+(z-a)^2}}\right) \tag{10.45}$$

The potential (10.45) can be used to generate the potential due to a semi-infinite line of sources by expanding (10.45) about the point $a \to -\infty$.

$$\begin{aligned}
&\lim_{a\to-\infty}\Phi(x,y,z,t;a,b)= \\
&\frac{\dot{S}(t)}{4\pi}\left(-Ln(2)+Ln\left(-\frac{1}{a}\right)+Ln\left(z-b+\sqrt{x^2+y^2+(z-b)^2}\right)\right)+ \\
&\frac{\dot{S}(t)y}{4\pi a}-\frac{\dot{S}(t)}{16\pi a^2}\left(x^2+y^2-2z^2\right)+\frac{\dot{S}(t)}{24\pi a^3}\left(3x^2z+3y^2z-2z^3\right)+O\left(\frac{1}{a^4}\right)
\end{aligned} \tag{10.46}$$

Taking the limit, $a \to -\infty$ in (10.46) the potential of a semi-infinite line of sources is

$$\lim_{a\to-\infty}\Phi(x,y,z,t;a,b)=\frac{\dot{S}(t)}{4\pi}Ln\left(z-b+\sqrt{x^2+y^2+(z-b)^2}\right)+\frac{\dot{S}(t)}{4\pi}\left(Ln\left(-\frac{1}{2a}\right)\right) \tag{10.47}$$

The additive constant which goes to infinity as $a \to -\infty$ is not surprising in view of the fact that the source distribution extends to infinity and the potential from any segment of this source distribution only dies off like $1/r$ where r is the distance from the segment. The singularity in the potential has no effect on the velocity field generated from $\bar{U}=\nabla\Phi$.

Let the line of sources extend to plus infinity by setting $a=-b$ in (10.45). Expand about the point $b \to \infty$. The result is

$$\begin{aligned}
&\lim_{b\to\infty}\Phi(x,y,z,t;b)= \\
&\frac{\dot{S}(t)}{4\pi}\left(-2Ln(2)+2Ln\left(\frac{1}{b}\right)+Ln\left(x^2+y^2\right)\right)-\frac{\dot{S}(t)}{8\pi b^2}\left(x^2+y^2-2z^2\right)+O\left(\frac{1}{b^4}\right)
\end{aligned} \tag{10.48}$$

When the limit $b \to \infty$ is applied the result is

$$\lim_{b\to\infty}\Phi(x,y,t;b)=\frac{\dot{S}(t)}{4\pi}Ln\left(x^2+y^2\right)+\frac{\dot{S}(t)}{2\pi}Ln\left(\frac{1}{2b}\right) \tag{10.49}$$

Again there is a logarithmic infinity in the potential when we add up an infinite line of sources in a three-dimensional world. Dropping the constant we recover the potential for a two-dimensional line source of area $Q(t)$.

$$\Phi(x,y,t)=\frac{Q(t)}{2\pi\rho}Ln\left(x^2+y^2\right)^{1/2} \tag{10.50}$$

The radial velocity generated by differentiating (10.50) with respect to r is

$$U_r = \frac{Q(t)}{2\pi\rho}\left(\frac{1}{r}\right) \qquad (10.51)$$

If the origin is enclosed by a circle and the area flux from the source is integrated the result is

$$\int_0^{2\pi} U_r r d\theta = \frac{Q(t)}{\rho} \qquad (10.52)$$

Using the same differential procedure we used in three-dimensions, the circularly symmetric source solution (10.50) can be used to generate the general solution of the two-dimensional Poisson equation for a compact distribution of sources. The result is

$$\Phi(x,y,t) = \frac{1}{2\pi\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(\bar{x}_s,t)\,Ln\left(\left|\bar{x}-\bar{x}_s\right|^{1/2}\right)dx_s\,dy_s \qquad (10.53)$$

Example 2 – Vector potential of a vortex monopole

The figure below shows a vortex monopole located at the origin with its counter-clockwise rotation axis aligned with the z axis.

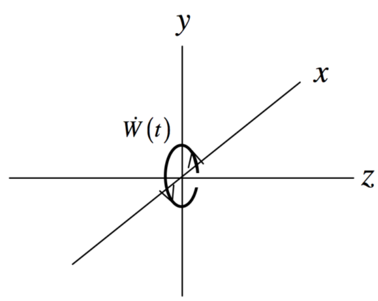


Figure 10.5 A vortex monopole

The vorticity source term is

$$\bar{\Omega}(\bar{x},t) = \left\{\ 0, 0, \dot{W}(t)\delta(x)\delta(y)\delta(z)\right\} \qquad (10.54)$$

Insert (10.54) into the general solution $\bar{A} = (0,0,A_z)$ where

$$A_z(\bar{x},t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\dot{W}(t)\delta(x_s)\delta(y_s)\delta(z_s)}{\left|\bar{x}-\bar{x}_s\right|}dx_s\,dy_s\,dz_s \qquad (10.55)$$

The solution is a vector point source

$$\bar{A} = \left\{ 0, 0, \frac{\dot{W}}{4\pi\left(x^2 + y^2 + z^2\right)^{1/2}} \right\} \tag{10.56}$$

where $\dot{W}$ is the z-component of the volume integrated strength of the vorticity source distribution.

$$\begin{gathered} \bar{\dot{W}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\Omega}\left(x_s, y_s, z_s\right) dx_s dy_s dz_s = \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\dot{W}} \delta\left(x_s\right)\delta\left(y_s\right)\delta\left(z_s\right) dx_s dy_s dz_s = \left\{0, 0, \dot{W}\right\} \end{gathered} \tag{10.57}$$

with units $\left[\bar{\dot{W}}\right] = Volume\,/\,Time = Circulation \times Length$. The corresponding velocity field is

$$\bar{U} = \left\{ -\frac{\dot{W}y}{4\pi\left(x^2 + y^2 + z^2\right)^{3/2}}, \frac{\dot{W}x}{4\pi\left(x^2 + y^2 + z^2\right)^{3/2}}, 0 \right\} \tag{10.58}$$

The circulation about a contour that encircles the z-axis is

$$\Gamma = \oint_C \bar{U} \bullet \hat{c}\, dC = \int_0^{2\pi} \left\{ -\frac{\dot{W}y}{2\pi r^3}, \frac{\dot{W}x}{2\pi r^3}, 0 \right\} \bullet \left\{ -\frac{y}{r}, \frac{x}{r}, 0 \right\} r d\theta = \frac{\dot{W}}{r} \tag{10.59}$$

The circulation of a monopole decays with radius from the source and there is no net circulation at infinity.

Example 2 – Vector potential of a line of vortex monopoles.

Shown below is a distribution of counter-clockwise vortex monopoles of uniform strength along the z-axis.

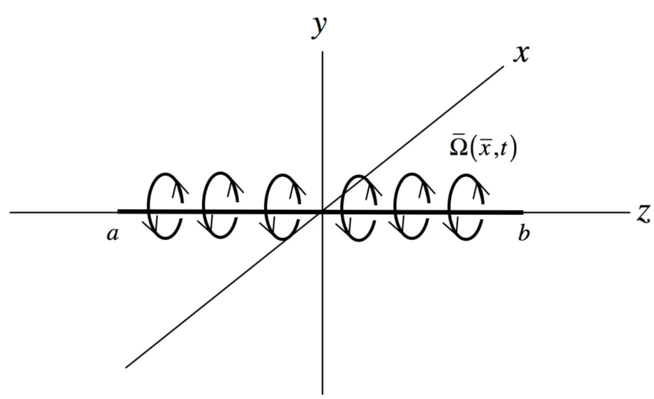


Figure 10.6 A line distribution of vortex monopoles

The vorticity source distribution for this case is

$$\bar{\Omega}(\bar{x},t) = \{0,0,\Gamma(t)\delta(x)\delta(y)u(b-z)u(z-a)\} \tag{10.60}$$

and the vector potential is $\bar{A} = \{0,0,A_z\}$ where

$$\begin{aligned} &A_z(x,y,z,t;a,b) = \\ &\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Gamma(t)\delta(x_s)\delta(y_s)u(b-z_s)u(z_s-a)}{\left((x-x_s)^2+(y-y_s)^2+(z-z_s)^2\right)^{1/2}}dx_s\,dy_s\,dz_s = \\ &\frac{1}{4\pi}\int_a^b\frac{\Gamma(t)}{\left(x^2+y^2+(z-z_s)^2\right)^{1/2}}dz_s = \frac{-\Gamma(t)}{4\pi}Ln\left(\frac{z-b+\sqrt{x^2+y^2+(z-b)^2}}{z-a+\sqrt{x^2+y^2+(z-a)^2}}\right) \end{aligned} \tag{10.61}$$

The velocity field is $\bar{U} = \nabla\times\bar{A}$.

$$\bar{U} = \frac{-\Gamma}{4\pi}\left(\begin{array}{c}\dfrac{1}{\sqrt{x^2+y^2+(z-b)^2}\left(z-b+\sqrt{x^2+y^2+(z-b)^2}\right)} - \\ \dfrac{1}{\sqrt{x^2+y^2+(z-a)^2}\left(z-a+\sqrt{x^2+y^2+(z-a)^2}\right)}\end{array}\right)\times\{y,-x,0\} \tag{10.62}$$

Let the vortex line extend equal distances from the origin, $a=-b$. The circulation about a contour that encircles the z-axis on the plane $z=0$ is

$$\Gamma_{circle} = \oint_C \bar{U}\bullet\hat{c}\,dC = \frac{\Gamma}{4\pi}\int_0^{2\pi}\left(\begin{array}{c}\dfrac{1}{\sqrt{x^2+y^2+b^2}\left(-b+\sqrt{x^2+y^2+b^2}\right)} - \\ \dfrac{1}{\sqrt{x^2+y^2+b^2}\left(b+\sqrt{x^2+y^2+b^2}\right)}\end{array}\right)\left(x^2+y^2\right)d\theta \tag{10.63}$$

Consider two limits of (10.63). In the first let the radius of the circle of integration become large.

$$\Gamma_{circle} = \lim_{R\to\infty}\frac{\Gamma}{4\pi}\int_0^{2\pi}\left(\begin{array}{c}\dfrac{1}{\sqrt{R^2+b^2}\left(-b+\sqrt{R^2+b^2}\right)} - \\ \dfrac{1}{\sqrt{R^2+b^2}\left(b+\sqrt{R^2+b^2}\right)}\end{array}\right)R^2 d\theta = \frac{\Gamma b}{R} \tag{10.64}$$

The circulation of a finite line of circulation decays with radius from the source and there is no net circulation at infinity. Now assume the radius of the circle of integration is finite and let the length of the line of circulation become infinite.

$$\Gamma_{circle} = \lim_{b\to\infty} \frac{\Gamma}{4\pi} \int_0^{2\pi} \left(\begin{array}{c} \dfrac{1}{\sqrt{R^2+b^2}\left(-b+\sqrt{R^2+b^2}\right)} - \\ \dfrac{1}{\sqrt{R^2+b^2}\left(b+\sqrt{R^2+b^2}\right)} \end{array} \right) R^2 d\theta = \Gamma \tag{10.65}$$

The circulation is constant independent of the radius of the circle of integration consistent with Helmholtz' laws for inviscid flow with vorticity.

