CHAPTER 7

SEVERAL FORMS OF THE EQUATIONS OF MOTION

7.1 THE NAVIER-STOKES EQUATIONS

Under the assumption of a Newtonian stress-rate-of-strain constitutive equation and a linear, thermally conductive medium, the equations of motion for compressible flow become the famous Navier-Stokes equations. In Cartesian coordinates,

$$\boxed{\begin{array}{c} \dfrac{\partial\rho}{\partial t} + \dfrac{\partial}{\partial x_i}(\rho U_i) = 0 \\[2em] \dfrac{\partial\rho U_i}{\partial t} + \dfrac{\partial}{\partial x_j}(\rho U_i U_j + P\delta_{ij}) - \rho G_i - \\[1em] \dfrac{\partial}{\partial x_j}(2\mu S_{ij} - ((2/3)\mu - \mu_v)\delta_{ij}S_{kk}) = 0 \\[2em] \dfrac{\partial\rho(e+k)}{\partial t} + \dfrac{\partial}{\partial x_i}(\rho U_i h_t - \kappa(\partial T/\partial x_i)) - \rho G_i U_i - \\[1em] \dfrac{\partial}{\partial x_i}(2\mu U_j S_{ij} - ((2/3)\mu - \mu_v)\delta_{ij} U_j S_{kk}) = 0 \end{array}} \tag{7.1}$$

The Navier-Stokes equations are the foundation of the science of fluid mechanics. With the inclusion of an equation of state, virtually all flow solving revolves around finding solutions of the Navier-Stokes equations. Most exceptions involve fluids where the relation between stress and rate-of-strain is nonlinear such as polymers, or where the equation of state is not very well understood (for example supersonic flow in water) or rarefied flows where the Boltzmann equation must be used to explicitly account for particle collisions.

The equations can take on many forms depending on what approximations or assumptions may be appropriate to a given flow. In addition, transforming the equations to different forms may enable one to gain insight into the nature of the solutions. It is essential to learn the many different forms of the equations and to become practiced in the manipulations used to transform them.

7.1.1 INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

If there are no body forces and the flow is incompressible, $\nabla \bullet \overline{U} = 0$ the Navier-Stokes equations reduce to what is probably their most familiar form.

$$\frac{\partial U_j}{\partial x_j} = 0$$

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial P}{\partial x_i} - \left(\frac{\mu}{\rho}\right)\frac{\partial^2 U_i}{\partial x_j \partial x_j} = 0 \tag{7.2}$$

$$\frac{\partial T}{\partial t} + U_j \frac{\partial T}{\partial x_j} - \left(\frac{\kappa}{\rho C}\right)\frac{\partial^2 T}{\partial x_j \partial x_j} - \frac{1}{2}\left(\frac{\mu}{\rho C}\right)\left(\frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j}\right)\left(\frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j}\right) = 0$$

where the internal energy is assumed to be $e = CT$ and C is the heat capacity of the material. The equation of state in this case is just $\rho = constant$. Notice that for incompressible flow the continuity and momentum equations are completely decoupled from the energy equation and can be solved separately. Once the velocity is known the energy equation can be solved for the temperature. The last term in the energy equation is always positive and represents a source of internal energy due to dissipation of kinetic energy by viscous friction. This will be discussed in much more detail in Chapter 7.

7.2 THE MOMENTUM EQUATION EXPRESSED IN TERMS OF VORTICITY

If the transport coefficients, μ and μ_v are assumed to be constant (a reasonable assumption if the Mach number is not too large), the compressible momentum equation can be written,

$$\frac{\partial\rho\bar{U}}{\partial t} + \nabla\bullet(\rho\bar{U}\bar{U}) + \nabla P + \rho\nabla\Psi - \mu\nabla^2\bar{U} - \left(\frac{1}{3}\mu + \mu_v\right)\nabla(\nabla\bullet\bar{U}) = 0 \tag{7.3}$$

where the body force is the gradient of a potential function $\bar{G} = -\nabla\Psi$. The vorticity is defined as the curl of the velocity.

$$\bar{\Omega} = \nabla\times\bar{U}. \tag{7.4}$$

If we use the vector identities

$$\begin{aligned} \bar{U}\bullet\nabla\bar{U} &= (\nabla\times\bar{U})\times\bar{U} + \nabla\left(\frac{\bar{U}\bullet\bar{U}}{2}\right) \\ \nabla\times(\nabla\times\bar{U}) &= \nabla(\nabla\bullet\bar{U}) - \nabla^2\bar{U} \end{aligned} \tag{7.5}$$

together with the continuity equation, the momentum equation can be written in the form,

$$\begin{aligned} &\rho\frac{\partial\bar{U}}{\partial t} + \rho(\bar{\Omega}\times\bar{U}) + \rho\nabla\left(\frac{\bar{U}\bullet\bar{U}}{2}\right) + \nabla P + \rho\nabla\Psi - \\ &\left(\frac{4}{3}\mu + \mu_v\right)\nabla(\nabla\bullet\bar{U}) + \mu\nabla\times\bar{\Omega} = 0 \end{aligned} . \tag{7.6}$$

If the flow is irrotational $\bar{\Omega} = 0$ (7.6) reduces to

$$\rho\frac{\partial\bar{U}}{\partial t} + \rho\nabla\left(\frac{\bar{U}\bullet\bar{U}}{2}\right) + \nabla P + \rho\nabla\Psi - \left(\frac{4}{3}\mu + \mu_v\right)\nabla(\nabla\bullet\bar{U}) = 0 \tag{7.7}$$

According to (7.7) viscous forces play very little role in momentum transport when the flow is irrotational. If the flow is compressible the effect of viscosity is only through a term that depends on $\nabla\bullet\bar{U}$ which is usually relatively small in flows at moderate Mach number. Remarkably, if the flow is incompressible the viscous term disappears altogether and viscosity plays no role whatsoever in momentum transport. Viscous stresses do act in the fluid but they generate no net momentum transfer in an irrotational flow. An example would be the flow over a wing outside the boundary layer near the surface of the wing and outside the viscous wake.

