CHAPTER 4

CONTROL VOLUMES, VECTOR CALCULUS

4.1 CONTROL VOLUME DEFINITION

The idea of the control volume is an extremely general concept used widely in fluid mechanics. In Chapter 1 we derived the equations for conservation of mass and momentum on a small cubic control volume fixed in space. In this chapter we will provide a general definition of the control volume and review some of the powerful mathematical tools of vector calculus used in conjunction with control volumes. This will enable us to re-derive in Chapter 5 the equations of motion on a general, arbitrarily moving, control volume suitable for a wide variety of applications.

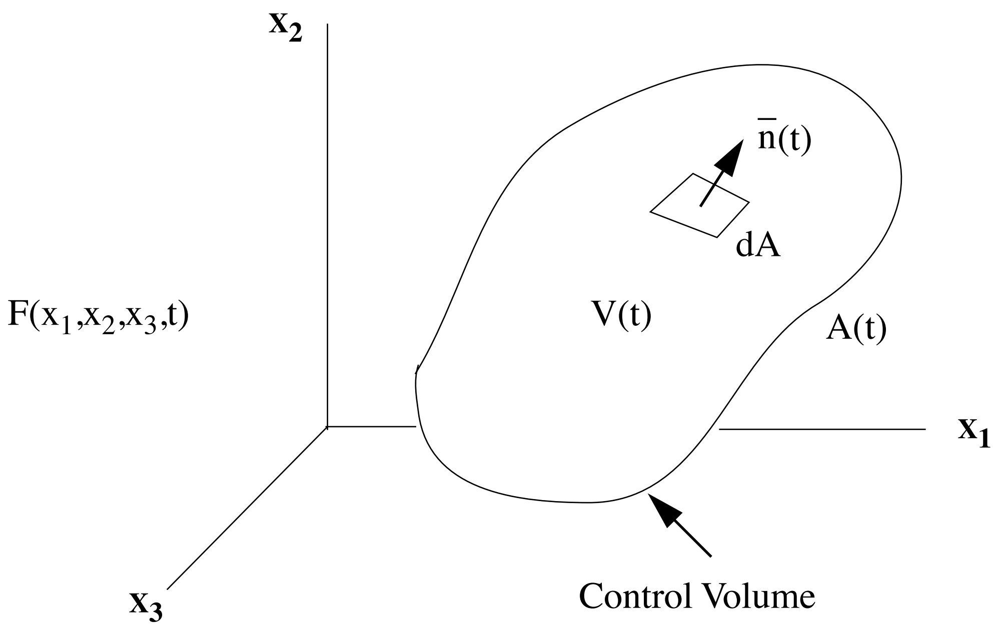

Figure 4.1   Control volume schematic

The control volume $V(t)$ is a closed, simply connected region in space which may be finite or infinitesimal in size. The size and geometry of the volume is selected to provide a convenient imaginary vessel for systematically accounting for the fluxes of various flow quantities related to the conservation of mass, momentum and energy. Part or all of the surface of the control volume may be moving and the motion can be arbitrary as long as the control volume is not torn apart.

$A(t)$ is the instantaneous surface and dA is an infinitesimal surface element.

$\bar{n}(t)$ is a unit vector normal to the surface that points outward from the interior of the control volume.

Let $F(x_1, x_2, x_3, t)$ be any field variable. It could be a scalar or a component of a vector or tensor.

4.2 VECTOR CALCULUS

The gradient operator is

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x_i} \tag{4.1}$$

where the subscript i on the right refers to the index of the vector component. As noted in Chapter 1 we will use this index notation to denote vectors and tensors throughout the text.

The gradient of a scalar function of space is written

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \frac{\partial F}{\partial x_i} \tag{4.2}$$

and the gradient of a vector function of space is

$$\nabla \bar{F} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}, \frac{\partial F_1}{\partial x_2}, \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1}, \frac{\partial F_2}{\partial x_2}, \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1}, \frac{\partial F_3}{\partial x_2}, \frac{\partial F_3}{\partial x_3} \end{pmatrix} = \frac{\partial F_i}{\partial x_j}. \tag{4.3}$$

The divergence of a vector is

$$\nabla \bullet \bar{F} = trace(\nabla \bar{F}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} \frac{\partial F_i}{\partial x_j} = \delta_{ij} \frac{\partial F_i}{\partial x_j} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = \frac{\partial F_i}{\partial x_i} \tag{4.4}$$

where the Kronecker unit tensor is defined as

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.5}$$

Equation (4.4) is the first time we have encountered a vector calculus relation involving two sets of repeated indices and the summation symbols have been included for clarity. For each index i there is a sum over j and the $i's$ are also summed. From here on we will use the Einstein convention and the summation symbols will be dropped. Note that multiplying by the Kronecker unit tensor and summing over both indices is equivalent to simply equating the two indices and summing as indicated in (4.4).

