CHAPTER 1

INTRODUCTION TO FLUID FLOW

1.1 INTRODUCTION

Fluid flows play a crucial role in a vast variety of natural phenomena and man-made systems. The life-cycles of stars, the creation of atmospheres, the sounds we hear, the vehicles we ride, the systems we build for flight, energy generation and propulsion all depend in an important way on the mechanics and thermodynamics of fluid flow. The purpose of this course is to introduce students in Aeronautics and Astronautics to the fundamental principles of fluid mechanics with emphasis on the development of the equations of motion as well as some of the analytical tools from calculus needed to solve practically important problems involving flows in channels along walls and over lifting bodies.

1.2 CONSERVATION OF MASS

Mass is neither created nor destroyed. This basic principle of classical physics is one of the fundamental laws governing fluid motion and is a good departure point for our introductory discussion.

Figure 1.1 below shows an infinitesimally small stationary, rectangular control volume $\Delta x \Delta y \Delta z$ through which a fluid is assumed to be moving. A control volume of this type with its surface fixed in space is called an *Eulerian* control volume. The fluid velocity vector has components $\overline{U} = (U, V, W)$ in the $\bar{x} = (x, y, z)$ directions and the fluid density is ρ. In a general, unsteady, compressible flow, all four flow variables may depend on position and time. The law of conservation of mass over this control volume is stated as

$$\left\{\begin{matrix} \textit{Rate of mass} \\ \textit{accumulation} \\ \textit{inside the control} \\ \textit{volume} \end{matrix}\right\} = \left\{\begin{matrix} \textit{Rate of mass} \\ \textit{flow} \\ \textit{into the control} \\ \textit{volume} \end{matrix}\right\} - \left\{\begin{matrix} \textit{Rate of mass} \\ \textit{flow} \\ \textit{out of the control} \\ \textit{volume} \end{matrix}\right\} \tag{1.1}$$

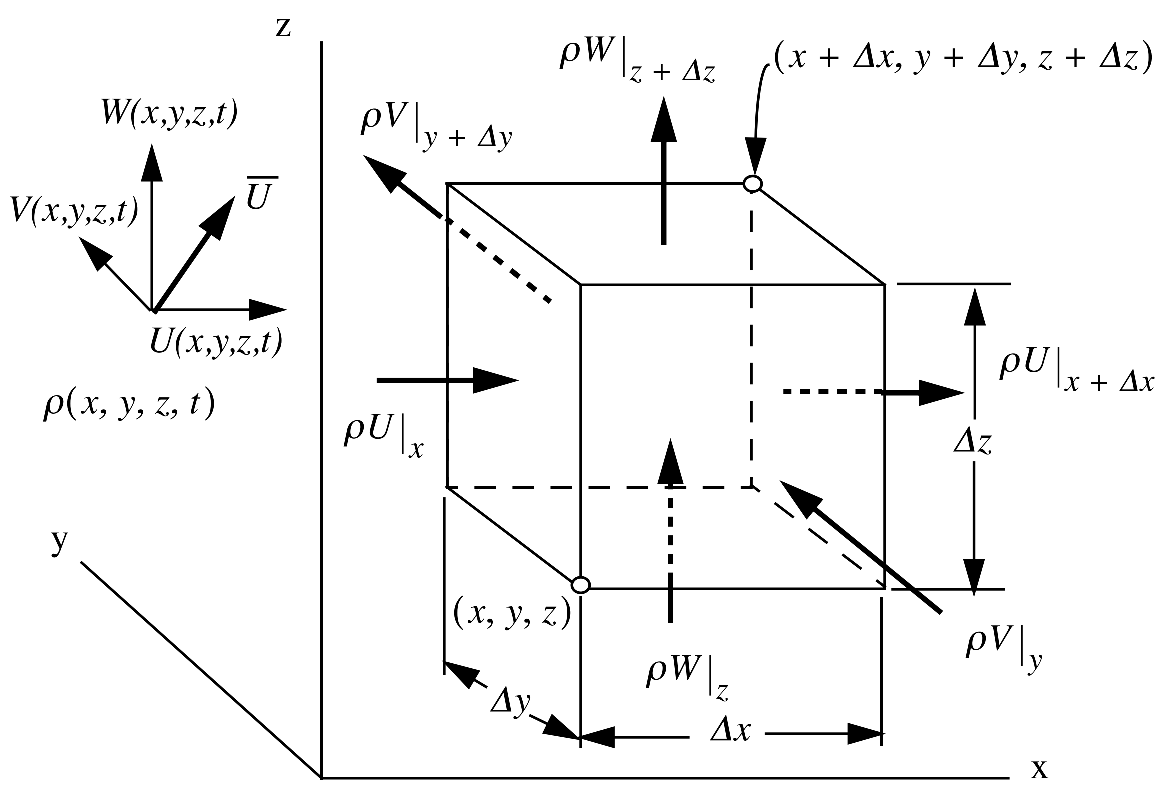


Figure 1.1 Fixed control volume in a moving fluid. The arrows shown denote fluxes of mass on the various faces of the control volume.