7.3 The momentum equation expressed in terms of the entropy and vorticity

In Chapter 2 we noted that when an equation of state is combined with Gibbs equation any thermodynamic variable can be expressed in terms of any two others. In an unsteady, three-dimensional flow of a continuous medium

$$s = s(h(x, y, z, t), P(x, y, z, t)). \tag{7.8}$$

Taking the gradient of (7.8) leads to

$$T\nabla s = \nabla h - \frac{\nabla P}{\rho} \quad . \tag{7.9}$$

which we derived in Chapter 2.

Equation (7.9) can be used to replace the gradient of the pressure in (7.6). Thus the momentum equation in terms of the entropy is

$$\begin{aligned}&\frac{\partial \bar{U}}{\partial t} + (\bar{\Omega} \times \bar{U}) + \nabla\left(h + \frac{U^2}{2} + \Psi\right) - T\nabla s - \\ &\left(\frac{1}{\rho}\right)\left(\frac{4}{3}\mu + \mu_v\right)\nabla(\nabla \bullet \bar{U}) + \left(\frac{\mu}{\rho}\right)\nabla \times \bar{\Omega} = 0\end{aligned} \quad . \tag{7.10}$$

7.3.1 Crocco's theorem

The inviscid form of the above equation with $\mu = 0, \mu_v = 0$ is

$$\frac{\partial \bar{U}}{\partial t} + (\bar{\Omega} \times \bar{U}) + \nabla\left(h + \frac{U^2}{2} + \Psi\right) - T\nabla s = 0, \tag{7.11}$$

Equation (7.11) is called *Crocco's theorem* and demonstrates the close relationship between vorticity and the gradient of the entropy in a compressible flow.

The relation

$$\bar{\Omega} \times \bar{U} = 0 \tag{7.12}$$

is satisfied if $\bar{U} = 0$ (a trivial situation), if the vorticity and velocity are parallel (this is called a Beltrami flow) or if the flow is irrotational, $\nabla \times \bar{U} = 0$.

7.3.2 The energy equation for inviscid, non-heat conducting flow

With the viscosities and thermal conductivity, κ, set to zero, the energy equation in (7.1) becomes

$$\frac{\partial\rho(e+k)}{\partial t}+\frac{\partial}{\partial x_i}(\rho U_i h_t)-\rho G_i U_i = 0. \tag{7.13}$$

If we are going to assume that the fluid is inviscid then it is consistent to also assume that the heat conductivity is zero since the underlying molecular collision mechanisms for both fluid properties are fundamentally the same in a gas. In Chapter 7 we will come to recognize that assuming the gas is inviscid and non-heat-conducting is equivalent to assuming that the entropy of a fluid element cannot change.

In terms of the stagnation enthalpy $h_t = h+k$ (7.13) can be written as

$$\frac{\partial\rho h_t}{\partial t}+\frac{\partial}{\partial x_i}(\rho U_i h_t)-\rho G_i U_i = \frac{\partial P}{\partial t}. \tag{7.14}$$

Use the continuity equation in (7.14) and $G_i = -\partial\Psi/\partial x_i$ to produce

$$\frac{\partial h_t}{\partial t}+U_i\frac{\partial}{\partial x_i}(h_t+\Psi) = \frac{1}{\rho}\frac{\partial P}{\partial t} \tag{7.15}$$

We developed the viscous, heat-conducting version of (7.15) previously in Chapter 5.

7.3.3 Steady flow

If the flow is steady, the inviscid energy equation (7.15) reduces to

$$U_i\frac{\partial}{\partial x_i}\left(h+\frac{U^2}{2}+\Psi\right) = \bar{U}\bullet\nabla\left(h+\frac{U^2}{2}+\Psi\right) = 0 \tag{7.16}$$

The quantity

$$h+\frac{U^2}{2}+\Psi = h_t+\Psi \tag{7.17}$$

is called the steady *Bernoulli integral* or Bernoulli function. In a general, inviscid, non-heat-conducting, steady flow the energy equation reduces to the statement (7.16) that the velocity field is normal to the gradient of the Bernoulli function.