Now return to the vector potential (10.61) let $a=-b$ and take the limit $b\to\infty$. The resulting vector potential for an infinite vortex line is $\overline{A}=\left(0,0,A_z\right)$ where

$$\lim_{b\to\infty} A_z\left(x,y,z,t;b\right) = \frac{-\Gamma(t)}{4\pi} Ln\left(x^2+y^2\right) - \frac{\Gamma(t)}{2\pi} Ln\left(\frac{1}{2b}\right) \tag{10.66}$$

Drop the singular constant. The fundamental source solution for the two-dimensional Poisson equation for the z component of the vector potential (aka the stream function) is

$$\Psi\left(x,y,t\right) = \frac{-\Gamma(t)}{2\pi} Ln\left(\left(x^2+y^2\right)^{1/2}\right) \tag{10.67}$$

Again we apply the same differential procedure we used in three-dimensions to two dimensions. The circularly symmetric source solution (10.67) is used to construct the general solution of the two-dimensional Poisson equation for a compact distribution of vorticity sources. The result is

$$\Psi\left(x,y,t\right) = \frac{-1}{2\pi\rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma\left(x_s,y_s,t\right) Ln\left(\left|\overline{x}-\overline{x}_s\right|^{1/2}\right) dx_s\, dy_s \tag{10.68}$$

Example 3 – Uniform flow past a sphere

The velocity potential of a dipole source of volume in a three-dimensional flow is

$$\Phi_{Dipole} = \frac{\kappa x}{\left(x^2+y^2+z^2\right)^{3/2}} \tag{10.69}$$

where κ is the strength of the dipole. The dipole added to a uniform flow generates the potential flow about a sphere in uniform flow.

$$\Phi_{Sphere} = \Phi_{Uniform\ Flow} + \Phi_{Dipole} = U_\infty x + \frac{\kappa x}{\left(x^2 + y^2 + z^2\right)^{3/2}} \tag{10.70}$$

The flow, including the singular flow inside the sphere is shown below.

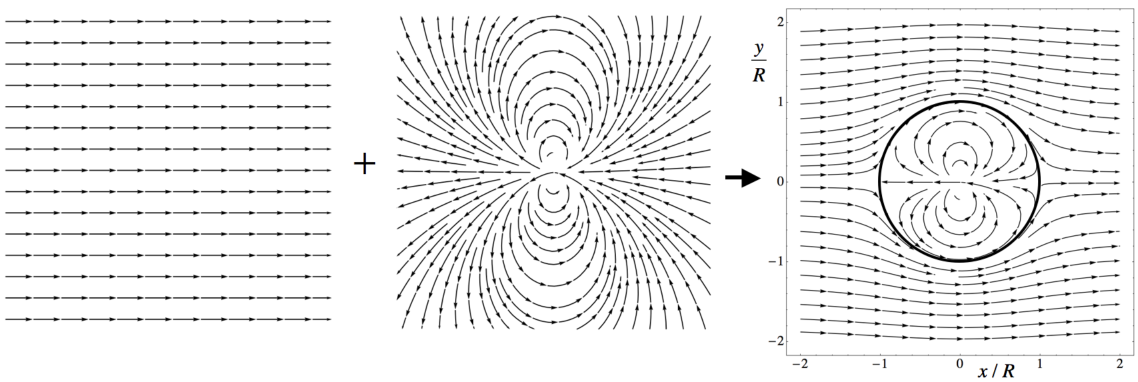


Figure 10.7 Potential flow past a sphere.

The velocity components are

$$\begin{aligned} U_x(x,y,z) &= U_\infty - \frac{3\kappa x^2}{r^5} + \frac{\kappa}{r^3} \\ U_y(x,y,z) &= -\frac{3\kappa xy}{r^5} \\ U_z(x,y,z) &= -\frac{3\kappa xz}{r^5} \end{aligned} \tag{10.71}$$

The radius of the sphere is determined from the positions of the stagnation points where $U_x = 0$.

$$R_{Sphere} = \left(\frac{2\kappa}{U_\infty}\right)^{1/3} \tag{10.72}$$

Note that the velocity disturbance at large distances from the sphere decays like $1/r^3$. Define the strength of the dipole as

$$\kappa = \frac{U_\infty}{2}\left(R_{Sphere}\right)^3 \tag{10.73}$$

Now the velocity potential in terms of the radius is

$$\Phi_{Sphere} = U_\infty x\left(1 + \frac{\left(R_{Sphere}\right)^3}{2\left(x^2 + y^2 + z^2\right)^{3/2}}\right) \tag{10.74}$$

and the velocities are

$$
\begin{aligned}
U_x(x,y,z) &= U_\infty\left(1-\frac{3\left(R_{Sphere}\right)^3 x^2}{2r^5}+\frac{\left(R_{Sphere}\right)^3}{2r^3}\right)\\
U_y(x,y,z) &= -U_\infty\frac{3\left(R_{Sphere}\right)^3 xy}{2r^5}\\
U_z(x,y,z) &= -U_\infty\frac{3\left(R_{Sphere}\right)^3 xz}{2r^5}
\end{aligned}
\tag{10.75}
$$

10.6 ELEMENTARY 2D POTENTIAL FLOWS

Any irrotational, incompressible, 2-D flow can be represented by, either the velocity potential and/or a stream function.

$$
\begin{aligned}
U &= \frac{\partial\Phi}{\partial x}=\frac{\partial\Psi}{\partial y}\\
V &= \frac{\partial\Phi}{\partial y}=-\frac{\partial\Psi}{\partial x}
\end{aligned}
\tag{10.76}
$$

The relations in (10.76) may be familiar as the Cauchy-Riemann equations from the theory of complex variables. Let

$$z = x+iy \tag{10.77}$$

The complex stream function is

$$W(z)=\Phi(x,y)+i\Psi(x,y) \tag{10.78}$$

An important representation of a complex variable due to Leonhard Euler is

$$z = x+iy = r\left(Cos(\theta)+iSin(\theta)\right)=re^{i\theta} \tag{10.79}$$

where

$$r=\left(x^2+y^2\right)^{1/2} \tag{10.80}$$

and

$$Tan(\theta)=\frac{y}{x} \tag{10.81}$$

Two-dimensional potential flows can be constructed from any analytic function of a complex variable, $W(z)$. From the Cauchy-Riemann conditions (10.76)

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=\frac{\partial^2\Psi}{\partial x\partial y}-\frac{\partial^2\Psi}{\partial y\partial x}=0 \tag{10.82}$$

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=-\frac{\partial^2\Phi}{\partial y\partial x}-\frac{\partial^2\Phi}{\partial x\partial y}=0 \tag{10.83}$$

Both the velocity potential and stream function satisfy Laplace's equation.

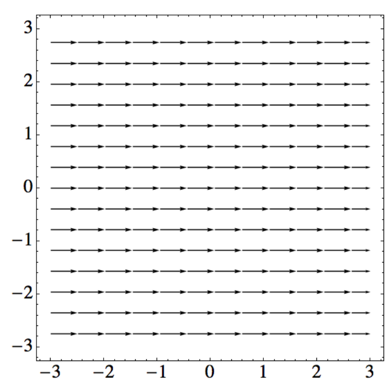
The derivative of the complex potential (the complex velocity) is independent of the path along which the derivative is taken.

$$\begin{aligned}\frac{dW}{dz}&=\frac{\partial\Phi}{\partial x}\frac{dx}{dz}+i\frac{\partial\Psi}{\partial x}\frac{dx}{dz}=U-iV\\ \frac{dW}{dz}&=\frac{\partial\Phi}{\partial y}\frac{dy}{dz}+i\frac{\partial\Psi}{\partial y}\frac{dy}{dz}=\frac{V}{i}+i\frac{U}{i}=U-iV\end{aligned} \tag{10.84}$$

Some Elementary Flows with their streamline patterns

1) Uniform flow in the x-direction

$$W=U_\infty z \qquad \Phi=U_\infty x \qquad \Psi=U_\infty y \tag{10.85}$$



2) A mass source at the origin

Here we solve the Poisson equation in two dimensions for the velocity potential with a point source of area at the origin.

$$\nabla^2\Phi=Q\delta(\bar{x}) \tag{10.86}$$

where Q is the strength of the area source.Use the Green's function solution

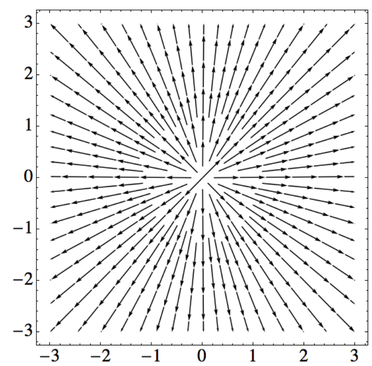
$$
\begin{aligned}
&\Phi(\bar{x}) = \frac{1}{2\pi}\int_A Q\delta(\bar{x}_s)\,Ln(|\bar{x}-\bar{x}_s|)dA = \\
&\frac{1}{2\pi}\int_0^{2\pi}\int_0^r Q\delta(r_s)\,Ln(|r-r_s|)drd\theta = \frac{Q}{2\pi}Ln(r)
\end{aligned}
\tag{10.87}
$$

The result (10.87) is the same one we derived earlier by taking the limit of a three dimensional line distribution of sources. On any circle surrounding the origin

$$
\int_0^{2\pi} U_r r d\theta = \int_0^{2\pi} \frac{Q}{2\pi r} r d\theta = Q \tag{10.88}
$$

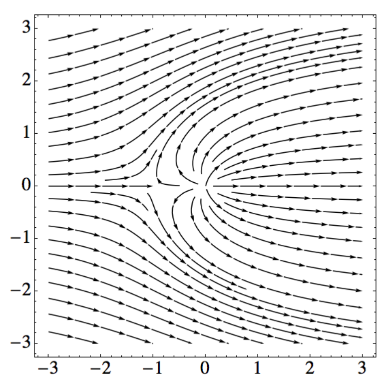
The potentials are

$$
W = \frac{Q}{2\pi}Ln(z) \qquad \Phi = \frac{Q}{2\pi}Ln(r) \qquad \Psi = \frac{Q}{2\pi}\theta \tag{10.89}
$$



3) Source at the origin plus uniform flow

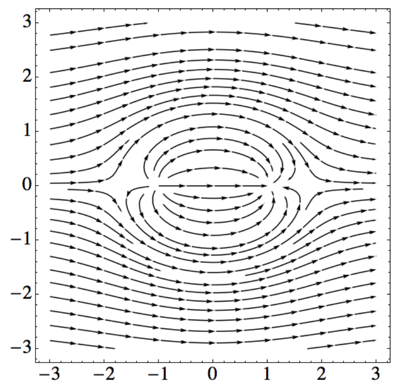
$$
W = U_\infty z + \frac{Q}{2\pi}Ln(z) \quad \Phi = U_\infty x + \frac{Q}{2\pi}Ln\left(x^2+y^2\right)^{1/2} \quad \Psi = U_\infty y + \frac{Q}{2\pi}ArcTan\left(\frac{y}{x}\right) \tag{10.90}
$$



4) Uniform flow plus a source at $x=-a$ and a sink of equal strength at $x=a$

$$\Phi = U_\infty x + \frac{Q}{2\pi} Ln\left(\left(x+a\right)^2 + y^2\right)^{1/2} - \frac{Q}{2\pi} Ln\left(\left(x-a\right)^2 + y^2\right)^{1/2} \quad (10.91)$$

$$\Psi = U_\infty y + \frac{Q}{2\pi} ArcTan\left(\frac{y}{x+a}\right) - \frac{Q}{2\pi} ArcTan\left(\frac{y}{x-a}\right) \quad (10.92)$$



5) Point vortex

Here we solve the Poisson equation for the stream function with a point source of circulation at the origin.