The dot (or inner) product of a vector and a tensor is

$$\bar{F} \bullet \nabla \bar{F} = F_j \frac{\partial F_i}{\partial x_j} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} F_1 \frac{\partial F_1}{\partial x_1} + F_2 \frac{\partial F_1}{\partial x_2} + F_3 \frac{\partial F_1}{\partial x_3} \\ F_1 \frac{\partial F_2}{\partial x_1} + F_2 \frac{\partial F_2}{\partial x_2} + F_3 \frac{\partial F_2}{\partial x_3} \\ F_1 \frac{\partial F_3}{\partial x_1} + F_2 \frac{\partial F_3}{\partial x_2} + F_3 \frac{\partial F_3}{\partial x_3} \end{pmatrix}. \tag{4.6}$$

The curl of a vector is defined by the skew-symmetric operation

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} = \tag{4.7}$$

$$\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right)\bar{e}_1 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right)\bar{e}_2 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)\bar{e}_3$$

or we can just write

$$\nabla \times \bar{F} = \left(\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right), \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right), \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)\right). \tag{4.8}$$

The quantities $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ are unit vectors in the three orthogonal coordinate directions. The curl can also be expressed using index notation.

$$(\nabla \times \bar{F})_i = \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j} \tag{4.9}$$

where the alternating unit tensor (also called the Levi-Civita tensor or the permutation tensor) is defined as

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if any two indices are the same} \\ 1, & ijk \text{ an even permutation } \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1, & ijk \text{ an odd permutation } \{(1,3,2),(3,2,1),(2,1,3)\} \end{cases} \tag{4.10}$$

The alternating unit tensor satisfies the following identities.

$$\begin{aligned} \delta_{ij}\varepsilon_{ijk} &= 0 \\ \varepsilon_{ipq}\varepsilon_{jpq} &= 2\delta_{ij} \\ \varepsilon_{ijk}\varepsilon_{ijk} &= 6 \\ \varepsilon_{ijk}\varepsilon_{pqk} &= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \end{aligned} \tag{4.11}$$

where δ_{ij} is the Kronecker unit tensor defined in (4.5).

4.2.1 USEFUL VECTOR IDENTITIES

Some useful vector identities involving first derivatives are as follows

$$
\begin{aligned}
\nabla(\psi\phi) &= \psi\nabla\phi + \phi\nabla\psi \\
\nabla \bullet (\phi\overline{U}) &= \phi\nabla \bullet \overline{U} + \overline{U} \bullet \nabla\phi \\
\nabla \times (\phi\overline{U}) &= \phi\nabla \times \overline{U} + \nabla\phi \times \overline{U} \\
\nabla(\overline{U} \bullet \overline{V}) &= (\overline{U} \bullet \nabla)\overline{V} + (\overline{V} \bullet \nabla)\overline{U} + \overline{U} \times (\nabla \times \overline{V}) + \overline{V} \times (\nabla \times \overline{U}) \\
\nabla \bullet (\overline{U} \times \overline{V}) &= \overline{V} \bullet \nabla \times \overline{U} - \overline{U} \bullet \nabla \times \overline{V} \\
\nabla \times (\overline{U} \times \overline{V}) &= \overline{U}(\nabla \bullet \overline{V}) + (\overline{V} \bullet \nabla)\overline{U} - \overline{V}(\nabla \bullet \overline{U}) - (\overline{U} \bullet \nabla)\overline{V}
\end{aligned}
\tag{4.12}
$$