In the absence of body forces, $Dh_t/Dt = 0$ in such a flow. The flow is necessarily adiabatic and the stagnation enthalpy of a fluid element is preserved. If the flow is fed from a reservoir where the enthalpy is everywhere uniform then, since h_t is preserved for each fluid element, the enthalpy remains uniform and $\nabla h_t = 0$ everywhere. In this case the momentum equation (Crocco's theorem) reduces to

$$(\bar{\Omega} \times \bar{U}) = T\nabla s. \tag{7.18}$$

If the flow is steady, inviscid and irrotational then both the stagnation enthalpy and the entropy are constant everywhere. A flow where $\nabla s = 0$ is called *homentropic*. In this instance both the momentum equation and energy equations reduce to the same result

$$h + \frac{U^2}{2} + \Psi = Constant \tag{7.19}$$

7.4 INVISCID, IRROTATIONAL, HOMENTROPIC FLOW

It is instructive to examine this case further. Again, assume there are no body forces $\Psi = 0$, set $\mu = \mu_v = 0$, $\kappa = 0$ in the energy equation and $\nabla \times \bar{U} = 0$ in the momentum equation (7.6). The governing equations become

$$\begin{gathered}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho U_k) = 0 \\
\rho \frac{\partial U_i}{\partial t} + \rho \frac{\partial}{\partial x_i}\left(\frac{U_k U_k}{2}\right) + \frac{\partial P}{\partial x_i} = 0. \\
\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}
\end{gathered} \tag{7.20}$$

In this approximation the energy equation reduces to the isentropic relation between pressure and density. These equations are the starting point for the development of inviscid flow theory as well as the theory for the propagation of sound.

7.4.1 Steady, inviscid, irrotational flow

If the flow is steady the equations (7.20) reduce to

$$\nabla \bullet (\rho \overline{U}) = 0$$
$$\nabla\left(\frac{\overline{U} \bullet \overline{U}}{2}\right) + \frac{\nabla P}{\rho} = 0 \quad . \tag{7.21}$$
$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

Take the gradient of the isentropic relation. The result is

$$\nabla P = a^2 \nabla \rho . \tag{7.22}$$

For an ideal gas

$$\nabla\left(\frac{P}{\rho}\right) = \frac{\nabla P}{\rho} - \frac{P}{\rho^2}\nabla\rho = \left(\frac{\gamma - 1}{\gamma}\right)\frac{\nabla P}{\rho} \tag{7.23}$$

where (7.22) is used. Using (7.23) the momentum equation now becomes

$$\nabla\left(\left(\frac{\gamma}{\gamma - 1}\right)\frac{P}{\rho} + \frac{\overline{U} \bullet \overline{U}}{2}\right) = 0 \tag{7.24}$$

The term in parentheses in (7.24) is the stagnation enthalpy. For this class of flows the momentum equation reduces to $\nabla h_t = 0$ and the entropy is the same everywhere as just noted in the previous section.

Substitute (7.22) into the continuity equation.

$$\overline{U} \bullet \frac{\nabla P}{\rho} + a^2 \nabla \bullet \overline{U} = 0 \tag{7.25}$$

Using (7.23) the continuity equation becomes

$$\overline{U} \bullet \nabla a^2 + (\gamma - 1)a^2 \nabla \bullet \overline{U} = 0 . \tag{7.26}$$

Using (7.24) we can write

$$\left(\frac{a^2}{\gamma - 1}\right) = h_t - \frac{\bar{U} \bullet \bar{U}}{2} \tag{7.27}$$

The continuity equation now becomes

$$(\gamma - 1)\left(h_t - \frac{\bar{U} \bullet \bar{U}}{2}\right) \nabla \bullet \bar{U} - \bar{U} \bullet \nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right) = 0 \tag{7.28}$$

The equations governing steady, inviscid, irrotational motion reduce to a single equation for the velocity vector $\bar{U}$.

7.5 THE VELOCITY POTENTIAL

The condition $\nabla \times \bar{U} = 0$ implies that the velocity can be expressed in terms of a scalar potential.

$$\bar{U} = \nabla \Phi . \tag{7.29}$$

Substitute (7.29) into (7.28). The result is the full potential equation for steady, irrotational flow.

$$(\gamma - 1)\left(h_t - \frac{\nabla\Phi \bullet \nabla\Phi}{2}\right) \nabla^2 \Phi - \nabla\Phi \bullet \nabla\left(\frac{\nabla\Phi \bullet \nabla\Phi}{2}\right) = 0 \tag{7.30}$$

The range of flows where (7.30) applies includes subsonic flow over bodies at Mach numbers below the critical Mach number at which shocks begin to form and supersonic flows that involve smooth expansion and compression such as the flow in a nozzle without shocks.

7.5.1 UNSTEADY POTENTIAL FLOW, THE UNSTEADY BERNOULLI INTEGRAL

The irrotational, unsteady momentum equation is, from (7.10),

$$\frac{\partial \bar{U}}{\partial t} + \nabla\left(h + \frac{U^2}{2} + \Psi\right) - T\nabla s - \left(\frac{1}{\rho}\right)\left(\frac{4}{3}\mu + \mu_v\right) \nabla(\nabla \bullet \bar{U}) = 0 . \tag{7.31}$$

Insert (7.29) into (7.31). The equation becomes

$$\nabla\left(\frac{\partial\Phi}{\partial t} + h + \frac{U^2}{2} + \Psi\right) - T\nabla s - \frac{1}{\rho}\left(\frac{4}{3}\mu + \mu_v\right)\nabla(\nabla^2\Phi) = 0 \quad . \tag{7.32}$$

If the flow is inviscid,

$$\nabla\left(\frac{\partial\Phi}{\partial t} + h + \frac{U^2}{2} + \Psi\right) - T\nabla s = 0 \,. \tag{7.33}$$

If the flow is inviscid and homentropic (homogeneously isentropic) with $\nabla s = 0$, the unsteady momentum equation reduces to

$$\nabla\left(\frac{\partial\Phi}{\partial t} + h + \frac{U^2}{2} + \Psi\right) = 0 \,. \tag{7.34}$$

The quantity

$$\boxed{\frac{\partial\Phi}{\partial t} + h + \frac{U^2}{2} + \Psi = F(t)} \tag{7.35}$$

which can be at most a function of time is called the *unsteady Bernoulli integral* and is widely applied in the analysis of unsteady, inviscid, compressible flows.