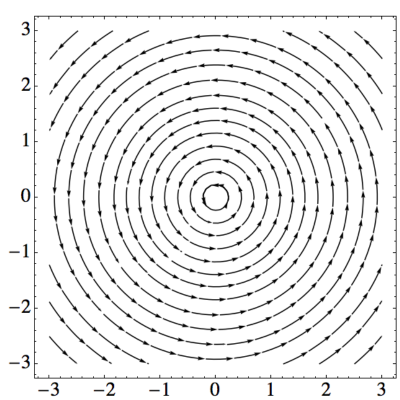
$$\nabla^2 \Psi = -\Gamma \delta(\overline{x}) \quad (10.93)$$

where Γ is the strength of the source. The Greens function solution is

$$\Psi(\overline{x}) = \frac{-1}{2\pi} \int_A \Gamma \delta(\overline{x}_s) Ln\left(\left|\overline{x} - \overline{x}_s\right|\right) dA = \frac{-1}{2\pi} \int_0^{2\pi} \int_0^r \Gamma \delta(r_s) Ln\left(\left|r - r_s\right|\right) dr d\theta = \frac{-\Gamma}{2\pi} Ln(r) \quad (10.94)$$

This is the same solution we derived earlier through a limiting process of allowing a finite vortex line become infinite. The potentials for a point vortex are

$$W = -\frac{i\Gamma}{2\pi} Ln(z) \qquad \Phi = \frac{\Gamma}{2\pi} \theta \qquad \Psi = -\frac{\Gamma}{2\pi} Ln(r) \quad (10.95)$$



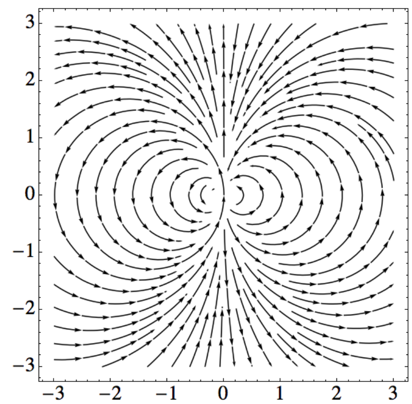
For any contour C surrounding the origin

$$\int_A \Omega \, dA = \int_A \nabla \times \bar{U} \, dA = \oint_C \bar{U} \hat{c} dC = \int_0^{2\pi} \frac{\Gamma}{2\pi r} r d\theta = \Gamma \qquad (10.96)$$

6) Vortex doublet

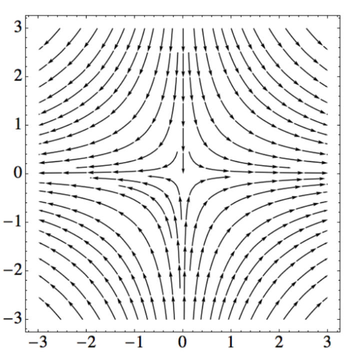
This is constructed from two point vortices of opposite circulation separated by the distance a. As they are brought together the strength $\lambda = a\Gamma$ is held constant.

$$W = \frac{\lambda}{2\pi}\left(\frac{i}{z}\right) = \frac{\lambda}{2\pi}\left(\frac{i}{r}\right)e^{-i\theta} \qquad \Phi = \frac{\lambda}{2\pi}\frac{Sin(\theta)}{r} \qquad \Psi = -\frac{\lambda}{2\pi}\frac{Cos(\theta)}{r} \qquad (10.97)$$



7) Stagnation point flow

$$W = Az^2 \qquad \Phi = A\left(x^2 - y^2\right) \qquad \Psi = 2Axy \qquad (10.98)$$

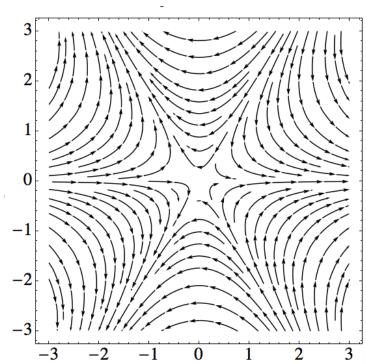


8) Flow in a corner

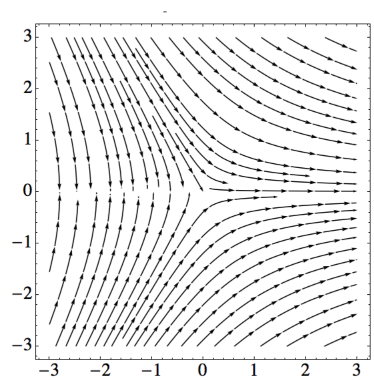
The potentials are ($n = 2$ is the stagnation point flow above).

$$W = Az^n = A\left(re^{i\theta}\right)^n \qquad \Phi = Ar^n Cos(n\theta) \qquad \Psi = Ar^n Sin(n\theta) \qquad (10.99)$$

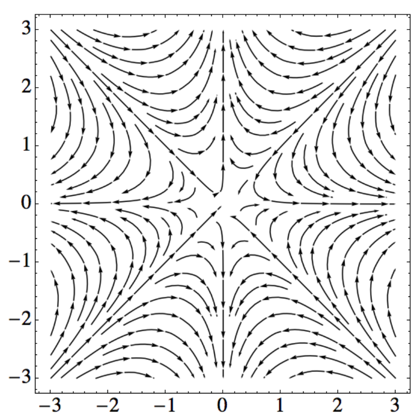
$n = 3$



$n = 3/2$



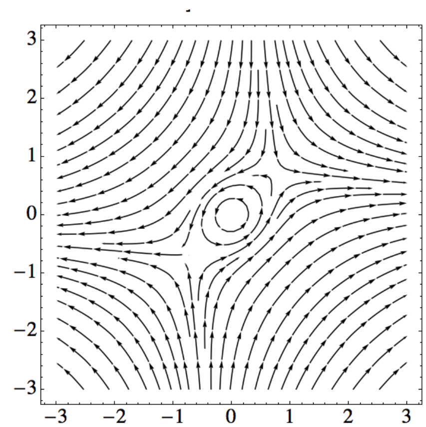
$n = 4$



9) Stagnation point flow plus vortex flow

Add together the potentials for a stagnation point flow and a point vortex.

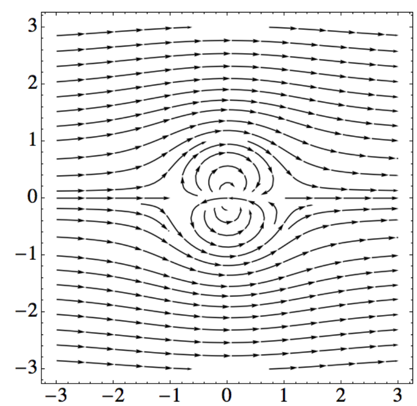
$$W = Az^2 - \frac{i\Gamma}{2\pi}Ln(z) \quad \Phi = A\left(x^2 - y^2\right) + \frac{\Gamma}{2\pi}\theta \quad \Psi = 2Axy - \frac{\Gamma}{2\pi}Ln(r) \tag{10.100}$$



10) Flow past a circular cylinder

Superpose a uniform flow with a dipole

$$W = U_\infty z + \frac{\kappa}{2\pi}\left(\frac{1}{z}\right) \quad \Phi = U_\infty x + \frac{\kappa}{2\pi}\left(\frac{x}{x^2+y^2}\right) \quad \Psi = U_\infty y + \frac{\kappa}{2\pi}\left(\frac{y}{x^2+y^2}\right) \tag{10.101}$$



The radius of the cylinder is

$$R = \left(\frac{\kappa}{2\pi U_\infty}\right)^{1/2} \tag{10.102}$$

and from the Bernoulli constant we get the pressure coefficient on the cylinder

$$\frac{P_\infty}{\rho} + \frac{1}{2}U_\infty^2 = \left(\frac{P}{\rho} + \frac{1}{2}U^2\right)_{R=\left(\frac{\kappa}{2\pi U_\infty}\right)^{1/2}} \tag{10.103}$$

$$C_p = \frac{P - P_\infty}{\frac{1}{2}\rho U_\infty^2} = \left(1 - \left(\frac{U}{U_\infty}\right)^2\right) =$$

$$\left(1 - \left(\left(U_\infty + \frac{\kappa}{2\pi}\left(\frac{1}{x^2 + y^2}\right) - \frac{\kappa}{2\pi}\left(\frac{2x^2}{\left(x^2 + y^2\right)^2}\right)\right)^2 + \left(\frac{\kappa}{2\pi}\right)^2\left(\frac{2xy}{\left(x^2 + y^2\right)^2}\right)^2\right)\right) =$$

$$C_p = \left(1 - \frac{1}{U_\infty^2}\left(\left(2U_\infty - U_\infty\left(\frac{2x^2}{R^2}\right)\right)^2 + R^4 U_\infty^2\left(\frac{2xy}{R^4}\right)^2\right)\right) = \qquad (10.104)$$

$$C_p = \left(1 - \left(4\left(1 - \left(\frac{x^2}{R^2}\right)\right)^2 + 4\left(\frac{xy}{R^2}\right)^2\right)\right) =$$

$$x = RCos(\theta), y = RSin(\theta)$$

$$C_p = 1 - 4Sin^2(\theta)$$

plotted below.

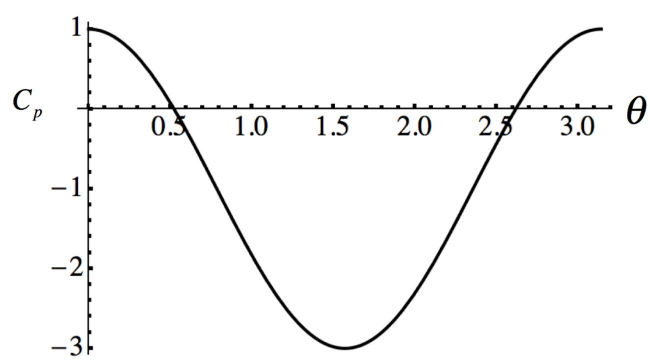


Fig 10.8 Pressure coefficient for irrotational flow past a circle.