Note that dot products involving the gradient operator are not always commutative. For example consider the difference between

$$
\begin{gathered}
\overline{U}(\nabla \bullet \overline{V}) = U_j \frac{\partial V_i}{\partial x_i} = \\
\left(U_1\left(\frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}\right), U_2\left(\frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}\right), U_3\left(\frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}\right)\right)
\end{gathered}
\tag{4.13}
$$

and

$$
\begin{gathered}
(\overline{V} \bullet \nabla)\overline{U} = V_i \frac{\partial U_j}{\partial x_i} = \\
\left(\left(V_1\frac{\partial U_1}{\partial x_1} + V_2\frac{\partial U_1}{\partial x_2} + V_3\frac{\partial U_1}{\partial x_3}\right), \left(V_1\frac{\partial U_2}{\partial x_1} + V_2\frac{\partial U_2}{\partial x_2} + V_3\frac{\partial U_2}{\partial x_3}\right), \right. . \\
\left.\left(V_1\frac{\partial U_3}{\partial x_1} + V_2\frac{\partial U_3}{\partial x_2} + V_3\frac{\partial U_3}{\partial x_3}\right)\right)
\end{gathered}
\tag{4.14}
$$

These two cases illustrate why the parentheses are needed and how useful the index notation can be in avoiding the kind of visual ambiguities that can arise in the use of vector notation.

Here are some more vector identities involving second derivatives

$$\nabla \bullet (\nabla F) = \nabla^2 F$$

$$\nabla \bullet (\nabla \bar{F}) = \nabla^2 \bar{F}$$

$$\nabla \bullet (\nabla \times \bar{F}) = 0 \tag{4.15}$$

$$\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \bullet \bar{F}) - \nabla^2 \bar{F}$$

where the Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \frac{\partial^2}{\partial x_i \partial x_i}. \tag{4.16}$$

The Laplacian is a scalar operator. The Laplacian of a scalar is a scalar

$$\nabla^2 F = \frac{\partial^2 F}{\partial x_i \partial x_i} \tag{4.17}$$

The Laplacian of a vector is a vector

$$\nabla^2 \bar{F} = \frac{\partial^2 F_j}{\partial x_i \partial x_i} \tag{4.18}$$

Note that the index that is summed is a dummy index and any repeated symbol will do, for example

$$\frac{\partial^2 F_j}{\partial x_k \partial x_k} \tag{4.19}$$

has the same meaning as the right hand side of (4.18).

4.3 Gauss' theorem

Gauss' theorem can be used to convert a volume integral involving the gradient operator to a surface integral involving the outward unit normal. For example

$$\int_V \nabla F \, dV = \int_A F \bar{n} dA \tag{4.20}$$

where F is a scalar. Here is another example

$$\int_V (\nabla \bullet \bar{F}) dV = \int_A \bar{F} \bullet \bar{n} dA \tag{4.21}$$

where $\bar{F}$ is a vector, and another

$$\int_V \frac{\partial F_{ij}}{\partial x_j} dV = \int_A F_{ij} n_j dA \tag{4.22}$$

where F_{ij} is a tensor. A volume integral involving the curl can be converted to a surface integral.

$$\int_V (\nabla \times \bar{F}) dV = \int_A \bar{n} \times \bar{F} dA \tag{4.23}$$