If the flow is a calorically perfect gas with $h = C_p T$, then (7.35) can be written

$$\frac{\partial\Phi}{\partial t} + \frac{\gamma}{\gamma - 1}\frac{P}{\rho} + \frac{U^2}{2} + \Psi = F(t) \tag{7.36}$$

Generally $F(t)$ can be taken to be a constant independent of time. The sort of unusual situation where it could be time dependent might occur in a closed wind tunnel where the pressure throughout the system was forced to change due to some overall volume change of the tunnel. Such changes can occur but they are usually negligible unless there is some intention to change the volume for a particular purpose, perhaps to exert some form of flow control.

In summary, the equations of inviscid, homentropic flow in terms of the velocity potential are

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} + \frac{\nabla\Phi \bullet \nabla\rho}{\rho} + \nabla^2\Phi = 0$$

$$\nabla\left(\frac{\partial\Phi}{\partial t} + \frac{\gamma}{\gamma-1}\frac{P}{\rho} + \frac{\nabla\Phi \bullet \nabla\Phi}{2} + \Psi\right) = 0 . \qquad (7.37)$$

$$\frac{P}{P_{ref}} = \left(\frac{\rho}{\rho_{ref}}\right)^{\gamma}$$

Note that the isentropic relation provides the needed equation to close the system in lieu of the energy equation. Equation (7.30) is the steady version of (7.37) all boiled down to one equation for the potential. We can reduce (7.37) to a single equation for the velocity potential as follows.

$$\frac{\partial\Phi}{\partial t} + \frac{\gamma}{\gamma-1}\frac{P}{\rho} + \frac{\nabla\Phi \bullet \nabla\Phi}{2} + \Psi = F(t) \qquad (7.38)$$

Replace the pressure in (7.38) with the density using the isentropic relation.

$$\rho = \left(\frac{(\rho_{ref})^{\gamma}}{P_{ref}}\left(\frac{\gamma-1}{\gamma}\right)\left(F(t) - \frac{\partial\Phi}{\partial t} - \frac{\nabla\Phi \bullet \nabla\Phi}{2} - \Psi\right)\right)^{\frac{1}{\gamma-1}}$$

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} = \frac{1}{\gamma-1}\left(\frac{dF(t)}{dt} - \Phi_{tt} - \nabla\Phi_t \bullet \nabla\Phi\right)\left(F(t) - \frac{\partial\Phi}{\partial t} - \frac{\nabla\Phi \bullet \nabla\Phi}{2} - \Psi\right)^{-1} \qquad (7.39)$$

$$\frac{1}{\rho}\nabla\rho = \frac{1}{\gamma-1}\left(-\nabla\Phi_t - \nabla\left(\frac{\nabla\Phi \bullet \nabla\Phi}{2}\right)\right)\left(F(t) - \frac{\partial\Phi}{\partial t} - \frac{\nabla\Phi \bullet \nabla\Phi}{2} - \Psi\right)^{-1}$$

where the body force potential is assumed to be only a function of space. Substitute (7.39) into the continuity equation. The full unsteady potential equation becomes

$$\left(\frac{dF(t)}{dt} - \Phi_{tt} - \nabla\Phi_t \bullet \nabla\Phi\right) + (\gamma-1)\left(F(t) - \frac{\partial\Phi}{\partial t} - \frac{\nabla\Phi \bullet \nabla\Phi}{2} - \Psi\right)\nabla^2\Phi -$$
$$\nabla\Phi \bullet \left(\nabla\Phi_t + \nabla\left(\frac{\nabla\Phi \bullet \nabla\Phi}{2}\right)\right) = 0 \qquad (7.40)$$

7.5.2 INCOMPRESSIBLE, IRROTATIONAL FLOW

If the flow is incompressible, $\nabla \bullet U = 0$, and irrotational, $\bar{\Omega} = 0$, the momentum equation in the absence of gravity $\Psi = 0$ reduces to,

$$\frac{\partial \bar{U}}{\partial t} + \nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right) + \nabla\left(\frac{P}{\rho}\right) = 0. \tag{7.41}$$

Note again that viscous forces, though present, have no effect on the momentum in an incompressible, irrotational flow. Since $\nabla \times \bar{U} = 0$ we can write $\bar{U} = \nabla\phi$ and in terms of the velocity potential, the momentum equation for incompressible, irrotational flow becomes

$$\nabla\left(\frac{\partial \Phi}{\partial t} + \frac{U^2}{2} + \frac{P}{\rho}\right) = 0. \tag{7.42}$$

The quantity in parentheses is the incompressible form of the Bernoulli integral and, as in the compressible case, is at most a function of time.

$$\frac{\partial \Phi}{\partial t} + \frac{U^2}{2} + \frac{P}{\rho} = F(t). \tag{7.43}$$

An incompressible, irrotational flow (steady or unsteady) is governed by Laplace's equation.

$$\nabla \bullet \bar{U} = \nabla^2 \Phi = 0 \tag{7.44}$$

which can be solved using a wide variety of well established techniques. Once the velocity potential Φ is known, the velocity field is generated by differentiation (7.29) and the pressure field is determined from the Bernoulli integral (7.43).