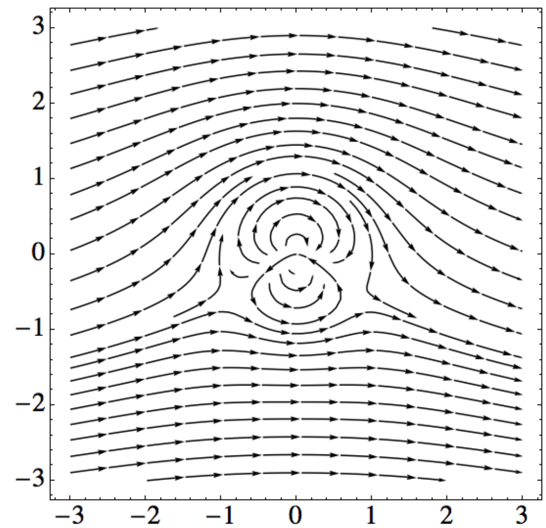
11) Superpose a uniform flow with a dipole and a vortex.

Take the circulation of the vortex to be in the clockwise direction.

$$W = U_\infty z + \frac{\kappa}{2\pi}\left(\frac{1}{z}\right) + \frac{i\Gamma}{2\pi}Ln(z)$$

$$\Phi = U_\infty x + \frac{\kappa}{2\pi}\left(\frac{x}{x^2 + y^2}\right) - \frac{\Gamma}{2\pi}\theta \qquad (10.105)$$

$$\Psi = U_\infty y + \frac{\kappa}{2\pi}\left(\frac{y}{x^2 + y^2}\right) + \frac{\Gamma}{2\pi}Ln(r)$$



Note that the radius of the circle is unchanged by the vortex but the forward and rearward stagnation points are moved symmetrically below the centerline.

10.7 Force on a rigid body translating in an inviscid fluid

10.7.1 Frames of reference

The figure below shows a rigid body translating in an inviscid incompressible fluid. The position of the body can be referenced either to coordinates fixed with respect to the body or coordinates fixed in space.

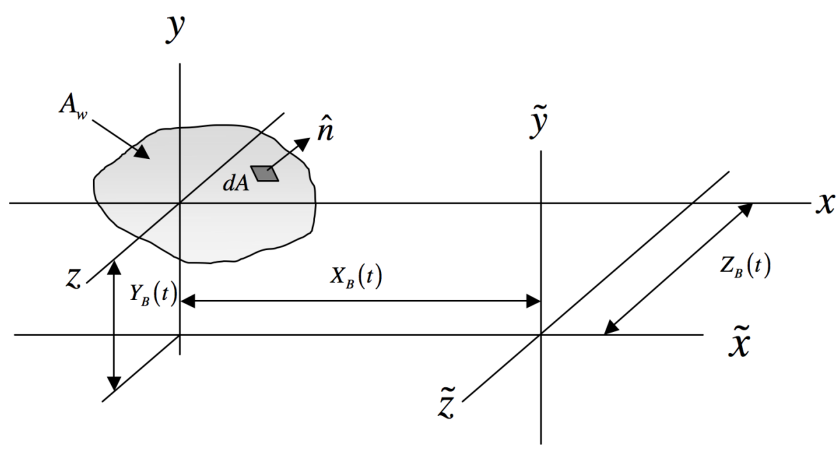


Figure 10.9 Rigid body translating in an inviscid fluid

The relationship between flow variables in the two coordinate systems is

$$
\begin{aligned}
&\tilde{x} = x + X_B(t) \\
&\tilde{y} = y + Y_B(t) \\
&\tilde{z} = z + Z_B(t) \\
&\tilde{t} = t \\
&\tilde{U}_{\tilde{x}} = U_x + \dot{X}_B(t) \\
&\tilde{U}_{\tilde{y}} = U_y + \dot{Y}_B(t) \\
&\tilde{U}_z = U_z + \dot{Z}_B(t) \\
&\frac{\tilde{P}}{\rho} = \frac{P}{\rho} - x\ddot{X}_B(t) - y\ddot{Y}_B(t) - z\ddot{Z}_B(t) \\
&\tilde{\Phi} = \Phi + x\dot{X}_B(t) + y\dot{Y}_B(t) + z\dot{Z}_B(t)
\end{aligned}
\tag{10.106}
$$

The body may be accelerating or decelerating. The transformation of pressure comes about because in a frame fixed to the body the observer would see the fluid at infinity accelerating. This acceleration would have to be produced by a uniform pressure gradient over the whole flow much like the hydrostatic pressure produced in a fluid subject to gravity. The pressure gradient transforms as

$$
\begin{aligned}
&\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial \tilde{x}} = \frac{1}{\rho}\frac{\partial P}{\partial x} - \ddot{X}_B(t) \\
&\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial \tilde{y}} = \frac{1}{\rho}\frac{\partial P}{\partial y} - \ddot{Y}_B(t) \\
&\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial \tilde{z}} = \frac{1}{\rho}\frac{\partial P}{\partial z} - \ddot{Z}_B(t)
\end{aligned}
\tag{10.107}
$$

The pressure gradient in the body-fixed frame induced by the motion of the coordinate system is $\left(\ddot{X}_B, \ddot{Y}_B, \ddot{Z}_B\right)$.

Our goal is to relate the movement of the body to the force that acts upon it in both the space-fixed and body-fixed frame of reference. The force is determined from the Bernoulli constant.

$$
\frac{\partial \Phi}{\partial t} + \frac{P}{\rho} + \frac{\bar{U} \bullet \bar{U}}{2} = f(t) \tag{10.108}
$$

To approach this problem we will use (10.106) to determine how (10.108) transforms between frames of reference. First, we need to work out the transformation of the time derivative of the potential. To transform partial derivatives one uses the perfect differential of Φ in tildaed coordinates.

$$d\tilde{\Phi} - \frac{\partial \tilde{\Phi}}{\partial \tilde{x}} d\tilde{x} - \frac{\partial \tilde{\Phi}}{\partial \tilde{y}} d\tilde{y} - \frac{\partial \tilde{\Phi}}{\partial \tilde{z}} d\tilde{z} - \frac{\partial \tilde{\Phi}}{\partial \tilde{t}} d\tilde{t} = 0 \tag{10.109}$$

Equation (10.109) is also called the contact condition on the derivatives of Φ. From (10.106) the differentials are

$$\begin{aligned}
d\tilde{\Phi} &= \left(\frac{\partial \Phi}{\partial x} + \dot{X}_B(t) \right) dx + \left(\frac{\partial \Phi}{\partial y} + \dot{Y}_B(t) \right) dy + \left(\frac{\partial \Phi}{\partial z} + \dot{Z}_B(t) \right) dz + \\
&\left(\frac{\partial \Phi}{\partial t} + x\ddot{X}_B(t) + y\ddot{Y}_B(t) + z\ddot{Z}_B(t) \right) dt \\
d\tilde{x} &= dx + \dot{X}_B(t) dt \\
d\tilde{y} &= dy + \dot{Y}_B(t) dt \\
d\tilde{z} &= dz + \dot{Z}_B(t) dt \\
d\tilde{t} &= dt
\end{aligned} \tag{10.110}$$

Substitute (10.110) into (10.109). and gather terms involving like differentials.

$$\begin{aligned}
&d\tilde{\Phi} - \frac{\partial \tilde{\Phi}}{\partial \tilde{x}} d\tilde{x} - \frac{\partial \tilde{\Phi}}{\partial \tilde{y}} d\tilde{y} - \frac{\partial \tilde{\Phi}}{\partial \tilde{z}} d\tilde{z} - \frac{\partial \tilde{\Phi}}{\partial \tilde{t}} d\tilde{t} = \\
&\left(\frac{\partial \Phi}{\partial x} + \dot{X}_B(t) - \frac{\partial \tilde{\Phi}}{\partial \tilde{x}} \right) dx + \left(\frac{\partial \Phi}{\partial y} + \dot{Y}_B(t) - \frac{\partial \tilde{\Phi}}{\partial \tilde{y}} \right) dy + \left(\frac{\partial \Phi}{\partial z} + \dot{Z}_B(t) - \frac{\partial \tilde{\Phi}}{\partial \tilde{z}} \right) dz + \\
&\left(\frac{\partial \Phi}{\partial t} + x\ddot{X}_B(t) + y\ddot{Y}_B(t) + z\ddot{Z}_B(t) - \dot{X}_B(t)\frac{\partial \tilde{\Phi}}{\partial \tilde{x}} - \dot{Y}_B(t)\frac{\partial \tilde{\Phi}}{\partial \tilde{y}} - \dot{Z}_B(t)\frac{\partial \tilde{\Phi}}{\partial \tilde{z}} - \frac{\partial \tilde{\Phi}}{\partial \tilde{t}} \right) dt = 0
\end{aligned} \tag{10.111}$$

Since the differentials $(dx, dy, dz, dt))$ are independent, the expressions in parentheses must all be zero. This leads to the following transformations of partial derivatives between tildaed and untildaed coordinates.

$$
\begin{aligned}
\frac{\partial\tilde{\Phi}}{\partial\tilde{x}} &= \frac{\partial\Phi}{\partial x} + \dot{X}_B(t) \\
\frac{\partial\tilde{\Phi}}{\partial\tilde{y}} &= \frac{\partial\Phi}{\partial y} + \dot{Y}_B(t) \\
\frac{\partial\tilde{\Phi}}{\partial\tilde{z}} &= \frac{\partial\Phi}{\partial z} + \dot{Z}_B(t) \\
\frac{\partial\tilde{\Phi}}{\partial\tilde{t}} &= \frac{\partial\Phi}{\partial t} + x\ddot{X}_B(t) + y\ddot{Y}_B(t) + z\ddot{Z}_B(t) - \dot{X}_B(t)\frac{\partial\tilde{\Phi}}{\partial\tilde{x}} - \dot{Y}_B(t)\frac{\partial\tilde{\Phi}}{\partial\tilde{y}} - \dot{Z}_B(t)\frac{\partial\tilde{\Phi}}{\partial\tilde{z}} \\
\frac{\partial\tilde{\Phi}}{\partial\tilde{t}} &= \frac{\partial\Phi}{\partial t} + x\ddot{X}_B(t) + y\ddot{Y}_B(t) + z\ddot{Z}_B(t) - \dot{X}_B(t)\frac{\partial\Phi}{\partial x} - \dot{Y}_B(t)\frac{\partial\Phi}{\partial y} - \dot{Z}_B(t)\frac{\partial\Phi}{\partial z} - \\
&\left(\dot{X}_B(t)^2 + \dot{Y}_B(t)^2 + \dot{Z}_B(t)^2\right)
\end{aligned}
\tag{10.112}
$$

Now we can transform the Bernoulli constant using (10.112) and (10.106).

$$
\begin{aligned}
&\frac{\partial\tilde{\Phi}}{\partial\tilde{t}} + \frac{\tilde{P}}{\rho} + \frac{1}{2}\left(\tilde{U}_{\tilde{x}}^2 + \tilde{U}_{\tilde{y}}^2 + \tilde{U}_{\tilde{z}}^2\right) = \\
&\frac{\partial\Phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\left(U_x^2 + U_y^2 + U_z^2\right) - \frac{1}{2}\left(\dot{X}_B(t)^2 + \dot{Y}_B(t)^2 + \dot{Z}_B(t)^2\right)
\end{aligned}
\tag{10.113}
$$