Consider a pair of faces perpendicular to the *x*-axis. The vector mass flux is $\rho\overline{U}$ with units *mass/area-time*. The rate of mass flow in through the face at x is the flux in the x-direction times the area $\rho U|_x \Delta y \Delta z$. The mass flow rate out through the face at $x + \Delta x$ is $\rho U|_{x + \Delta x}\Delta y \Delta z$. Similar expressions apply to the other two pairs of faces. The rate of mass accumulation inside the volume is $\Delta x \Delta y \Delta z(\partial\rho / \partial t)$ and this is equal to the sum of the mass fluxes over the six faces of the control volume. The mass balance (1.1) is expressed mathematically as

$$\begin{gathered}\Delta x \Delta y \Delta z\left(\frac{\partial \rho}{\partial t}\right) = \Delta y \Delta z \rho U|_x - \Delta y \Delta z \rho U|_{x + \Delta x} + \\ \Delta x \Delta z \rho V|_y - \Delta x \Delta z \rho V|_{y + \Delta y} + \Delta x \Delta y \rho W|_z - \Delta x \Delta y \rho W|_{z + \Delta z}\end{gathered} \tag{1.2}$$

which can be rearranged to read

$$\begin{gathered}\Delta x \Delta y \Delta z\left(\frac{\partial \rho}{\partial t}\right) + \Delta y \Delta z(\rho U|_{x + \Delta x} - \rho U|_x) + \\ \Delta x \Delta z(\rho V|_{y + \Delta y} - \rho V|_y) + \Delta x \Delta y(\rho W|_{z + \Delta z} - \rho W|_z) = 0\end{gathered} \tag{1.3}$$

Divide (1.3) through by the infinitesimal volume $\Delta x \Delta y \Delta z$.

$$\frac{\partial \rho}{\partial t} + \frac{\rho U|_{x + \Delta x} - \rho U|_x}{\Delta x} + \frac{\rho V|_{y + \Delta y} - \rho V|_y}{\Delta y} + \frac{\rho W|_{z + \Delta z} - \rho W|_z}{\Delta z} = 0 \tag{1.4}$$

Let $(\Delta x \rightarrow 0, \Delta y \rightarrow 0, \Delta z \rightarrow 0)$. In this limit (1.4) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z} = 0 \tag{1.5}$$

Equation (1.5) is called the continuity equation and is the general partial differential equation for conservation of mass for *any* moving, continuous medium (continuum). The continuum might be a compressible gas, a liquid or a moving solid such as glacier ice or the rock crust of the Earth.

1.2.1 Incompressible flow

The compressibility of a medium becomes important when the speed of a body begins to approach the speed of sound in the medium. At speeds much lower than the speed of sound the disturbance created by a body pushing aside the fluid is too small to significantly change the thermal energy of the fluid and fluid behaves as if it is incompressible $\rho = constant$. In this limit (1.5) reduces to

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0. \tag{1.6}$$

Note that equation (1.6) applies to both steady and unsteady incompressible flow.

1.2.2 Index notation and the Einstein convention

For convenience, vector components are often written with subscripts. This is called *index notation* and one makes the following replacements.

$$\begin{aligned} (x, y, z) &\rightarrow (x_1, x_2, x_3) \\ (U, V, W) &\rightarrow (U_1, U_2, U_3) \end{aligned} \tag{1.7}$$

In index notation equation (1.5) is concisely written in the form

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial(\rho U_i)}{\partial x_i} = 0 \tag{1.8}$$

where the subscript refers to the i*-th* vector component.

Vector calculus is an essential tool for developing the equations that govern compressible flow and summed products such as (1.8) arise often. Notice that the sum in (1.8) involves a repeated index. The theory of Relativity is another area where such sums arise often and when Albert Einstein was developing the special and general theory he too recognized that such sums always involve an index that is repeated twice but never three times or more. In effect the presence of repeated indices *implies a summation process* and the summation symbol can be dropped. To save effort and space Einstein did just that and the understanding that a repeated index denotes a sum has been known as the Einstein convention ever since. Using the Einstein convention (1.8) becomes

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U_i)}{\partial x_i} = 0} \tag{1.9}$$

Remember, the rule of thumb is that a single index denotes a vector component and a repeated index represents a sum. Three or more indices the same means that there is a mistake in the equation somewhere! The upper limit of the sum is 1, 2 or 3 depending on the number of space dimensions in the problem. In the notation of vector calculus (1.9) is written

$$\frac{\partial\rho}{\partial t} + \nabla \bullet (\rho\overline{U}) = 0 \tag{1.10}$$

and (1.6) is $\nabla \bullet \overline{U} = 0$. Vector notation has the advantage of being concise and independent of the choice of coordinates but is somewhat abstract. The main advantage of index notation is that it expresses precisely what differentiation and summation processes are being done in a particular coordinate system. In Cartesian coordinates, the gradient vector operator is

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{1.11}$$

The continuity equation as well as the rest of the equations of fluid flow are given in cylindrical and spherical polar coordinates in Appendix 2.

