

CHAPTER 8

ENTROPY GENERATION AND TRANSPORT

8.1 CONVECTIVE FORM OF THE GIBBS' EQUATION

In this chapter we will address two questions.

- 1) How is Gibbs equation related to the energy conservation equation?
- 2) How is the entropy of a fluid affected by its motion?

We have touched on the latter questions several times in this text but without a rigorous analysis. In this chapter the precise form of the flow properties that represent the sources of entropy will be derived from first principles.

In this endeavor we will use the Gibbs equation on a fluid element.

$$T \frac{Ds}{Dt} = \frac{De}{Dt} + P \frac{Dv}{Dt}. \quad (8.1)$$

and the energy conservation equation

$$\frac{\partial \rho(e + k)}{\partial t} + \nabla \cdot \left(\rho \bar{U} \left(e + \frac{P}{\rho} + k \right) - \bar{\tau} \cdot \bar{U} + \bar{Q} \right) - \rho \bar{G} \cdot \bar{U} = 0. \quad (8.2)$$

8.2 THE KINETIC ENERGY EQUATION

To answer these questions we first need to form a conservation equation for the kinetic energy. This can be accomplished by manipulating the momentum and continuity equations. Multiply the momentum equation by U_i (sum over i).

$$U_i \frac{\partial \rho U_i}{\partial t} + U_i \frac{\partial}{\partial x_j} (\rho U_i U_j + P \delta_{ij} - \tau_{ij}) - \rho U_i G_i = 0. \quad (8.3)$$

First let's look at the temporal and convective terms in this equation.

$$U_i \frac{\partial(\rho U_i)}{\partial t} + U_i \frac{\partial(\rho U_i U_j)}{\partial x_j} =$$

$$\underline{U_i U_i \frac{\partial \rho}{\partial t}} + \underline{U_i \rho \frac{\partial U_i}{\partial t}} + \underline{U_i U_i U_j \frac{\partial \rho}{\partial x_j}} + \underline{\rho U_i U_i \frac{\partial U_j}{\partial x_j}} + \underline{\rho U_i U_j \frac{\partial U_i}{\partial x_j}} \quad (8.4)$$

The sum of the underlined terms is zero by continuity. Thus

$$U_i \frac{\partial(\rho U_i)}{\partial t} + U_i \frac{\partial(\rho U_i U_j)}{\partial x_j} = \rho \frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_j} \quad (8.5)$$

where, $k = U_i U_i / 2$. Use continuity again in the form,

$$k \frac{\partial \rho}{\partial t} + k U_j \frac{\partial \rho}{\partial x_j} + k \rho \frac{\partial U_j}{\partial x_j} = 0. \quad (8.6)$$

Add the above two equations to get the temporal-convective part of the kinetic energy equation.

$$U_i \frac{\partial(\rho U_i)}{\partial t} + U_i \frac{\partial(\rho U_i U_j)}{\partial x_j} = \frac{\partial(\rho k)}{\partial t} + \frac{\partial}{\partial x_j}(\rho k U_j) \quad (8.7)$$

Now let's rearrange the pressure term.

$$U_i \frac{\partial(P \delta_{ij})}{\partial x_j} = \frac{\partial(P U_j)}{\partial x_j} - P \frac{\partial U_j}{\partial x_j} \quad (8.8)$$

and the viscous stress term.

$$U_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial(U_i \tau_{ij})}{\partial x_j} - \tau_{ij} \frac{\partial U_i}{\partial x_j}. \quad (8.9)$$

Finally, the kinetic energy equation is

$$\boxed{\frac{\partial(\rho k)}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j k + P U_j - U_i \tau_{ij}) = \rho U_j G_j + P \frac{\partial U_j}{\partial x_j} - \tau_{ij} \frac{\partial U_i}{\partial x_j}} \quad (8.10)$$

8.3 INTERNAL ENERGY

When the kinetic energy equation (8.10) is subtracted from the energy equation (8.2) the result is the conservation equation for internal energy.

$$\boxed{\frac{\partial \rho e}{\partial t} + \frac{\partial}{\partial x_j} (\rho U_j e + Q_j) = -P \frac{\partial U_j}{\partial x_j} + \tau_{ij} \frac{\partial U_i}{\partial x_j}}. \quad (8.11)$$

Note that both of these equations are in the general conservation form

$$\frac{\partial (\quad)}{\partial t} + \nabla \cdot (\quad) = \text{sources}. \quad (8.12)$$

The series of steps used to generate these equations are pretty standard and illustrate that the equations of motion can be (and often are) manipulated to create a conservation equation for practically any variable we wish.