4.4 STOKES' THEOREM

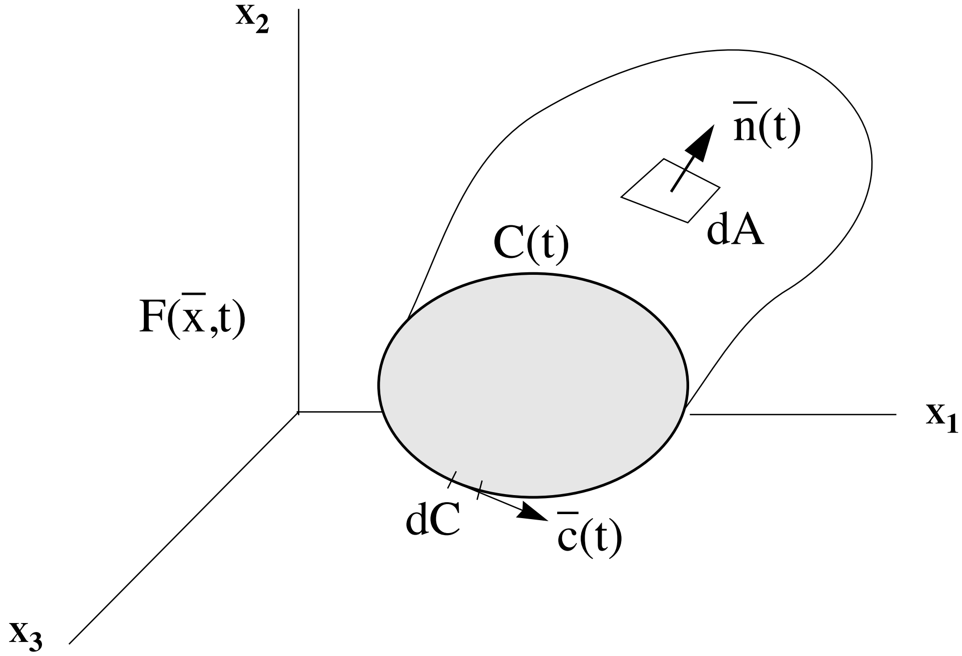

Figure 4.2 Unclosed control volume used to define Stokes' theorem.

Stokes' theorem can be used to convert a surface integral involving the curl of a vector to a line integral involving the unit tangent over the boundary of an unclosed control volume.

Although we defined the control volume to be closed there are instances when an unclosed control volume such as that depicted above (looking kind of like a sock!) can be useful.

The differential length dC is an infinitesimal line segment along the bounding curve $C(t)$ with unit tangent $\bar{c}(t)$. Stokes theorem tells us that

$$\int_A (\nabla \times \bar{F}) \bullet \bar{n} dA = \oint_C \bar{F} \bullet \bar{c} dC$$
$$\int_A \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j} n_i dA = \oint_C F_k c_k dC \qquad (4.24)$$

and

$$\int_A (\bar{n} \times \nabla F) dA = \oint_C F \bar{c} dC . \tag{4.25}$$

Gauss' theorem is used repeatedly in the derivation of the equations of motion. Stokes theorem comes up in the theory of rotational flow particularly in the development of wing theory.

4.5 PROBLEMS

Problem 1 - Working in Cartesian coordinates and using index notation, prove each of the following the vector identities

$$\nabla \bullet (\rho \bar{F}) = \bar{F} \bullet \nabla \rho + \rho \nabla \bullet \bar{F} \tag{4.26}$$

$$\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \bullet \bar{F}) - \nabla^2 \bar{F} \tag{4.27}$$

$$\bar{F} \bullet \nabla \bar{F} = (\nabla \times \bar{F}) \times \bar{F} + \nabla\left(\frac{\bar{F} \bullet \bar{F}}{2}\right) \tag{4.28}$$

Problem 2 - Let $\bar{e}_i$, $\bar{e}_j$ and $\bar{e}_k$ be the unit vectors in a right hand orthogonal coordinate system. Show that

$$\varepsilon_{ijk} = \bar{e}_i \bullet (\bar{e}_j \times \bar{e}_k) \tag{4.29}$$

Problem 3 - Demonstrate Stokes' theorem by integration of the curl of some smooth vector field variable over a square boundary.

Problem 4 - Find a unit vector normal to each of the following surfaces.

i) $x + y + z = 2$

ii) $ax^2 + by^2 + cz^2 = 1$

iii) $xyz = 1$

Problem 5 - Show that the unit vector normal to the plane

$$ax + by + cz = d \tag{4.30}$$

has the components

$$\bar{n} = \left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right) \tag{4.31}$$

Why doesn't $\bar{n}$ depend on d?

Problem 6 - Verify Gauss' theorem

$$\int_V (\nabla \bullet \bar{F}) dV = \int_A \bar{F} \bullet \bar{n} dA \tag{4.32}$$

in each of the following cases,

i) $\bar{F} = (x, y, z)$ and V is a cube of side b aligned with the x, y, z axes,

ii) $\bar{F} = \bar{n}_r r^2$ where $\bar{n}_r$ is a unit vector in the radial direction and V is a sphere of radius b surrounding the origin and $r^2 = x^2 + y^2 + z^2$.

Problem 6 - Verify Stokes' theorem

$$\int_A (\nabla \times \bar{F}) \bullet \bar{n} dA = \oint_C \bar{F} \bullet \bar{c} dC \tag{4.33}$$

where $\bar{F} = (x, y, z)$ and A is the surface of a cube of side b aligned with the x, y, z axes. The open face of the cube has an outward normal aligned with the positive x-axis.