1.3 Particle paths, streamlines and streaklines in 2-D steady flow

Let's begin with a study of fluid flow in two dimensions. Figure 1.2 shows the theoretically computed flow over a planar, lifting airfoil in steady, inviscid (non-viscous) flow. The theory used to determine the flow assumes that the flow is irrotational

$$\nabla \times \overline{U} = 0 \tag{1.12}$$

and that the flow speed is very low so that (1.6) holds.

$$\nabla \bullet \overline{U} = 0. \tag{1.13}$$

A vector field that satisfies (1.12) can always be represented as the gradient of a scalar potential function $\phi = \Phi(x, y)$ therefore

$$\overline{U} = \nabla\Phi. \tag{1.14}$$

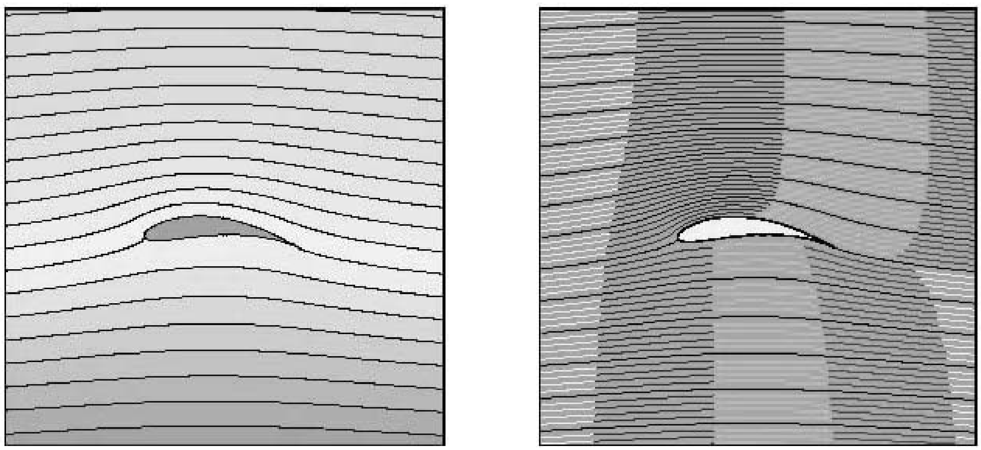
or, in terms of components

$$(U, V) = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}\right) \tag{1.15}$$

When (1.14) is substituted into (1.13) the result is Laplace's equation

$$\nabla \bullet \nabla \Phi = \nabla^2 \Phi = 0. \tag{1.16}$$

Figure 1.2 depicts the solution of equation (1.16) over the airfoil shown.



(a) (b)

Figure 1.2 Flow over a 2-D lifting wing; (a) streamlines, (b) streaklines.

The boundary conditions on the airfoil are that the flow is allowed to slip tangent to the surface but cannot penetrate the surface and the flow is required to leave the sharp trailing edge of the airfoil smoothly. The latter requirement is the famous Kutta condition. In a real airfoil viscous friction prevents the flow from going around the sharp trailing edge. By enforcing the Kutta condition, the solution of (1.16) is able to mimic the effects of viscosity in real airfoils thus allowing the lift to be determined. Once the scalar potential is found the velocity field is generated using (1.14). The streamlines shown in Figure 1.2(a) are simply lines that are everywhere parallel to the velocity vector field.

A careful examination of the picture reveals quite a bit of information. First, note that the mass flow rate between any two streamlines lines is fixed (fluid cannot cross streamlines). As the streamlines approach the leading edge of the airfoil the separation between lines increases indicating a deceleration of the fluid. Near the point of maximum wing thickness on the upper surface, the streamlines come closer together indicating a speed-up of the fluid to greater than the free stream speed. Below the airfoil the streamlines move apart indicating that the fluid slows down. Downstream of the airfoil trailing edge, the flow speed recovers to the freestream value. Notice that the largest velocity changes occur near the wing. In the upper and lower parts of the picture, far away from the wing, the flow is deflected upward by the wing but the distance between streamlines changes little and the corresponding flow speed change is relatively small.

Figure 1.2 (b) depicts streaklines in the flow over the airfoil. These are produced numerically the same way one would produce dye lines in a real flow. Fluid elements that pass through a given point upstream of the airfoil are marked forming a streak. In the figure, alternating bands of fluid are marked light and dark. The flow pattern produced by the streaklines is identical to the streamline pattern but there is one added piece of information. The streakline pattern depicts the integrated effect of the flow velocity on the *position* of the fluid elements that constitute the streak.

One of the popular explanations of how an airfoil produces lift is that adjacent fluid elements on either side of the stagnation streamline ahead of the wing must meet at the trailing edge at the same time. The reasoning then goes as follows; since a particle that travels above the wing must travel farther than the one below, it must travel faster thus reducing the pressure over the wing and producing lift. Figure 1.2 (b) clearly shows that this explanation is completely erroneous. The fluid below the wing is substantially retarded compared to the fluid that passes above the wing.

The figures below show two more streamline examples in a slightly compressible situation with the effects of viscosity included. These are computations of the flow over a wing flap at a free stream speed of approximately 30% of the speed of sound. The flow conditions are the same in both cases except that in the right-hand picture the trailing edge of the main wing (visible in the upper left corner of the picture) has attached to it a small vertical flap called a Gurney flap.