The source term

$$P \frac{\partial U_j}{\partial x_j} = P \nabla \cdot \bar{U} \quad (8.13)$$

appears in both the kinetic and internal energy equations with opposite sign. It can be either positive or negative depending on whether the fluid is expanding or being compressed. Recall the continuity equation in the form

$$\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \bar{U}. \quad (8.14)$$

Multiply by the pressure

$$P \nabla \cdot \bar{U} = -\frac{PD\rho}{\rho Dt} = \rho \left(P \frac{Dv}{Dt} \right). \quad (8.15)$$

The term in parentheses we recognize as the work term in the Gibbs equation (the whole term is actually the work per second per unit volume). This term represents a reversible exchange between internal and kinetic energy due to the work of compression or expansion of a fluid element.

8.4 VISCIOUS DISSIPATION OF KINETIC ENERGY

Now let's look at the term

$$\varepsilon = \tau_{ij} \frac{\partial U_i}{\partial x_j}. \quad (8.16)$$

This needs to be worked a little further. Substitute the expression for the Newtonian stress tensor and decompose the velocity gradient into a symmetric and antisymmetric part.

$$\varepsilon = \left(2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_v \right) \delta_{ij} S_{kk} \right) (S_{ij} + W_{ij}) \quad (8.17)$$

The sum over the antisymmetric part is zero. Thus

$$\varepsilon = 2\mu(S_{ij}S_{ij}) - \frac{2}{3}\mu(S_{ii}S_{kk}) + \mu_v(S_{ii}S_{kk}). \quad (8.18)$$

We can write the first two terms as a square.

$$\varepsilon = 2\mu \left(S_{ij} - \frac{1}{3}\delta_{ij}S_{kk} \right) \left(S_{ij} - \frac{1}{3}\delta_{ij}S_{kk} \right) + \mu_v(S_{ii}S_{kk}) \quad (8.19)$$

The scalar quantity, ε , is the viscous dissipation of kinetic energy and it is clear from the fact that (8.19) is a sum of squares that it is always positive. The dissipation term appears as a sink in the kinetic energy equation and as a source in the internal energy equation. It is the irreversible conversion of kinetic energy to internal energy due to viscous friction.

We can attach two physical interpretations to Stokes' hypothesis; the assumption that $\mu_v = 0$. Recall that the mean normal stress is

$$\sigma_{mean} = (1/3)\sigma_{ii} = -P + \mu_v S_{kk}. \quad (8.20)$$

In terms of flow forces, Stokes' hypothesis implies that, in a viscous fluid, the mean normal stress is everywhere equal to the pressure.

We can give a second interpretation to Stokes' hypothesis in terms of flow energy by noting that

$$\text{trace}(S_{ij} - (1/3)\delta_{ij}S_{kk}) = 0. \quad (8.21)$$

Under Stokes' hypothesis the second term in the dissipation expression (8.19) is zero and the only contributor to kinetic energy dissipation is the anisotropic (off diagonal) part of the rate of strain tensor.

8.5 ENTROPY

We are now in a position to say something about the entropy of the system. Recall the Gibbs equation.

$$\rho \frac{De}{Dt} - \left(\frac{P}{\rho}\right) \frac{D\rho}{Dt} = \rho T \frac{Ds}{Dt} \quad (8.22)$$

Write the internal energy equation,

$$\frac{\partial \rho e}{\partial t} + \frac{\partial}{\partial x_j} (\rho U_j e + Q_j) = -P \frac{\partial U_j}{\partial x_j} + \varepsilon \quad (8.23)$$

in terms of the substantial derivative.

$$\rho \frac{De}{Dt} + P \frac{\partial U_j}{\partial x_j} = -\frac{\partial Q_j}{\partial x_j} + \varepsilon. \quad (8.24)$$

We have already written down the continuity equation in the form required to replace the second term on the left hand side.

$$\left(\frac{P}{\rho}\right) \frac{D\rho}{Dt} = -P \frac{\partial U_j}{\partial x_j} \quad (8.25)$$

The energy equation now becomes

$$\rho \frac{De}{Dt} - \left(\frac{P}{\rho}\right) \frac{D\rho}{Dt} = -\frac{\partial Q_j}{\partial x_j} + \varepsilon. \quad (8.26)$$