7.6 THE VORTICITY EQUATION

Earlier we cast the momentum equation in terms of the vorticity. Now let's derive a conservation equation for the vorticity itself. If we take the curl of the momentum equation the result is

$$\begin{aligned} \nabla \times \Bigg(\frac{\partial \bar{U}}{\partial t} + (\bar{\Omega} \times \bar{U}) + \nabla\left(h + \frac{U^2}{2} + \Psi\right) - T\nabla s - \\ \left(\frac{1}{\rho}\right)\left(\frac{4}{3}\mu + \mu_v\right)\nabla(\nabla \bullet \bar{U}) + \left(\frac{\mu}{\rho}\right)\nabla \times \bar{\Omega}\Bigg) = 0 \end{aligned} \tag{7.45}$$

or

$$\frac{\partial\bar{\Omega}}{\partial t} + \nabla\times(\bar{\Omega}\times\bar{U}) - \nabla\times(T\nabla s) - \left(\frac{4}{3}\mu + \mu_v\right)\nabla\times\left(\left(\frac{1}{\rho}\right)\nabla(\nabla\bullet\bar{U})\right) + \mu\nabla\times\left(\left(\frac{1}{\rho}\right)\nabla\times\bar{\Omega}\right) = 0 \quad . \tag{7.46}$$

Using vector identities, (7.46) can be rearranged to read

$$\frac{\partial\bar{\Omega}}{\partial t} + \bar{U}\bullet\nabla\bar{\Omega} = (\bar{\Omega}\bullet\nabla)\bar{U} - \bar{\Omega}\nabla\bullet\bar{U} + \nabla T\times\nabla s + \left(\frac{4}{3}\mu + \mu_v\right)\nabla\times\left(\left(\frac{1}{\rho}\right)\nabla(\nabla\bullet\bar{U})\right) - \mu\nabla\times\left(\left(\frac{1}{\rho}\right)\nabla\times\bar{\Omega}\right) \quad . \tag{7.47}$$

For inviscid, homentropic flow, $\nabla s = 0, \mu = 0, \mu_v = 0$, the vorticity equation reduces to

$$\frac{D\bar{\Omega}}{Dt} = \frac{\partial\bar{\Omega}}{\partial t} + \bar{U}\bullet\nabla\bar{\Omega} = (\bar{\Omega}\bullet\nabla)\bar{U} - \bar{\Omega}(\nabla\bullet\bar{U})\,. \tag{7.48}$$

Equation (7.48) is interpreted to mean that, in an inviscid fluid, if the flow is initially irrotational it will always remain irrotational. Conversely, if the flow has vorticity to begin with, then that vorticity can be convected or amplified through stretching or volume change but cannot dissappear.

If the flow is incompressible and viscous, the vorticity equation is

$$\frac{D\bar{\Omega}}{Dt} = \frac{\partial\bar{\Omega}}{\partial t} + \bar{U}\bullet\nabla\bar{\Omega} = (\bar{\Omega}\bullet\nabla)\bar{U} + \frac{\mu}{\rho}\nabla^2\bar{\Omega}\,. \tag{7.49}$$

For planar, two-dimensional flow, $(\bar{\Omega}\bullet\nabla)\bar{U} = 0$ and there is only one non-zero component of the vorticity, Ω_z which satisfies the diffusion equation

$$\frac{D\Omega_z}{Dt} = \left(\frac{\mu}{\rho}\right)\left(\frac{\partial^2\Omega_z}{\partial x^2} + \frac{\partial^2\Omega_z}{\partial y^2}\right)\,. \tag{7.50}$$

This equation is of the same form as the incompressible equation for the transport of the temperature. See the energy equation in (7.2) with the viscous dissipation term neglected.

$$\frac{DT}{Dt} = \frac{\kappa}{\rho}\nabla^2 T \tag{7.51}$$

In two dimensions vorticity diffuses like a scalar such as temperature or concentration. But note that vorticity can be positive or negative and where mixing between opposite signs occurs, vorticity can disappear through diffusion. In three-dimensional flow, the vorticity can be amplified or reduced due to vortex stretching or compression arising from the nonlinear term $\bar{\Omega} \bullet \nabla\bar{U}$.

7.7 FLUID FLOW IN THREE DIMENSIONS, THE DUAL STREAM FUNCTION

The equations for particle paths in a three-dimensional, steady fluid flow are

$$\frac{dx}{dt} = U(x, y, z)\ ; \qquad \frac{dy}{dt} = V(x, y, z)\ ; \qquad \frac{dz}{dt} = W(x, y, z)\,. \tag{7.52}$$