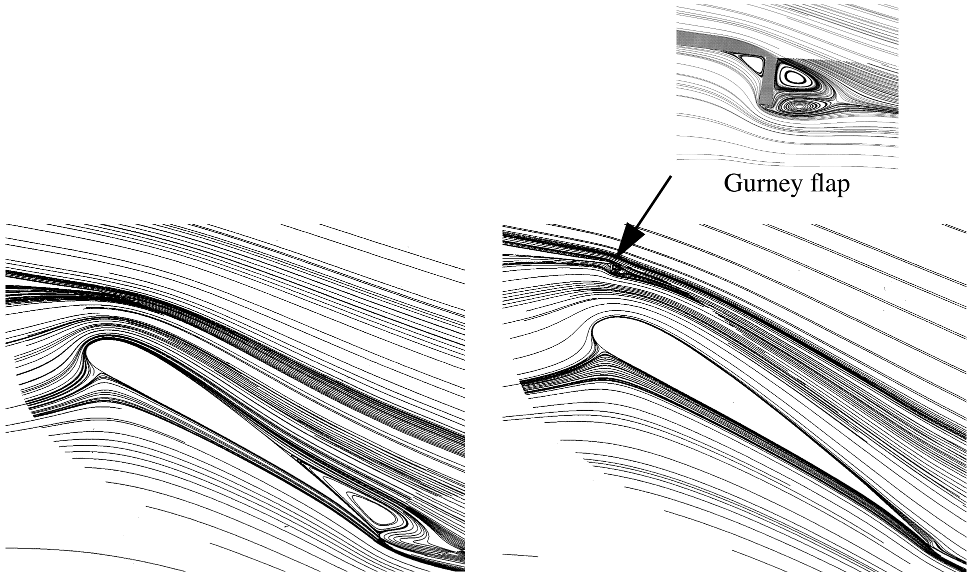


Figure 1.3 Computed streamlines over a wing flap.

The Gurney flap was developed by Dan Gurney in an effort to improve the speed of his new IRL race car prior to the start of the 1971 racing season in Phoenix. The car was disappointingly slow and after several days of testing, the driver Bobby Unser challenged Gurney to find a solution. Gurney decided to add a spoiler to the rear wing by riveting a length of aluminum right-angle to the trailing edge. When Unser tested the car it was just as slow as before and so everyone felt that the idea was a failure. Later Unser confided to Gurney that the reason the car was slow was that the down-force from the flap on the rear wheels was so large that the car was understeering badly. It was clear that with some adjustments to increase the down-force on the front wings the car could be made much faster. For a while the Gurney team deflected questions with the fiction that the flap was purely to increase the structural strength of the wing but eventually the competition got wise and the Gurney flap was widely adopted.

While the Gurney flap increases both the lift and drag of a single wing the effect on a two-element airfoil is to cause the flow over the flap to reattach as shown in Figure 1.3 thereby increasing lift and *reducing* the overall drag of the wing-flap system. This allows the system to be effective at much higher flap angles.

Figure 1.3 was produced using the computed velocity field. Once the velocity field is known, the user places computational "particles" at selected grid points and integrates the trajectory of the particles forward in time. The choice of initial points depends on the amount of detail one wants to visualize and this accounts for the uneven distribution of streamlines in these figures. The integration traces out the trajectories of the particles. Since the flow is steady this process effectively traces out the streamlines. Changes in flow speed can be seen in terms of the convergence and divergence of streamlines, just as in the incompressible case, although the variation of fluid density over the field complicates this interpretation slightly. We shall return to this last point shortly.

1.3.1 Analysis of particle paths and streamlines

Let's learn how to analyze particle paths and streamlines theoretically. The figure below shows a typical trajectory in space of a fluid element moving under the action of a two-dimensional *steady* velocity field.

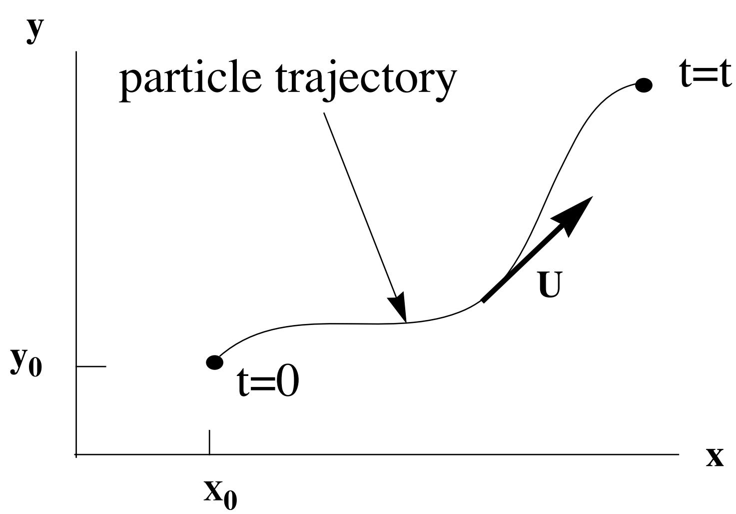


Figure 1.4 Particle trajectory in a 2-D, steady flow field.