Comparing this form of the energy equation with the Gibbs equation, we see that the left-hand-side corresponds to the entropy term

$$\rho T \frac{Ds}{Dt} = -\frac{\partial Q_j}{\partial x_j} + \varepsilon. \quad (8.27)$$

Let's put this in the form of a conservation equation for the entropy. The approach is to use the continuity equation yet again!

$$\begin{aligned}\rho \frac{\partial s}{\partial t} + \rho U_j \frac{\partial s}{\partial x_j} &= -\frac{1}{T} \frac{\partial Q_j}{\partial x_j} + \frac{\varepsilon}{T} \\ s \frac{\partial \rho}{\partial t} + s U_j \frac{\partial \rho}{\partial x_j} + s \rho \frac{\partial U_j}{\partial x_j} &= 0\end{aligned}\tag{8.28}$$

Add the two to get

$$\frac{\partial \rho s}{\partial t} + \frac{\partial}{\partial x_j} (\rho U_j s) = -\frac{1}{T} \frac{\partial Q_j}{\partial x_j} + \frac{\varepsilon}{T}.\tag{8.29}$$

The heat flux term can be rearranged into a divergence term and a squared term.

$$\frac{\partial}{\partial x_j} \left(\frac{Q_j}{T} \right) = \frac{1}{T} \frac{\partial Q_j}{\partial x_j} - \frac{Q_j}{T^2} \frac{\partial T}{\partial x_j}\tag{8.30}$$

For a linear conducting material, the heat flux is

$$Q_j = -k \frac{\partial T}{\partial x_j}.\tag{8.31}$$

Let

$$\boxed{Y = \frac{k}{T} \left(\frac{\partial T}{\partial x_j} \frac{\partial T}{\partial x_j} \right)}.\tag{8.32}$$

Finally, the conservation equation for the entropy becomes

$$\boxed{\frac{\partial \rho s}{\partial t} + \frac{\partial}{\partial x_j} \left(\rho U_j s - \frac{k}{T} \frac{\partial T}{\partial x_j} \right) = \frac{Y + \varepsilon}{T}}.\tag{8.33}$$

The integral form of the entropy transport equation on a general control volume is

$$\frac{D}{Dt} \int_{V(t)} \rho s dV + \int_{A(t)} \left(\rho (U_j - U_{Aj}) s - \frac{k}{T} \frac{\partial T}{\partial x_j} \right) n_j dA = \int_{V(t)} \left(\frac{Y + \varepsilon}{T} \right) dV\tag{8.34}$$

The right hand side of (8.33) is the entropy source term and is always positive. This remarkable result provides an explicit expression (in terms of squared temperature gradients and squared velocity gradients) for the irreversible changes in entropy that occur in a compressible flow. Notice that the body force term contributes nothing to the entropy.

Consider a closed adiabatic box containing a viscous, heat conducting fluid subject to the no slip condition on the walls. The fluid is stirred by a fan that is then turned off and the flow is allowed to settle.

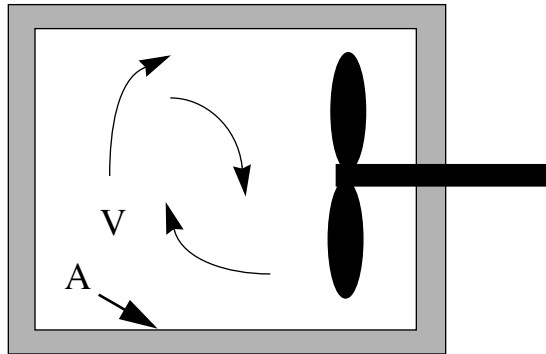


Figure 8.1 Fluid stirred by a fan in an adiabatic box.