The particle paths are determined by integrating (7.52)

$$x = f(\tilde{x}, \tilde{y}, \tilde{z}, t)\ ; \quad y = g(\tilde{x}, \tilde{y}, \tilde{z}, t)\ ; \quad z = h(\tilde{x}, \tilde{y}, \tilde{z}, t) \tag{7.53}$$

where $(\tilde{x}, \tilde{y}, \tilde{z})$ is the vector coordinate of a particle at $t = 0$. Elimination of t among these three relations leads to *two* infinite families of integral surfaces; the dual streamfunction surfaces

$$\psi^1 = \Psi^1(x, y, z)\ ; \qquad \psi^2 = \Psi^2(x, y, z)\,. \tag{7.54}$$

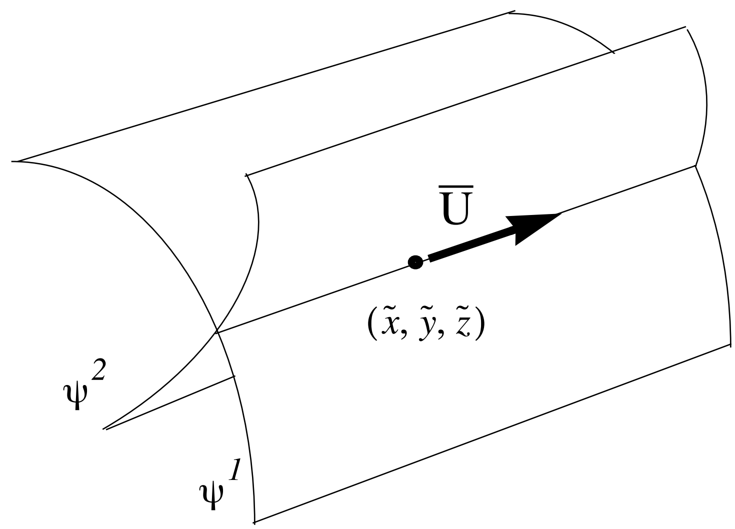
The total differential of the stream function is

$$d\psi = \frac{\partial\Psi}{\partial x}dx + \frac{\partial\Psi}{\partial y}dy + \frac{\partial\Psi}{\partial z}dz \tag{7.55}$$

Use (7.52) to replace the differentials in (7.55) and note that on a stream surface $d\psi = 0$. The streamfunctions (7.54) are each integrals of the first order PDE

$$\bar{U} \cdot \nabla\Psi^j = U\frac{\partial\Psi^j}{\partial x} + V\frac{\partial\Psi^j}{\partial y} + W\frac{\partial\Psi^j}{\partial z} = 0\ ; \qquad j = 1, 2\,. \tag{7.56}$$

A given initial point, $(\tilde{x}, \tilde{y}, \tilde{z})$, defines two streamsurfaces. The velocity vector through the point lies along the intersection of the surfaces as implied by (7.56) and shown below .



Given the dual streamfunctions, the velocity field can be reconstructed from

$$\overline{U} = \nabla \Psi^1 \times \nabla \Psi^2 \tag{7.57}$$

7.8 THE VECTOR POTENTIAL

The velocity field of an incompressible flow can be represented as the curl of a vector potential, $\overline{A}$ also called the vector stream function.

$$\overline{U} = \nabla \times \overline{A} . \tag{7.58}$$

Use of the vector potential in (7.58) guarantees that the incompressibility condition $\nabla \bullet \overline{U} = 0$ is satisfied. In two-dimensional flow only the out-of-plane component of $\overline{A}$ is nonzero and this corresponds to the stream function discussed in Chapter 1. Take the curl of (7.58).

$$\nabla \times \overline{U} = \nabla \times \nabla \times \overline{A} \tag{7.59}$$

Now use the vector identity

$$\nabla(\nabla \bullet \overline{A}) - \nabla^2 \overline{A} = \nabla \times (\nabla \times \overline{A}) \tag{7.60}$$

Equation (7.59) becomes

$$\overline{\Omega} = \nabla(\nabla \bullet \overline{A}) - \nabla^2 \overline{A} \tag{7.61}$$

7.8.1 SELECTION OF A COULOMB GAUGE

There is a certain arbitrariness to the vector potential and we can take advantage of this to simplify (7.61). Suppose we add the gradient of a scalar function to the vector potential. Let

$$\bar{A}' = \bar{A} + \nabla f \tag{7.62}$$

Since $\nabla \times \nabla f \equiv 0$ the velocity field generated by $\bar{A}'$ or $\bar{A}$ is identically the same.

$$\nabla \times \bar{A}' = \nabla \times \bar{A} \tag{7.63}$$

Imagine that we are given $\bar{A}'$. We can always find a function f such that

$$\nabla \bullet \bar{A}' = \nabla^2 f \tag{7.64}$$

which implies that

$$\nabla \bullet \bar{A} = 0 \tag{7.65}$$

This is called choosing a *Coloumb* gauge for $\bar{A}$. In effect we are free to impose the condition (7.65) on $\bar{A}$ without affecting the velocity field generated from $\bar{A}$ when we take the curl.

Using (7.65) in (7.61) generates the Poisson equation for the vector potential.

$$\nabla^2 \bar{A} = -\bar{\Omega}. \tag{7.66}$$

In order for (7.66) to be useful one has to be given the vorticity field $\bar{\Omega}$. The vector potential is then determined using standard techniques for solving equations of Poisson type. The velocity field is then generated by taking the curl.