The equations that determine the trajectory are:

$$\begin{aligned}\frac{dx(t)}{dt} &= U(x(t), y(t)) \\ \frac{dy(t)}{dt} &= V(x(t), y(t))\end{aligned} . \tag{1.17}$$

where U and V are the flow velocity components in the x and y directions respectively.

The velocity field is assumed to be a smooth function of position. Formally, these equations are solved by integrating the velocity field forward in time.

$$x(t) = x_0 + \int_0^t U(x(t), y(t))dt$$
$$y(t) = y_0 + \int_0^t V(x(t), y(t))dt \tag{1.18}$$

The result is a set of parametric functions for the particle coordinates x and y in terms of the time, t, along a particle path

$$x(t) = F(x_0, y_0, t)\ ; \qquad y(t) = G(x_0, y_0, t)\,. \tag{1.19}$$

The solution of (1.17) can also be expressed as a family of lines derived by eliminating t between the functions F and G in (1.19).

$$\psi = \Psi(x, y)\,. \tag{1.20}$$

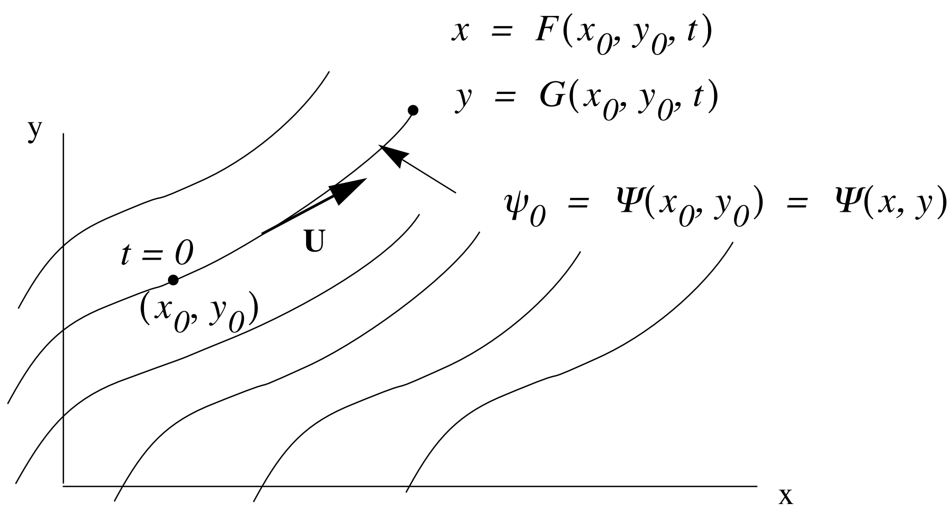


Figure 1.5 Streamlines in steady flow. The value of a particular streamline is determined by the coordinates of a point on the streamline.

This is essentially how the streamlines observed in Figure 1.2(a) and Figure 1.3 are generated. The value of a particular streamline is determined by the initial conditions

$$\psi_0 = \Psi(x_0, y_0). \tag{1.21}$$

This is the situation depicted schematically in Figure 1.5.

The streaklines in Figure 1.2(b) were generated by shading a segment of fluid elements that pass through an initial point (x_0, y_0) during a fixed interval in time. Selecting a vertical line of initial points well upstream of the airfoil leads to the bands shown in the figure. The length of a segment is directly related to the velocity history of the fluid particles that make up the segment.

The stream function can also be determined as the solution of the first order ordinary differential equation obtained by eliminating dt between the two particle path equations in (1.17),

$$\frac{dy}{dx} = \frac{V(x, y)}{U(x, y)}. \tag{1.22}$$

The differential of $\Psi(x, y)$ is

$$d\psi = \frac{\partial \boldsymbol{\Psi}}{\partial x}dx + \frac{\partial \boldsymbol{\Psi}}{\partial y}dy. \tag{1.23}$$

If we use (1.17) to replace the differentials dx and dy in (1.23) the result is

$$d\psi = \left(U(x, y)\frac{\partial \boldsymbol{\Psi}}{\partial x} + V(x, y)\frac{\partial \boldsymbol{\Psi}}{\partial y}\right)dt. \tag{1.24}$$

On a line of constant $\psi = \psi_0$ the differential $d\psi = 0$ and so for nonzero dt, the right hand side of (1.24) can be zero only if the expression in parentheses is zero.

Thus the stream function, $\Psi(x, y)$, can be determined in two ways; either as the solution of a linear, first order PDE,

$$\boldsymbol{U} \bullet \nabla \boldsymbol{\Psi} = U(x, y)\frac{\partial \boldsymbol{\Psi}}{\partial x} + V(x, y)\frac{\partial \boldsymbol{\Psi}}{\partial y} = 0. \tag{1.25}$$

or as the solution of the ODE (1.22) which we can write in the form,

$$-V(x, y)dx + U(x, y)dy = 0. \tag{1.26}$$

Equation (1.25) is the mathematical expression of the statement that streamlines are parallel to the velocity vector field.

1.3.2 The integrating factor

On a line of constant $\psi = \psi_0$ the differential $d\psi = 0$ and

$$\frac{\partial\Psi}{\partial x}dx + \frac{\partial\Psi}{\partial y}dy = 0. \tag{1.27}$$

A very important question concerns the relationship between the differential expressions, (1.26) and (1.27). Are they the same? In particular, can (1.26) be regarded as a perfect differential?