In the space-fixed frame the Bernoulli constant is equal to the pressure at infinity

$$
\frac{\partial\tilde{\Phi}}{\partial\tilde{t}} + \frac{\tilde{P}}{\rho} + \frac{1}{2}\left(\tilde{U}_{\tilde{x}}^2 + \tilde{U}_{\tilde{y}}^2 + \tilde{U}_{\tilde{z}}^2\right) = \frac{P_\infty}{\rho} \tag{10.114}
$$

An observer in the body-fixed frame will see a time dependent Bernoulli constant that includes the velocities at infinity generated by the motion of the body.

$$
\frac{\partial\Phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\left(U_x^2 + U_y^2 + U_z^2\right) = \frac{P_\infty}{\rho} + \frac{1}{2}\left(\dot{X}_B(t)^2 + \dot{Y}_B(t)^2 + \dot{Z}_B(t)^2\right) \tag{10.115}
$$

Solve (10.115) for the pressure in the body-fixed frame.

$$
\frac{P}{\rho} - \frac{P_\infty}{\rho} = \frac{1}{2}\left(\dot{X}_B(t)^2 + \dot{Y}_B(t)^2 + \dot{Z}_B(t)^2\right) - \frac{\partial\Phi}{\partial t} - \frac{1}{2}\bar{U}\bullet\bar{U} \tag{10.116}
$$

The transformation of the vector velocity is

$$
\bar{\tilde{U}} = \bar{U} + \bar{U}_B \tag{10.117}
$$

where

$$\bar{U}_B = \left\{ \dot{X}(t), \dot{Y}(t), \dot{Z}_B \right\} \tag{10.118}$$

10.7.2 BOUNDARY CONDITION ON THE BODY.

The body is defined by the function

$$G(x, y, z, t) = 0 \tag{10.119}$$

with outward normal

$$\hat{n} = \frac{\nabla F}{|F|} \tag{10.120}$$

A fluid element on the surface of the body remains on the surface except at a point of attachment or detachment. This statement can be expressed mathematically as

$$\frac{DG}{Dt} = \frac{\partial G}{\partial t} + \nabla\Phi \bullet \nabla G = 0 \quad \text{on} \quad G(x, y, z, t) = 0 \tag{10.121}$$

If the flow is inviscid then at every point on the surface of the body the component of the fluid velocity normal to the body is equal to the normal component of the body velocity. This condition can be stated as

$$\frac{\partial \Phi}{\partial n} = \nabla\Phi \bullet \hat{n} = \bar{U}_B \bullet \hat{n} \quad \text{on} \quad G(x, y, z, t) = 0 \tag{10.122}$$

where $\bar{U}_B = \left(\dot{X}_B(t), \dot{Y}_B(t), \dot{Z}_B(t) \right)$ is the velocity of the body. The surface condition can also be written as

$$\nabla\Phi \bullet \nabla G = \bar{U}_B \bullet \nabla G \tag{10.123}$$

10.7.3 RELATION BETWEEN THE FORCE ACTING ON THE BODY AND THE POTENTIAL

The force vector acting on the surface of the body is

$$\frac{\bar{F}}{\rho} = -\int_{A_w} \left(\frac{P}{\rho} - \frac{P_\infty}{\rho} \right) \hat{n}\, dA = \int_{A_w} \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} \bar{U} \bullet \bar{U} - \frac{1}{2} \left(\dot{X}_B(t)^2 + \dot{Y}_B(t)^2 + \dot{Z}_B(t)^2 \right) \right) \hat{n}\, dA \tag{10.124}$$

Since the body is not moving in the body-fixed frame

$$\int_{A_w} \frac{\partial \Phi}{\partial t} \hat{n} dA = \frac{d}{dt} \int_{A_w} \Phi \hat{n} dA \tag{10.125}$$

Also note that

$$\int_{A_w} \hat{n}\, dA = 0 \tag{10.126}$$

The force on the body (10.124) reduces to

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dA + \int_{A_w}\left(\frac{\bar{U}\bullet\bar{U}}{2}\right)\hat{n}\, dA \tag{10.127}$$

Let the velocity vector in the space-fixed coordinates be denoted $\tilde{\bar{U}} = \bar{u}$. From (10.106)

$$\bar{U} = \bar{u} - \dot{X}_B \tag{10.128}$$

Substitute (10.128) into (10.127). The force on the body (10.127) becomes

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dA + \int_{A_w}\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}\, dA - \int_{A_w}\left(\bar{U}_B(t)\bullet\bar{u}\right)\hat{n}\, dA \tag{10.129}$$

The identity $\bar{A}\times\left(\bar{B}\times C\right) = \left(\bar{A}\bullet\bar{C}\right)\bar{B} - \left(\bar{A}\bullet\bar{B}\right)\bar{C}$ can be used to rearrange the second term on the right of (10.129). Let

$$\left(\bar{U}_B\bullet\bar{u}\right)\hat{n} = \left(\bar{U}_B\bullet\hat{n}\right)\bar{u} + \bar{U}_B\times\left(\hat{n}\times\bar{u}\right) \tag{10.130}$$

Now

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dA + \int_{A_w}\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}\, dA - \int_{A_w}\left(\bar{U}_B(t)\bullet\hat{n}\right)\bar{u}\, dA - \bar{U}_B(t)\times\int_{A_w}\left(\hat{n}\times\bar{u}\right)dA \tag{10.131}$$

Consider the control volume shown below

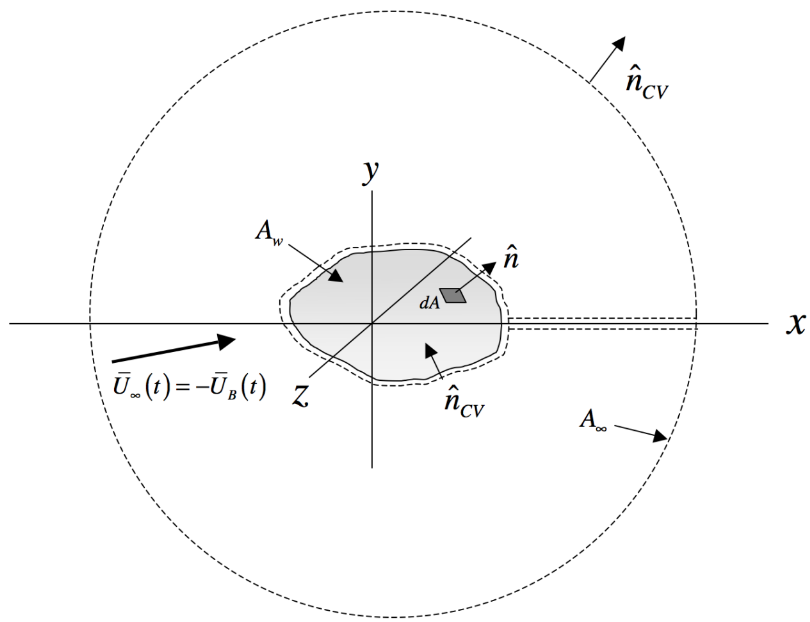


Figure 10.10 Control volume surrounding a rigid body translating in an inviscid fluid.

The surface area of the control volume comprises the surface far from the body A_∞, the surface of the solid body A_w and a narrow tube along the x-axis that makes the volume simply connected. The velocity vector coming at the body is $U_\infty(t)$. Note that on the body the outward normal of the control volume, $\hat{n}_{CV}$, points into the body and is opposite to the direction of $\hat{n}$.

Consider the second term on the right of (10.131) which is the volume integral of the kinetic energy in the space-fixed frame.

$$\int_V \nabla\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)dV = \int_A \left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}_{CV}dV = \int_{A_w}\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}_{CV}dV + \int_{A_\infty}\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}_{CV}dV \qquad (10.132)$$

The last integral in (10.132) vanishes as $r \to \infty$ because $u \sim 1/r^3$. Now

$$\begin{aligned} \int_{A_w}\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}_{CV}dV &= \int_V \nabla\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)dV \\ \int_{A_w}\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}dV &= -\int_V \nabla\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)dV = -\int_V (\bar{u}\bullet\nabla\bar{u})dV \end{aligned} \qquad (10.133)$$

where $\nabla\bullet\bar{u} = 0$ has been used.

Now consider the third term on the right of (10.131), the integral

$$\int_{A_w} \left(\bar{U}_B(t) \bullet \hat{n}\right) \bar{u}\, dA \qquad (10.134)$$

The boundary condition on the surface of the body is $\bar{U} \bullet \hat{n} = 0$. Using (10.128) this becomes

$$\bar{U} \bullet \hat{n} = \left(\bar{u} - \bar{U}_B\right) \bullet \hat{n} = 0 \Rightarrow \bar{U}_B \bullet \hat{n} = \bar{u} \bullet \hat{n} \qquad (10.135)$$

Using (10.135) the surface integral (10.134) can be expressed as

$$\int_{A_w} \left(\bar{U}_B(t) \bullet \hat{n}\right) \bar{u}\, dA = \int_{A_w} (\bar{u} \bullet \hat{n}) \bar{u}\, dA = \int_{A_w} (\bar{u}\,\bar{u}) \bullet \hat{n}\, dA \qquad (10.136)$$

Now consider the volume integral

$$\int_V \nabla \bullet (\bar{u}\,\bar{u})\, dV = \int_A (\bar{u}\,\bar{u}) \bullet \hat{n}_{CV} dA = \int_{A_w} (\bar{u}\,\bar{u}) \bullet \hat{n}_{CV} dA + \int_{A_\infty} (\bar{u}\,\bar{u}) \bullet \hat{n}_{CV} dA \qquad (10.137)$$

Again, since $\bar{u} \sim 1/r^3$ the integral on the right of (10.137) vanishes as $r \to \infty$ so

$$\int_{A_w} (\bar{u}\,\bar{u}) \bullet \hat{n} dA = -\int_V \nabla \bullet (\bar{u}\,\bar{u})\, dV \qquad (10.138)$$

where $\hat{n} = -\hat{n}_{CV}$ has been used. Expand the volume integral in (10.138) using $\nabla \bullet \bar{u} = 0$ again.

$$\int_{A_w} \left(\bar{U}_B(t) \bullet \hat{n}\right) \bar{u}\, dA = \int_{A_w} (\bar{u}\,\bar{u}) \bullet \hat{n} dA = -\int_V \nabla \bullet (\bar{u}\,\bar{u})\, dV = -\int_V \bar{u} \bullet \nabla(\bar{u})\, dV \qquad (10.139)$$

Using (10.133) and (10.139) we can state that

$$\int_{A_w} \left(\frac{\bar{u} \bullet \bar{u}}{2}\right) \hat{n}\, dA - \int_{A_w} \left(\bar{U}_B(t) \bullet \hat{n}\right) \bar{u}\, dA = -\int_V \bar{u} \bullet \nabla(\bar{u})\, dV + \int_V \bar{u} \bullet \nabla(\bar{u})\, dV = 0 \qquad (10.140)$$

The force on the body is

$$\frac{\bar{F}}{\rho} = \frac{d}{dt} \int_{A_w} \Phi \hat{n} dA - \bar{U}_B(t) \times \int_{A_w} (\hat{n} \times \bar{u})\, dA \qquad (10.141)$$