The entropy of the fluid in the box behaves according to

$$\frac{d}{dt} \int_V \rho s dV + \int_A \left(\rho U_j s - \frac{k}{T} \frac{\partial T}{\partial x_j} \right) n_j dA = \int_V \left(\frac{\gamma + \varepsilon}{T} \right) dV . \quad (8.35)$$

The surface integral is zero by the no-slip condition and the assumption of adiabaticity. The volume integral of the density times the intensive entropy is the extensive entropy.

$$S = \int_V \rho s dV \quad (8.36)$$

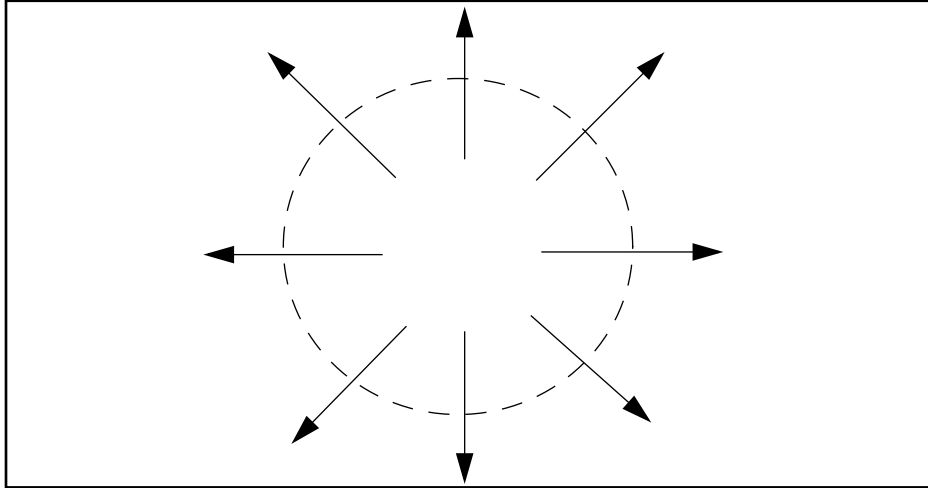
Thus as the closed, adiabatic, system comes to rest, the entropy continues to increase until all the gradients in velocity and temperature have vanished.

$$\frac{dS}{dt} = \int_V \left(\frac{\gamma + \varepsilon}{T} \right) dV > 0 \quad (8.37)$$

8.6 PROBLEMS

Problem 1 - Show that (8.18) can be expressed in the form (8.19).

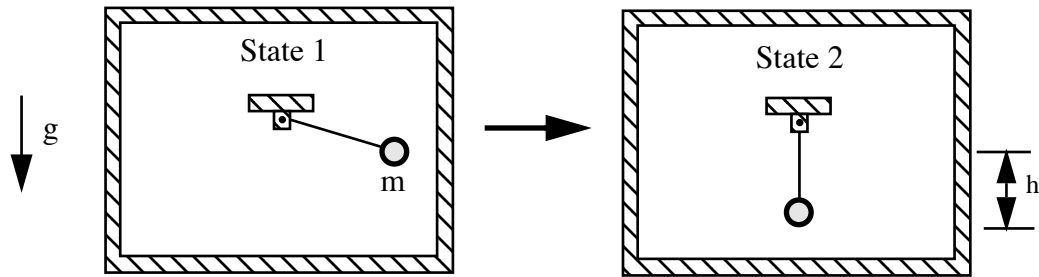
Problem 2 - Imagine a spherically symmetric flow in the absence of shear.



Let the fluid velocity in the radial direction be $f(r)$. Work out the expression for the kinetic energy dissipation in terms of f and the two viscosities μ , μ_v . Suppose the fluid is a monatomic gas for which $\mu_v = 0$. Does there exist a function f for which the dissipation is zero? Refer to Appendix 2 for the dissipation function in spherical polar coordinates.

Problem 3 - Suppose the fluid in the box shown in Section 7.5 is Air initially at 300K and one atmosphere. Let the tip velocity of the propeller be 50 m/sec and the typical boundary layer thickness over the surface of the propeller be 1 millimeter. The surface area of the propeller is 1 m^2 and the volume of the box is 1 m^3 . Roughly estimate the rate at which the temperature of the air in the box increases due to viscous dissipation.

Problem 4 - A simple pendulum of mass $m = 1.0\text{ kilogram}$ is placed inside a rigid adiabatic container filled with 0.1 kilograms of Helium gas initially at rest (state 1). The gas pressure is one atmosphere and the temperature is 300°K . The acceleration due to gravity is $g = 9.8\text{ meters/sec}^2$.



At $t = 0$ the pendulum is released from an initial displacement height $h = 0.1$ meters and begins to oscillate and stir the gas. Eventually the pendulum and gas all come to rest (state 2).

- 1) Assume all the potential energy of the pendulum is converted to internal energy of the gas. What is the change in temperature of the Helium in going from state 1 to state 2?
- 2) What is the change in entropy per unit mass of the Helium in going from state 1 to state 2?