The test for a perfect differential is to compare cross derivatives. Equation (1.26) is a perfect differential if and only if,

$$-\frac{\partial V}{\partial y} = \frac{\partial U}{\partial x}. \tag{1.28}$$

If the integrability condition (1.28) is satisfied then there must exist some function $\Psi(x, y)$ such that $V = -\partial\Psi/\partial x$ and $U = \partial\Psi/\partial y$ and the expression (1.26) can be equated to the differential of this function $d\psi$. In this case it is easy to solve for $\Psi(x, y)$ by direct integration of (1.26).

But for an arbitrary choice of the functions $U(x, y)$ and $V(x, y)$, the answer to the above question is in general no! In other words, for general V and U

$$\frac{V}{U} = \frac{-\partial\Psi/\partial x}{\partial\Psi/\partial y} \tag{1.29}$$

but

$$V \neq -\frac{\partial\Psi}{\partial x} \qquad U \neq \frac{\partial\Psi}{\partial y}. \tag{1.30}$$

In order to convert equation (1.26) to a perfect differential it must be multiplied by an *integrating factor*. In general there is no systematic method for finding the integrating factor and so the analytic solution of (1.26) for general functions U and V remains a difficult unsolved problem in mathematics. However we shall see later that in the case of fluid flow the integrating factor can be identified using the equation for conservation of mass.

It was shown by the German mathematician Johann Pfaff in the early 1800's that an integrating factor for the expression $-Vdx + Udy$ always exists. That is, for any choice of smooth functions U and V, there always exists a function $M(x, y)$ such that,

$$d\psi = -M(x, y)V(x, y)dx + M(x, y)U(x, y)dy \tag{1.31}$$

and the partial derivatives are

$$\left.\begin{aligned} \frac{\partial \Psi}{\partial x} &= -M(x, y)V(x, y) \\ \frac{\partial \Psi}{\partial y} &= M(x, y)U(x, y) \end{aligned}\right\} \tag{1.32}$$

A differential expression like $-Vdx + Udy$ is often called a Pfaffian form. In the language of differential geometry it is called a differential 1-form. Pfaffian forms of higher dimension, say $Adx + Bdy + Cdz$ are often encountered in physics and with certain restrictions on $A(x, y, z)$, $B(x, y, z)$ and $C(x, y, z)$ an integrating factor can be found. But, the existence of the integrating factor is only assured unconditionally in two dimensions.

Pfaff's theorem can be understood by noting that the flow *patterns* formed by the vector fields with local slopes V/U and $(MV)/(MU)$ are identical whereas the local flow *speeds* are not since the lengths of individual vectors differ by the factor $M(x, y)$. Since the solution of (1.17) clearly exists (we could just integrate the equations forward in time numerically) then the integrating factor must also exist.

Perfect differentials and integrating factors are generally covered in an upper level undergraduate course in calculus. Unfortunately the presentation is often cursory and tends to get passed over fairly quickly, often without making the deep connection that exists to the analysis of physical problems. In this chapter we see the connection to the theory of fluid flow. In Chapter 2 we will find that the concept of a perfect differential is one of the key tools needed to develop the laws of thermodynamics.

The following sample problem is worked out in quite a bit of detail to help strengthen your understanding.

1.3.3 Sample Problem - The integrating factor and reconstruction of a function from its perfect differential

Solve the first order ordinary differential equation

$$\frac{dy}{dx} = \frac{y}{x}H(xy) \tag{1.33}$$

where H is an arbitrary smooth function. Rearrange (1.33) to read

$$-yH(xy)dx + xdy = 0. \tag{1.34}$$

This differential expression fails the cross derivative test

$$\frac{\partial(-yH(xy))}{\partial y} \neq \frac{\partial(x)}{\partial x} \tag{1.35}$$

and is not a perfect differential. The integrating factor for (1.34) is known to be

$$M = \frac{1}{xy + xyH(xy)} \tag{1.36}$$

Multiplying (1.34) by (1.36) converts it to a perfect differential.

$$d\psi = -\frac{yH(xy)}{xy + xyH(xy)}dx + \frac{x}{xy + xyH(xy)}dy \tag{1.37}$$

We now know the partial derivatives of the solution

$$\left.\begin{aligned}\frac{\partial\Psi}{\partial x} &= -\frac{yH(xy)}{xy + xyH(xy)} \\ \frac{\partial\Psi}{\partial y} &= \frac{x}{xy + xyH(xy)}\end{aligned}\right\} \tag{1.38}$$

The expression (1.37) can now be integrated by integrating either partial derivative with the appropriate variable held fixed. Let's start by integrating the partial derivative with respect to y.

$$\psi = \int^{y} \frac{x}{xy + xyH(xy)}\bigg|_{x\ constant} dy + f(x) \tag{1.39}$$

Note that we have to include a "constant" of integration that can be at most a function of x. Let $\alpha = xy$. The solution can now be expressed as

$$\psi = \int^{xy} \frac{1}{\alpha(1 + H(\alpha))} d\alpha + f(x) \tag{1.40}$$

Now differentiate (1.40) with respect to x and equate to the known partial derivative with respect to x in (1.38).