The velocity vector of the body motion is opposite to the velocity vector of the fluid approaching the body.

$$\bar{U}_B(t) = -\bar{U}_\infty(t) \qquad (10.142)$$

and so we can finally express the force on the body in the body-fixed frame.

$$\frac{\bar{F}}{\rho}=\frac{d}{dt}\int_{A_w}\Phi\hat{n}dA+\bar{U}_\infty(t)\times\int_{A_w}(\hat{n}\times\bar{u})dA \tag{10.143}$$

The second term on the right of (10.143) can be expressed in terms of the velocity in the body fixed frame using $\bar{u}=\bar{U}+\bar{U}_B=\bar{U}-\bar{U}_\infty$.

$$\begin{aligned}
\frac{\bar{F}}{\rho}&=\frac{d}{dt}\int_{A_w}\Phi\hat{n}dA+\bar{U}_\infty(t)\times\int_{A_w}\left(\hat{n}\times\left(\bar{U}-U_\infty\right)\right)dA=\\
\frac{\bar{F}}{\rho}&=\frac{d}{dt}\int_{A_w}\Phi\hat{n}dA+\bar{U}_\infty(t)\times\int_{A_w}\left(\hat{n}\times\bar{U}\right)dA-\bar{U}_\infty(t)\times\int_{A_w}(\hat{n})dA\times\bar{U}_\infty(t)\\
\frac{\bar{F}}{\rho}&=\frac{d}{dt}\int_{A_w}\Phi\hat{n}dA+\bar{U}_\infty(t)\times\int_{A_w}\left(\hat{n}\times\bar{U}\right)dA
\end{aligned} \tag{10.144}$$

where (10.126) has been used to eliminate the rightmost term in the middle relation (10.144). Finally the force on the body by the flow in the body-fixed frame is

$$\frac{\bar{F}}{\rho}=\frac{d}{dt}\int_{A_w}\Phi\hat{n}dA-\bar{U}_\infty(t)\times\int_{A_w}(\nabla\Phi\times\hat{n})dA \tag{10.145}$$

There are a couple of interesting features of this result. The first is that, if the flow is steady, the force by the flow on the body is perpendicular to the free stream velocity vector; there can be lift but no drag. The second is that if the flow is unsteady so that the first integral on the right of (10.145) is nonzero then there can be a drag force on the body.

10.8 Force on a 2-D rigid body

Let the body be a two dimensional cylinder where $A_w\to C_w$.

$$\frac{\bar{F}}{\rho(oneunitofspan)}=\frac{d}{dt}\int_{C_w}\Phi\hat{n}dC-\bar{U}_\infty(t)\times\oint_{C_w}(\nabla\Phi\times\hat{n})dC \tag{10.146}$$

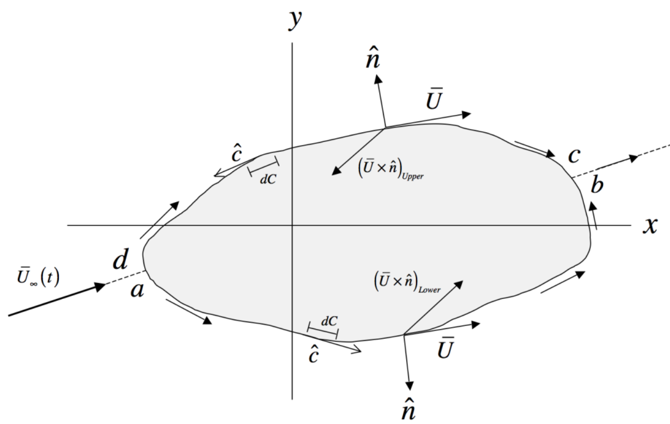


Figure 10.11 Circulation about a two-dimensional rigid body translating in an inviscid fluid.

The flow splits as it negotiates the body and so the integral breaks up into two parts

$$\frac{\bar{F}}{\rho(oneunitofspan)} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dl - \bar{U}_\infty(t)\times\left(\int_a^b (\bar{U}\times\hat{n})_{Lower}\, dl + \int_c^d (\bar{U}\times\hat{n})_{Upper}\, dl\right) \quad (10.147)$$

The surface normal vector is also normal to the surface velocity vector. The cross products on the upper and lower surfaces are

$$(\bar{U}\times\hat{n})_{Upper} = |\bar{U}|\hat{k} \quad (10.148)$$

$$(\bar{U}\times\hat{n})_{Lower} = -|\bar{U}|\hat{k} \quad (10.149)$$

Now

$$\frac{\bar{F}}{\rho(oneunitofspan)} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dl - \bar{U}_\infty(t)\times\left(\int_a^b |\bar{U}|\, dl - \int_c^d |\bar{U}|\, dl\right)\hat{k} \quad (10.150)$$

The integral of the absolute velocity in (10.150) is the same as the cyclic integral of the velocity, the circulation, where the positive direction along the contour is in the *counterclockwise* direction as shown in the figure above.

$$\begin{aligned} &\int_a^b |\bar{U}|\, dl - \int_c^d |\bar{U}|\, dl = \oint_C \bar{U}\bullet\hat{c}\, dC \\ &\Gamma(t) = \oint_C \bar{U}\bullet\hat{c}\, dC \end{aligned} \quad (10.151)$$

In two dimensions the force by the flow on the body is

$$\frac{\bar{F}}{\rho(oneunitofspan)} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dl - \bar{U}_\infty(t)\times\Gamma(t)\hat{k} \qquad (10.152)$$

This is the sought after relationship between the force that acts on a body and the circulation about the body.

10.9 VIRTUAL MASS

Let's take a look at a case where the first term in the force equation (10.141) plays an important role. Recall the solution for the flow past a sphere discussed in section 10.5. The flow, is shown again below.

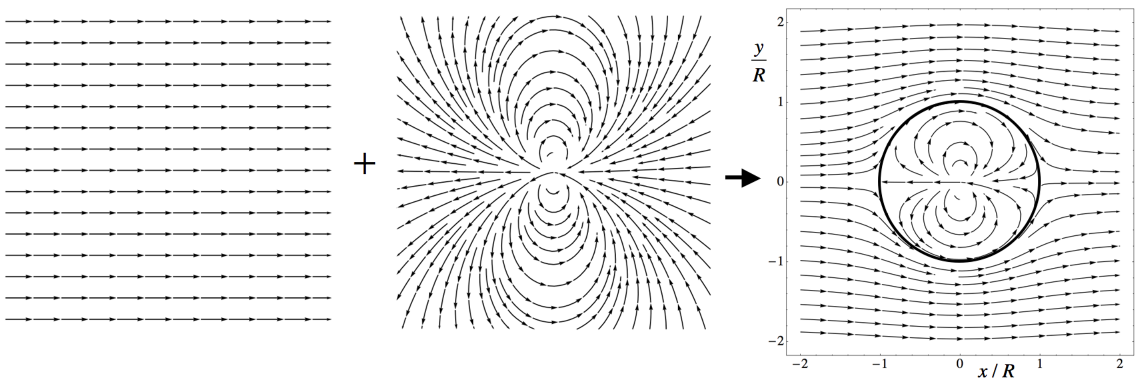


Figure 10.12 Potential flow past a sphere.

The force on the sphere is

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dA - \bar{U}_\infty(t)\times\int_{A_w}(\nabla\Phi\times\hat{n})dA \qquad (10.153)$$

There is no circulation about any contour that surrounds the sphere and so the second term in (10.153) is zero. If the flow past the sphere is steady then the first term in (10.153) is also zero. Now let the free stream velocity to be a function of time. The potential is

$$\Phi_{Sphere} = U_\infty f(t)x\left(1+\frac{(R_{Sphere})^3}{2(x^2+y^2+z^2)^{3/2}}\right) \qquad (10.154)$$

The radius of the sphere is constant which implies that the strength of the dipole in (10.72) is also proportional to the same function of time. According to (10.153) there will be a force in the x-direction on the sphere.

$$\frac{F_x}{\rho} = \frac{d}{dt}\int_{A_w} \Phi n_x dA \tag{10.155}$$

where $n_x = Cos(\theta)$. Carry out the integration in (10.155).

$$\begin{aligned}
&\frac{F_x}{\rho} = \frac{d}{dt}\int_{A_w} \Phi n_x dA = \\
&U_\infty \frac{df}{dt}\int_0^{2\pi}\int_0^{\pi} R_{Sphere} Cos(\theta)\left(1 + \frac{\left(R_{Sphere}\right)^3}{2\left(R_{Sphere}\right)^3}\right) Cos(\theta)\left(R_{Sphere}\right)^2 Sin(\theta)\, d\theta\, d\phi = \\
&U_\infty \frac{df}{dt} 2\pi\left(R_{Sphere}\right)^3
\end{aligned} \tag{10.156}$$

Now consider the rest frame of the fluid with the sphere translating to the left with velocity $\dot{X}_B(t) = -U_\infty f(t)$. The velocity potential in the space-fixed (tildaed) frame is related to the velocity potential in the body-fixed (untildaed) frame by

$$\begin{aligned}
&\tilde{\Phi}(\tilde{x},\tilde{y},\tilde{z},\tilde{t}) = \Phi(x,y,z,t) + x\dot{X}_B(t) \\
&\tilde{x} = x + X_B(t) \\
&\tilde{y} = y \\
&\tilde{z} = z \\
&\tilde{t} = t \\
&\tilde{\Phi}(\tilde{x},\tilde{y},\tilde{z},\tilde{t}) = \Phi\left(\tilde{x} - X_B(\tilde{t}),\tilde{y},\tilde{z},\tilde{t}\right) + \left(\tilde{x} - X_B(\tilde{t})\right)\dot{X}_B(\tilde{t}) = \\
&\Phi\left(\tilde{x} - X_B(\tilde{t}),\tilde{y},\tilde{z},\tilde{t}\right) - U_\infty f(\tilde{t})\left(\tilde{x} - X_B(\tilde{t})\right)
\end{aligned} \tag{10.157}$$

The force on the body in the space-fixed frame is

$$\frac{\tilde{\tilde{F}}}{\rho} = \frac{d}{dt}\int_{\tilde{A}_w} \tilde{\Phi}(\tilde{x},\tilde{y},\tilde{z},\tilde{t})\tilde{\tilde{n}} d\tilde{A} \tag{10.158}$$

Use the vector potential of the sphere flow in (10.157) and substitute into (10.158). The x-component of the force on the sphere in the space-fixed frame is

$$\begin{aligned}
&\frac{\tilde{F}_x}{\rho} = \frac{d}{dt}\int_{A_w}\left(U_\infty f(t)\left(\tilde{x} - X_B(\tilde{t})\right)\left(\frac{\left(R_{Sphere}\right)^3}{2\left(\left(\tilde{x} - X_B(\tilde{t})\right)^2 + \tilde{y}^2 + \tilde{z}^2\right)^{3/2}}\right)\right)\tilde{n}_x d\tilde{A} = \\
&\frac{U_\infty}{2}\frac{df}{dt}\int_{A_w}\left(\tilde{x} - X_B(\tilde{t})\right)\tilde{n}_x d\tilde{A}
\end{aligned} \tag{10.159}$$

Now integrate (10.159) with $\tilde{n}_x = Cos\left(\tilde{\theta}\right)$ and $\tilde{x} - X_B\left(\tilde{t}\right) = R_{Sphere}Cos\left(\tilde{\theta}\right)$.

$$\frac{\tilde{F}_x}{\rho} = \frac{U_\infty\left(R_{Sphere}\right)^3}{2}\frac{df}{dt}\int_0^\pi\int_0^{2\pi} Cos^2\left(\tilde{\theta}\right)Sin\left(\tilde{\theta}\right)d\phi d\theta = \frac{2\pi}{3}\left(R_{Sphere}\right)^3 U_\infty \frac{df}{dt} \qquad (10.160)$$

In the body-fixed frame the force on the sphere is

$$\frac{F_x}{\rho} = 2\pi\left(R_{Sphere}\right)^3 U_\infty \frac{df}{dt} \qquad (10.161)$$

Whereas, in the space-fixed frame the force on the sphere is

$$\frac{\tilde{F}_x}{\rho} = \frac{2\pi}{3}\left(R_{Sphere}\right)^3 U_\infty \frac{df}{dt} \qquad (10.162)$$

or

$$\frac{\tilde{F}_x}{\left(U_\infty \dfrac{df}{dt}\right)} = \frac{2\pi}{3}\rho\left(R_{Sphere}\right)^3 \qquad (10.163)$$

The quantity on the right of (10.163) is called the virtual mass of the sphere. It is equal to one-half of the mass of fluid displaced by the sphere.