The vector potential is directly related to the dual streamfunctions discussed in the previous section.

$$\bar{A} = \Psi^1 \nabla \Psi^2 = -\Psi^2 \nabla \Psi^1 . \tag{7.67}$$

7.9 INCOMPRESSIBLE FLOW WITH MASS AND VORTICITY SOURCES

Consider an incompressible flow constructed from a distribution of mass sources, $Q(\boldsymbol{x}, t)$ and distributed sources of vorticity, $\overline{\Omega}(\boldsymbol{x}, t)$. The velocity field is constructed from a linear superposition of two fields.

$$\overline{U} = \overline{U}_{sources} + \overline{U}_{vortices}. \tag{7.68}$$

The velocity field generated by the mass sources is irrotational and that generated by the vorticity sources is divergence free. The continuity equation for such a flow now has a source term

$$\nabla \bullet \overline{U} = \nabla \bullet \overline{U}_{sources} = Q(\boldsymbol{x}, t) \tag{7.69}$$

and the curl of the velocity is

$$\nabla \times \overline{U} = \nabla \times \overline{U}_{vortices} = \overline{\Omega}(\boldsymbol{x}, t) \tag{7.70}$$

In terms of the potentials the velocity field is

$$\overline{U} = \nabla\Phi + \nabla \times \overline{A}. \tag{7.71}$$

The potentials satisfy the system of Poisson equations

$$\left.\begin{aligned} \nabla^2 \Phi &= Q(\boldsymbol{x}, t) \\ \nabla^2 \overline{A} &= -\overline{\Omega}(\boldsymbol{x}, t) \end{aligned}\right\}. \tag{7.72}$$

This approach allows one to construct fairly complex flow fields that can be rotational while retaining the simplicity of working in terms of potentials governed by linear equations (7.72) and the associated law of superposition (7.68). Notice that the theory of potential flow is exactly analogous to the theory of potentials in electricity and magnetism. The mass sources coincide with the distribution of electric charges and the vorticity coincides with the electric currents.

7.10 TURBULENT FLOW

Turbulent flows are characterized by unsteady eddying motions with a wide range of scales superimposed on a slowly changing or steady mean flow. The eddies mix fluid rapidly and are responsible for much higher rates of mass, momentum and heat transport than would occur if the flow was laminar. The primary difficulty with analyzing or numerically computing turbulent flows stems from the fact that, at high Reynolds number, the the range of scales involved can be very large with velocity gradients at the finest scale, where most kinetic energy dissipation occurs, of order $\sqrt{R_e}$ larger than the gradients at the largest scale that drive the flow. Computations designed to fully simulate the flow must be resolved on such a fine grid that often only a few simulations can be carried out even at moderate Reynolds number. Very high Reynolds numbers are still beyond today's largest computers. Experiments designed to measure a turbulent flow are faced with a similar challenge requiring extremely small measurement volumes as well as the ability to measure over a large field comparable to the overall size of the flow.

Recall the equations of motion, repeated here for convenience

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i) = 0 \\
&\frac{\partial \rho U_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_i U_j + P\delta_{ij}) - \rho G_i - \\
&\frac{\partial}{\partial x_j}(2\mu S_{ij} - ((2/3)\mu - \mu_v)\delta_{ij}S_{kk}) = 0 \\
&\frac{\partial \rho(e+k)}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i h_t - \kappa(\partial T/\partial x_i)) - \rho G_i U_i - \\
&\frac{\partial}{\partial x_i}(2\mu U_j S_{ij} - ((2/3)\mu - \mu_v)\delta_{ij}U_j S_{kk}) = 0
\end{aligned}
\tag{7.73}
$$

In this section we will discuss an alternative to fully resolving the unsteady flow introduced by Osborne Reynolds in 1895 in which each flow variable is decomposed into a mean and fluctuating part. When the equations of motion are suitably averaged, the result is a new set of equations that relate the mean flow to the correlated part of the fluctuating motion.

Imagine an ensemble of realizations of the time dependent flow that all satisfy the same set of initial and boundary conditions, perhaps in the form of a series of laboratory experiments repeated over and over. Decompose each flow variable into an ensemble mean and a fluctuation

$$Q(x, y, z, t) = \bar{Q}(x, y, z, t) + Q'(x, y, z, t) \tag{7.74}$$

where Q is a generic variable. The mean is defined by

$$\bar{Q} = \frac{1}{N}\sum_{n=1}^{N} Q_n . \tag{7.75}$$

where the number of realizations of the flow is N and the index refers to the nth member of the ensemble. All members of the ensemble are aligned to have the same time origin and evolve with the same clock which is reset each time a new realization is initiated.

Apply the decomposition and averaging to a linear term in the equations of motion such as

$$\overline{\frac{\partial}{\partial x_j}(P\delta_{ij})} = \overline{\frac{\partial}{\partial x_j}((\bar{P}+P')\delta_{ij})} = \frac{\partial}{\partial x_j}(\bar{P}\delta_{ij}) + \overline{\frac{\partial}{\partial x_j}(P'\delta_{ij})} = \frac{\partial}{\partial x_j}(\bar{P}\delta_{ij}) \tag{7.76}$$

Any term that is linear in the fluctuation is zero. Nonlinear terms are more complex. The decomposition of the density and velocity is

$$\begin{aligned} \rho &= \bar{\rho} + \rho' \\ U &= \bar{U} + U' \\ V &= \bar{V} + V' \\ W &= \bar{W} + W' \end{aligned} \tag{7.77}$$

Consider averaging the momentum flux term ρUV .

$$\begin{gathered} \overline{\rho UV} = \overline{(\bar{\rho}+\rho')(\bar{U}+U')(\bar{V}+V')} = \\ \bar{\rho}\bar{U}\bar{V} + \bar{\rho'}\bar{U}\bar{V} + \bar{\rho}\bar{U'}\bar{V} + \bar{\rho}\bar{U}\bar{V'} + \overline{\rho'U'}\bar{V} + \overline{\rho'V'}\bar{U} + \bar{\rho}\overline{U'V'} + \overline{\rho'U'V'} \end{gathered} \tag{7.78}$$