$$\begin{gathered}\frac{\partial}{\partial x}\left(\int^{xy} \frac{1}{\alpha(1 + H(\alpha))} d\alpha + f(x)\right) = -\frac{yH(xy)}{xy + xyH(xy)} \\ \frac{y}{xy + xyH(xy)} + \frac{df}{dx} = -\frac{yH(xy)}{xy + xyH(xy)} \\ \frac{df}{dx} = -\frac{1}{x}\end{gathered} \tag{1.41}$$

This step allows us to solve for f. Finally the exact solution of (1.33) is

$$\psi = \int^{xy} \frac{1}{\alpha(1 + H(\alpha))} d\alpha - \ln x \tag{1.42}$$

Note that we had to know the integrating factor in order to solve the problem.

Sometimes what appears to be a trick will work. For example you can also get to the solution (1.42) by simply simply defining a new variable $\alpha = xy$. This allows (1.33) to be solved by separation of variables. All such tricks are equivalent to finding the integrating factor.

In the general case the required integrating factor is not known. For example the equation

$$\frac{dy}{dx} = \frac{xy + y + y^2}{2x^2 - xy} \tag{1.43}$$

comes up in the study of viscous boundary layer flow along a flat plate. We know an integrating factor exists but it has never been found.

Fortunately when it comes to streamline patterns in two dimensions, conservation of mass can be used to determine the integrating factor.

1.3.4 Incompressible flow in 2 dimensions

The flow of an incompressible fluid in 2-D is constrained by the continuity equation (1.6) that in two dimensions reduces to

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \tag{1.44}$$

This is exactly the condition (1.28) and so in this case the stream function and the velocity field are related by

$$U = \frac{\partial \Psi}{\partial y} \; ; \qquad V = -\frac{\partial \Psi}{\partial x} \tag{1.45}$$

and $-Vdx + Udy$ is guaranteed to be a perfect differential. The integrating factor is one. Therefore one can always write for 2-D incompressible flow

$$d\psi = -Vdx + Udy. \tag{1.46}$$

If U and V are known functions, the stream function is determined analytically by integration of (1.46). In practice the easiest way to assign values to the streamlines is to integrate (1.46) at some reference state well upstream of the airfoil where the velocity is uniform $(U, V) = (U_{\infty}, 0)$.

1.3.5 Incompressible, irrotational flow in two dimensions

If the flow is incompressible and irrotational then the velocity field is described by both a stream function (1.45) and a potential function (1.15). The two functions are related by

$$\begin{aligned} \frac{\partial \Psi}{\partial y} &= \frac{\partial \Phi}{\partial x} \\ -\frac{\partial \Psi}{\partial x} &= \frac{\partial \Phi}{\partial y} \end{aligned} \tag{1.47}$$

These are the well known Cauchy-Riemann equations from the theory of complex variables. Solutions of Laplace's equation (1.16) which is the equation of motion for this class of flows can be determined using the powerful methods of complex analysis. Interestingly, the flow can be solved just from the continuity equation (1.13) without the use of the equation for conservation of momentum. The irrotationality condition (1.12) essentially supplants the need for the momentum equation.

1.3.6 Compressible flow in 2 dimensions

The continuity equation for the steady flow of a compressible fluid in two dimensions is

$$\frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V) = 0 \tag{1.48}$$

In this case, the required integrating factor for (1.26) is the density $\rho(x, y)$ and we can write

$$d\psi = -\rho V dx + \rho U dy \tag{1.49}$$

The stream function in a compressible flow is proportional to the mass flux with units *mass/area-sec* and the convergence and divergence of lines in the flow over the flap shown in Figure 1.3 is a reflection of variations in mass flux over different parts of the flow field.

1.4 PARTICLE PATHS IN THREE DIMENSIONS

Figure 1.6 shows the trajectory in space traced out by a particle under the action of a general three-dimensional, unsteady flow, $\overline{U}(\bar{x}, t)$.

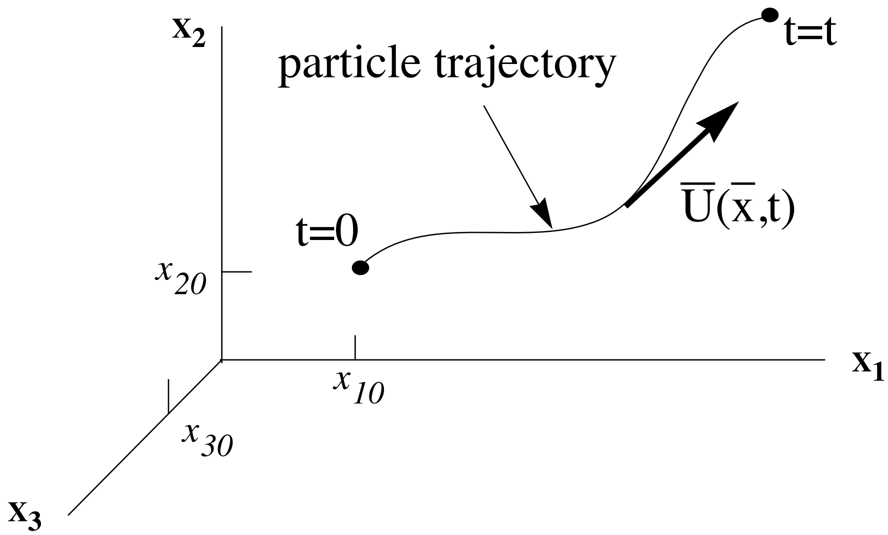


Figure 1.6 Particle trajectory in three dimensions.