Why the discrepancy between (10.161) and (10.162)? To understand the difference we need to go back to the transformation of the pressure and velocity potential in (10.106).

$$\begin{aligned} \frac{\tilde{P}}{\rho} &= \frac{P}{\rho} - x\ddot{X}_B\left(t\right) - y\ddot{Y}_B\left(t\right) - z\ddot{Z}_B\left(t\right) \\ \tilde{\Phi} &= \Phi + x\dot{X}_B\left(t\right) + y\dot{Y}_B\left(t\right) + z\dot{Z}_B\left(t\right) \end{aligned} \qquad (10.164)$$

In the rest frame of the fluid the pressure at infinity is a constant P_∞. Whereas in the rest frame of the sphere there is a pressure gradient in the free stream (10.107) that produces an effective buoyancy force on the sphere that has been included in (10.164) and the resultant force (10.161). But is this force real?

Consider the experiment depicted below.

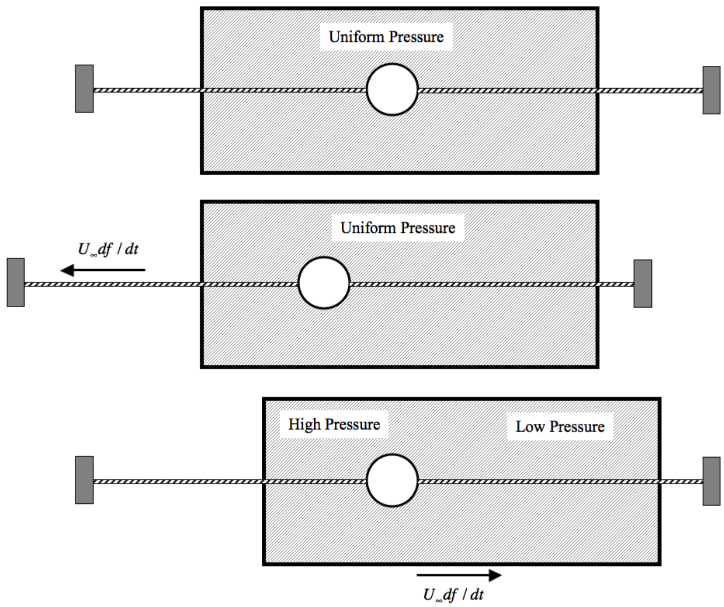


Figure 10.13 Accelerated sphere versus accelerated fluid

A sphere is inside a rigid container filled with an incompressible, inviscid fluid. A cable attached to the sphere can be used to accelerate the sphere or the sphere can be held at rest while the container and the fluid inside is accelerated. Pressure sensors on the sphere can be used to measure the force exerted by the fluid on the sphere. In the top diagram the system is all is at rest, the fluid is at pressure P_∞ and there is no net force on the sphere. In the middle figure the sphere is accelerated to the left at $\dot{U}_\infty$ and the force required to maintain the acceleration (the tension in the cable) is given by (10.162). In the bottom figure the sphere is held in place by the cable while the container and fluid is accelerated to the right at $\dot{U}_\infty$. The leftward force exerted by the cable to maintain the position of the sphere is given by (10.161).

When the whole fluid is accelerated a pressure gradient develops similar to the pressure gradient that would be developed in a gravitational field. If the sphere is detached from the cable and free to move, the buoyancy force would accelerate the sphere. If the density of the sphere is lower than the surrounding fluid then it would accelerate to the right faster than the container. If the density of the sphere is higher than the fluid then it would drift to the left relative to the container. If the sphere density is exactly the same as the fluid then the buoyancy force would balance exactly with the inertia of the sphere and the net force on the sphere would be given by (10.162)

10.10 FLOW DUE TO FORCE APPLIED TO A VISCOUS, INCOMPRESSIBLE FLUID

Now we would like to use some of the concepts from potential theory to understand the flow created by a system of forces acting on a viscous flow. Figure 10.8 depicts a localized system of forces $\overline{F}(\overline{x},t)$ acting on a viscous, incompressible fluid in a three-dimensional, unbounded region.

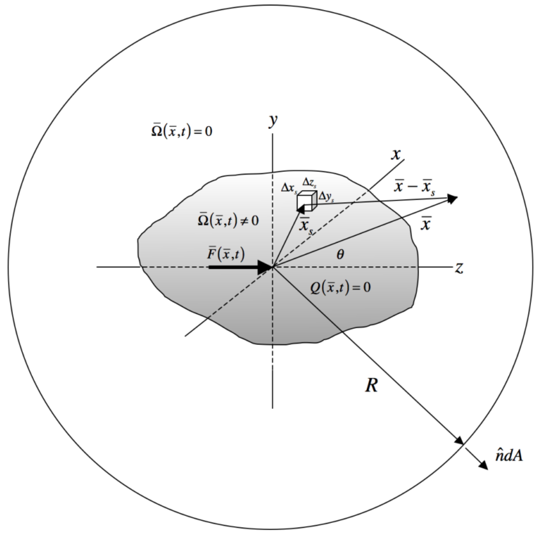


Figure 10.14 System of forces acting on an infinite viscous incompressible fluid

There are no mass sources, $Q(\bar{x},t)=0$. The forces are initiated at time $t=0$ and after a finite time has elapsed, a finite region of non-zero vorticity $\bar{\Omega}(\bar{x},t)$ has grown out from the origin through convection and diffusion. The flow may be laminar or turbulent. Beyond this region the vorticity is zero. The momentum equation for incompressible flow with an applied force is

$$\frac{\partial \bar{U}}{\partial t}+\nabla\bullet\left(\bar{U}\bar{U}\right)+\nabla\left(\frac{P}{\rho}\right)+\nu\left(\nabla\times\bar{\Omega}\right)=\frac{\bar{F}(\bar{x},t)}{\rho} \qquad (10.165)$$

Let's work out the volume integral of the momentum generated since the onset of the distribution of forces.

$$\bar{H}(t)=\int_V \bar{U}(\bar{x},t)\,dV \qquad (10.166)$$

Integrate (10.165) and use Gauss' theorem to convert volume integrals to surface integrals.

$$\frac{dH(t)}{dt}+\int_A\left(\bar{U}\bar{U}\right)\bullet\hat{n}\,dA+\int_A\left(\frac{P}{\rho}\right)\hat{n}\,dA+\nu\int\hat{n}\times\bar{\Omega}\,dA=\int_V\frac{\bar{F}(\bar{x},t)}{\rho}dV \qquad (10.167)$$

The viscous term is clearly zero since the vorticity at the surface of the control volume is zero. The only contributions to the integrated momentum come from the convective and pressure terms. To progress further we need to determine the nature of the velocity and pressure terms in the inviscid region at large radius from the sources. Use Gauss' theorem to write (10.166) as a surface integral of the vector potential.

$$\overline{H}(t)=\int_V \overline{U}(\overline{x},t)\,dV=\int_V \nabla\times\overline{A}(\overline{x},t)\,dV=\int_A \hat{n}\times\overline{A}(\overline{x},t)\,dA \qquad (10.168)$$

Over the spherical control volume shown in figure 10.8

$$\overline{H}(t)=R^2\int_0^{\pi}\int_0^{2\pi}\left(\frac{\overline{x}}{R}\right)\times\overline{A}(\overline{x},t)\,Sin(\theta)\,d\theta d\phi \qquad (10.169)$$

where the outward normal vector at the radius R is

$$\hat{n}=\frac{\overline{x}}{R}=\hat{i}Sin(\theta)Cos(\phi)+\hat{j}Sin(\theta)Sin(\phi)+\hat{k}Cos(\theta) \qquad (10.170)$$

and $(\hat{i},\hat{j},\hat{k})$ are unit vectors in the (x,y,z) respectively. Substitute into (10.169) the expression for $\overline{A}(\overline{x},t)$ from the general integral solution of the Poisson equation and exchange the order of integration.

$$\overline{H}(t)=\frac{R}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^{2\pi}\int_0^{\pi}\frac{\overline{x}}{|\overline{x}-\overline{x}_s|}Sin(\theta)\,d\theta d\phi\right)\times\overline{\Omega}(\overline{x}_s,t)\,dx_s\,dy_s\,dz_s \qquad (10.171)$$

Note that

$$\int_0^{2\pi}\int_0^{\pi}\frac{\overline{x}}{|\overline{x}-\overline{x}_s|}Sin(\theta)\,d\theta d\phi=\frac{4}{3}\pi\left(\frac{\overline{x}_s}{R}\right) \qquad (10.172)$$

This result can be demonstrated by recognizing that we can choose to rotate the coordinate axes to align the source vector $\overline{x}_s$ with the z axis, $\hat{x}_s=\hat{i}(0)+\hat{j}(0)+\hat{k}z_s$.

Now

$$\overline{H}(t)=\frac{2}{3}\overline{I}(t) \qquad (10.173)$$

where

$$\overline{I}(t)=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{x}\times\overline{\Omega}(\overline{x},t)\,dx\,dy\,dz \qquad (10.174)$$

is called the impulse of the vorticity distribution. Notice that the integral of the momentum is fully converged over just the vorticity distribution. Potential flow motions beyond the vortical region do not contribute to the total momentum of the fluid!