As in (7.76), averages of terms that are linear in the fluctuations are zero by the definition of the mean,

$$\overline{\rho UV} = \bar{\rho}\bar{U}\bar{V} + (\overline{\rho' U'}\bar{V} + \overline{\rho' V'}\bar{U} + \bar{\rho}\overline{U'V'} + \overline{\rho' U'V'}) = \bar{\rho}\bar{U}\bar{V} + \tau_{xy}\big|_{turbulent} \tag{7.79}$$

The terms in parentheses can be thought of as effective stresses due to the background fluctuations. Similar terms arise when generating the turbulent normal stresses in compressible flow

$$\begin{aligned}\overline{\rho UU} &= \bar{\rho}\bar{U}\bar{U} + (2\overline{\rho' U'}\bar{U} + \bar{\rho}\overline{U'U'} + \overline{\rho' U'U'}) = \bar{\rho}\bar{U}\bar{U} + \tau_{xx}\big|_{turbulent} \\ \overline{\rho VV} &= \bar{\rho}\bar{V}\bar{V} + (2\overline{\rho' V'}\bar{V} + \bar{\rho}\overline{V'V'} + \overline{\rho' V'V'}) = \bar{\rho}\bar{V}\bar{V} + \tau_{yy}\big|_{turbulent}\end{aligned} \tag{7.80}$$

Time dependent terms are ensemble averaged in the same way.

$$\begin{aligned}\overline{\frac{\partial\rho}{\partial t}} &= \frac{1}{N}\sum_{n=1}^{N}\left.\frac{\partial\rho}{\partial t}\right|_n = \frac{\partial\bar{\rho}}{\partial t} + \overline{\frac{\partial\rho'}{\partial t}} = \frac{\partial\bar{\rho}}{\partial t} \\ \overline{\frac{\partial\rho U_i}{\partial t}} &= \frac{1}{N}\sum_{n=1}^{N}\left.\frac{\partial\rho U_i}{\partial t}\right|_n = \frac{\partial\bar{\rho}\bar{U}_i}{\partial t} + \overline{\frac{\partial\rho'\bar{U}_i}{\partial t}} + \overline{\frac{\partial\bar{\rho}U'_i}{\partial t}} + \overline{\frac{\partial\rho' U'_i}{\partial t}} = \frac{\partial\bar{\rho}\bar{U}_i}{\partial t} + \frac{\partial\overline{\rho' U'_i}}{\partial t}\end{aligned} \tag{7.81}$$

Since fluid properties such as μ, μ_v and κ are temperature dependent they must also be included in the decomposition and averaging process.

7.10.1 TURBULENT INCOMPRESSIBLE, ISOTHERMAL FLOW

If the flow is incompressible, the density is constant. If the temperature is also constant the equations of motion simplify to

$$\begin{aligned}&\frac{\partial U_j}{\partial x_j} = 0 \\ &\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_j}{\partial x_j} + \frac{1}{\rho}\frac{\partial P}{\partial x_i} - \left(\frac{\mu}{\rho}\right)\frac{\partial^2 U_i}{\partial x_j \partial x_j} = 0\end{aligned} \tag{7.82}$$

Substitute the Reynolds decomposition (7.74) into (7.82) and ensemble average the equations.

$$\frac{\partial \bar{U}_j}{\partial x_j} = 0$$

$$\frac{\partial \bar{U}_i}{\partial t} + \frac{\partial \bar{U}_i \bar{U}_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \left(\frac{\mu}{\rho}\right)\frac{\partial^2 \bar{U}_i}{\partial x_j \partial x_j} - \frac{\partial \overline{U'_i U'_j}}{\partial x_j} \tag{7.83}$$

In incompressible flow the only nonlinear term is the convective term and the only term involving the fluctuations is the last term on the right which can be viewed as an effective turbulent stress term, the Reynolds stress.

$$\tau_{ij}\Big|_{turbulent} = \rho \overline{U'_i U'_j} \tag{7.84}$$

The advantage of this approach is that the entire unsteady motion does not need to be resolved, only the correlations (7.84) are needed. The fundamental difficulty is that the Reynolds stresses constitute an additional set of unknowns and the equations (7.83) are unclosed. The field of turbulence modeling is basically all about searching for model equations that can be used to relate the Reynolds stresses to the mean flow so the system (7.83) can be closed.

In free shear flows the laminar stress term in (7.83), $\mu\partial^2 \bar{U}_i / \partial x_j \partial x_j$ is generally much smaller than the Reynolds stress term and some simplification can be achieved by neglecting the viscous term in the momentum equation altogether. In wall bounded flows this cannot be done. The Reynolds stresses tend to dominate the flow away from the wall but near the wall the velocity fluctuations are damped to zero and the viscous stress term is responsible for the friction on the wall. We will have more to say about the flow near a wall in Chapter 8.

7.11 PROBLEMS

Problem 1 - Derive equation (7.6) beginning with the Navier-Stokes equations. Do the same for equation (7.47).

Problem 2 - Show that for homentropic flow of an ideal gas $\nabla p = a^2 \nabla \rho$ where a is the local speed of sound.