10.11 MULTIPOLE EXPANSION OF THE POISSON EQUATION IN THE FAR FIELD

Return to the Poisson equation solution for the vector potential.

$$\bar{A}(\bar{x},t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{\Omega}(\bar{x}_s,t)}{|\bar{x}-\bar{x}_s|}dx_s\,dy_s\,dz_s \tag{10.175}$$

We are interested in the far field disturbance caused by the vorticity distribution. To this end we can expand the Green's function in (10.175) for $\bar{x}_s << \bar{x}$. The result is

$$\lim_{\bar{x}_s/R\to 0}\bar{A}(\bar{x},t) =$$

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\Omega}(\bar{x}_s,t)\left(\frac{1}{r}+\frac{\bar{x}_s\bullet\bar{x}}{r^3}+\frac{\left(\frac{3}{2}(\bar{x}_s\bar{x}_s):\left(\bar{x}\,\bar{x}-\frac{1}{3}\bar{\bar{I}}\,r^2\right)\right)}{r^5}+O\left(\frac{1}{r^4}\right)+...\right)dx_s\,dy_s\,dz_s$$

(10.176)

where $r=\left(x^2+y^2+z^2\right)^{1/2}$. The limiting behavior of the vector potential can now be expressed as

$$\lim_{\bar{x}_s\to 0}\bar{A}(\bar{x},t)=\frac{\bar{M}_0}{4\pi r}+\frac{\bar{\bar{M}}_1\bullet\bar{x}}{4\pi r^3}+\frac{3\bar{\bar{\bar{M}}}_2:\left(\bar{x}\,\bar{x}-\frac{1}{3}\bar{\bar{I}}\,r^2\right)}{8\pi r^5}+... \tag{10.177}$$

where

$$\bar{M}_0=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\Omega}(\bar{x}_s,t)dx_s\,dy_s\,dz_s \tag{10.178}$$

$$\bar{\bar{M}}_1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\Omega}(\bar{x}_s,t)\bar{x}_s\,dx_s\,dy_s\,dz_s \tag{10.179}$$

$$\bar{\bar{\bar{M}}}_2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\Omega}(\bar{x}_s,t)(\bar{x}_s\bar{x}_s)dx_s\,dy_s\,dz_s \tag{10.180}$$

are the vector monopole, tensor dipole and third order tensor quadrapole moments of the vorticity distribution. In index notation the far field vector potential is

$$\lim_{\bar{x}_s/R\to 0}\bar{A}(\bar{x},t) =$$

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Omega_i(\bar{x}_s,t)\left(\frac{1}{r}+\frac{x_{sj}x_j}{r^3}+\frac{\left(\frac{3}{2}(x_{sj}x_{sk})\left(x_j\,x_k-\frac{\delta_{jk}}{3}r^2\right)\right)}{r^5}+O\left(\frac{1}{r^4}\right)+...\right)dx_s\,dy_s\,dz_s$$

(10.181)

where δ_{jk} is the Kronecker unit tensor. Note that the order of terms left-to-right is important. Now we can use the expansion of the vector potential to determine the integrated momentum.

$$\bar{H}(t) = R^2 \int_A \left(\frac{\bar{x}}{R}\right) \times \left(\frac{\bar{M}_0}{4\pi R} + \frac{\bar{\bar{M}}_1 \bullet \bar{x}}{4\pi R^3} + \frac{\frac{3}{2}\bar{\bar{\bar{M}}}_2 : \left(\bar{x}\,\bar{x} - \frac{1}{3}\bar{\bar{I}}\, r^2\right)}{4\pi R^5} + \ldots \right) Sin(\theta)\, d\theta d\phi \quad (10.182)$$

When the integration is carried out the monopole and quadrapole terms are zero. The dipole term integrates as follows.

$$\bar{H}(t) = \frac{1}{4\pi} \int_A \hat{n} \times \left(\bar{\bar{M}}_1 \bullet \hat{n} \right) Sin(\theta)\, d\theta d\phi \quad (10.183)$$

Recall $\hat{n} = \frac{\bar{x}}{R} = \hat{i} Sin(\theta) Cos(\phi) + \hat{j} Sin(\theta) Sin(\phi) + \hat{k} Cos(\theta)$. The dipole term can be related to the vortex impulse.

$$\bar{\bar{M}}_1 \bullet \bar{x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\Omega_i x_{sj} x_j\right) dx_s\, dy_s\, dz_s = -\frac{1}{2}\int_V \left(\bar{x} \times \left(\bar{x}_s \times \bar{\Omega}\right)\right)_i d\bar{x}_s = -\bar{x} \times \bar{I}(t) \quad (10.184)$$

Now the far-field vector potential is

$$\lim_{\bar{x}_s / R \to 0} \bar{A}(\bar{x}, t) = -\frac{\bar{x} \times \bar{I}(t)}{4\pi r^3} \quad (10.185)$$

and the volume integral of the momentum is

$$\bar{H}(t) = \frac{2}{3}\left\{I_x, I_y, I_z\right\} \quad (10.186)$$

The far field velocity from the multi-pole expansion of the vector potential is

$$\bar{U}(\bar{x}, t) = \frac{1}{4\pi}\left(\nabla \times \left(\frac{\bar{M}_0}{r}\right)\right) - \frac{1}{4\pi}\nabla \times \left(\frac{\bar{x} \times \bar{I}(t)}{r^3}\right) + \frac{3}{8\pi}\nabla \times \left(\frac{\bar{\bar{\bar{M}}}_2 : \left(\bar{x}\,\bar{x} - \frac{1}{3}\bar{\bar{I}}\,\mathrm{R}^2\right)}{r^5}\right) + \ldots$$

(10.187)

The first term in (10.187) is the velocity field generated by vortex monopoles, zero in this case. The velocity generated by the dipole term can also be written

$$\bar{U}(\bar{x},t) = -\frac{1}{4\pi}\nabla\left(\frac{\bar{x}\bullet\bar{I}(t)}{r^3}\right) \tag{10.188}$$

In the far field, the convective term in (10.165) behaves as

$$\nabla\bullet\left(\bar{U}\bar{U}\right) \sim \frac{1}{r^5} \tag{10.189}$$

As $R \to \infty$ the integral of the convective term in (10.167) goes to zero. Now

$$\frac{dH(t)}{dt} + \int_A \left(\frac{P}{\rho}\right)\hat{n}\,dA = \int_V \frac{\bar{F}(\bar{x},t)}{\rho}dV \tag{10.190}$$

At large values of r the vorticity is zero, the convective term is vanishingly small, and the equations of motion reduce to

$$\frac{\partial\bar{U}}{\partial t} + \nabla\left(\frac{P}{\rho}\right) = 0 \tag{10.191}$$

Equation (10.191) can be used with (10.188) to solve for the pressure field.

$$\nabla\left(\frac{P}{\rho} - \frac{1}{4\pi r^3}\bar{x}\bullet\frac{d\bar{I}(t)}{dt}\right) = 0 \tag{10.192}$$

and so the far field pressure is

$$\frac{P}{\rho} = \frac{1}{4\pi r^3}\bar{x}\bullet\frac{d\bar{I}(t)}{dt} \tag{10.193}$$

Now we can complete the integration of the momentum equation (10.190). The pressure integral is

$$\int_A \left(\frac{P}{\rho}\right)\hat{n}\,dA = \frac{1}{4\pi}\int_A \left(\left(\frac{\bar{x}}{r^3}\right)\bullet\frac{d\bar{I}(t)}{dt}\right)\hat{n}dA = \frac{1}{4\pi}\left(\int_A \bar{x}\,\bar{x}dA\right)\bullet\frac{d\bar{I}(t)}{dt} = \frac{1}{3}\frac{d\bar{I}(t)}{dt} \tag{10.194}$$

Finally

$$\frac{2}{3}\frac{d\bar{I}}{dt} + \frac{1}{3}\frac{d\bar{I}}{dt} = \int_V \frac{\bar{F}(\bar{x},t)}{\rho}dV \tag{10.195}$$

and

$$\frac{d\bar{I}}{dt} = \int_V \left(\frac{\bar{F}(\bar{x},t)}{\rho} \right) dV \tag{10.196}$$

which is the mechanical definition of impulse. The applied forces generate the impulse, 2/3 is contained in the directed motion of the fluid and 1/3 is removed by the pressure field at infinity that opposes the motion.

Example – A point force located at the origin pointed in the z *- direction.*

Let

$$\frac{\bar{F}(\bar{x},t)}{\rho} = \left\{ 0,0,\frac{J(t)}{\rho}\delta(x)\delta(y)\delta(z) \right\} \tag{10.197}$$

The time derivative of the impulse:

$$\frac{d\bar{I}}{dt} = \int_V \left(\frac{\bar{F}(\bar{x},t)}{\rho} \right) dV = \left\{ 0,0,\int_V \frac{J(t)}{\rho}\delta(x)\delta(y)\delta(z)dV \right\} \tag{10.198}$$

the impulse:

$$\bar{I}(t) = \left\{ 0,0,\frac{1}{\rho}\int_0^t J(t)dt \right\} \tag{10.199}$$

The pressure in the far field:

$$\frac{P}{\rho} = \frac{1}{4\pi}\left(\frac{\bar{x}}{r^3} \right) \bullet \left\{ 0,0,\frac{J(t)}{\rho}\int_V \delta(x)\delta(y)\delta(z)dV \right\} = \frac{J(t)}{4\pi\rho}\left(\frac{z}{r^3} \right) \tag{10.200}$$

and the velocity in the far field:

$$\bar{U}(\bar{x},t) = -\left(\frac{1}{4\pi\rho}\int_0^t J(t')dt' \right)\nabla\left(\frac{z}{r^3} \right) = \left(\frac{1}{4\pi\rho}\int_0^t J(t')dt' \right)\left\{ \frac{3xz}{r^5},\frac{3yz}{r^5},\frac{2z^2 - x^2 - y^2}{r^5} \right\} \tag{10.201}$$

The force creates a viscous jet with its main flow directed along the positive x-axis. The jet can be used to study the Reynolds number dependence of the starting vortex that forms when the force is turned on.

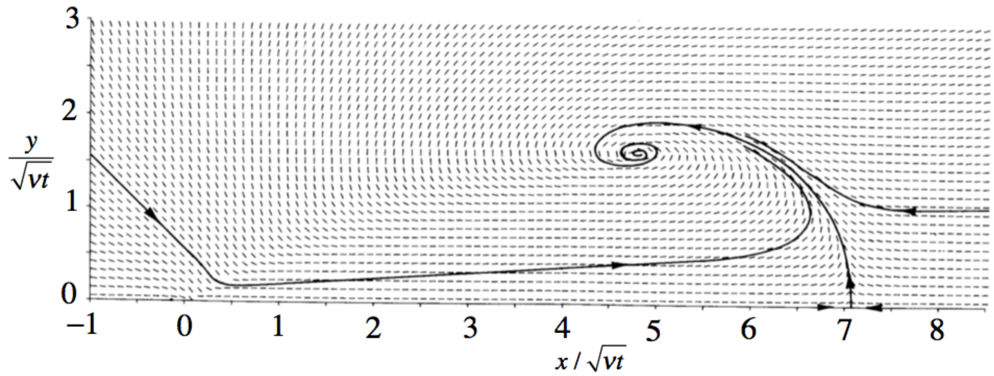


Figure 10.15 Numerical calculation of the starting vortex generated by an impulsively started viscous jet at a Reynolds number of 25.

Figure 10.15 depicts a computation of the unsteady flow generated when a point force at the origin is turned on and held constant in a viscous fluid. The figure shows the particle paths in a cut passing through the axis of the jet. The computation is carried out at a jet Reynolds number $R_e = J/(\rho\nu) = 25$.