Using index notation, the equations governing the motion of the particle are:

$$\frac{dx_i(t)}{dt} = U_i(x_1(t), x_2(t), x_3(t), t) \quad ; \quad i = 1, 2, 3 \tag{1.50}$$

Formally, these equations are solved by integrating the velocity field.

$$x_i(t) = x_{i0} + \int_0^t U_i(x_1(t), x_2(t), x_3(t), t)dt \quad ; \quad i = 1, 2, 3 \tag{1.51}$$

We shall return to the discussion of flow patterns in both 2 and 3 dimensions when we discuss the kinematics of flow fields in Chapter 4.

1.5 THE SUBSTANTIAL DERIVATIVE

The acceleration of a particle is

$$\frac{d^2 x_i(t)}{dt^2} = \frac{d}{dt} U_i(x_1(t), x_2(t), x_3(t), t) = \frac{\partial U_i}{\partial t} + \frac{\partial U_i}{\partial x_k}\frac{dx_k}{dt}. \tag{1.52}$$

Remember, according to the Einstein convention, the repeated index denotes a sum over $k = 1, 2, 3$. Use (1.50) to replace dx_k/dt by U_k in (1.52). The result is called the substantial or material derivative and is usually denoted by D/Dt.

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \overline{U} \bullet \nabla(\ ) . \tag{1.53}$$

The substantial derivative of the velocity is

$$\frac{DU_i}{Dt} = \frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} . \tag{1.54}$$

The time derivative of any flow variable evaluated on a fluid element is given by a similar formula. For example the time rate of change of the density $\rho(x_1(t), x_2(t), x_3(t), t)$ of a given fluid particle is:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + U_k \frac{\partial \rho}{\partial x_k} \tag{1.55}$$

The substantial derivative is the time derivative of some property of a fluid element referred to a fixed frame of reference within which the fluid element moves as shown in Figure 1.6 above.

1.5.1 Frames of reference

Occasionally it is necessary to transform variables between a fixed and moving set of coordinates as shown in Figure 1.7. The transformation of position and velocity is

$$\begin{aligned} x' &= x - X(t) \\ y' &= y - Y(t) \\ z' &= z - Z(t) \\ U' &= U - \dot{X}(t) \\ V' &= V - \dot{Y}(t) \\ W' &= W - \dot{Z}(t) \end{aligned} \tag{1.56}$$

where $\bar{X} = (X(t), Y(t), Z(t))$ is the displacement of the moving frame in three coordinate directions and $d\bar{X}/dt = (\dot{X}(t), \dot{Y}(t), \dot{Z}(t))$ is the velocity of the frame. Note that in general, the frame may be accelerating.

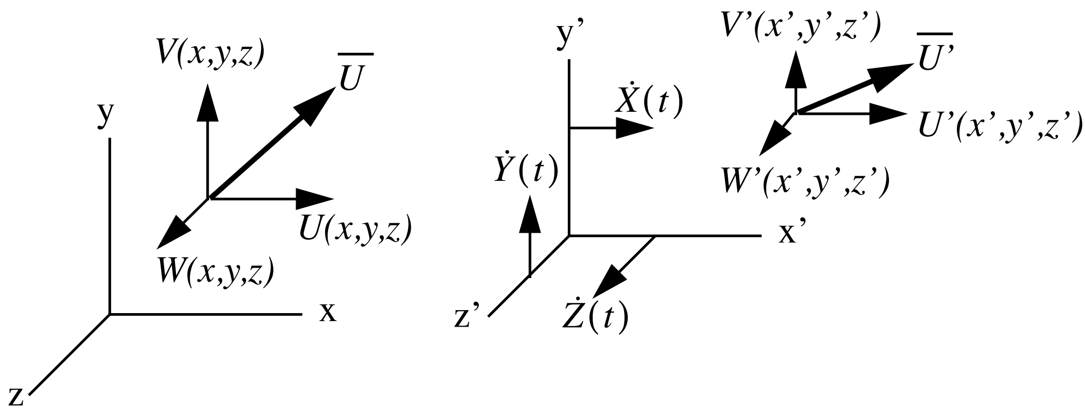


Figure 1.7   Fixed and moving frames of reference

The substantial derivative of some property of a fluid element, (1.54), (1.55), is sometimes referred to as the "derivative moving with the fluid element". Do not interpret this as a transformation to a coordinate system that is attached to the fluid element. If for some reason it is actually desirable to make such a transformation the velocity of the particle in that frame is zero and the substantial derivative of velocity is also zero. The substantial derivative of a variable such as the density of the particle at the origin of the moving coordinates is $D\rho/Dt = \partial\rho/\partial t$.

Both the momentum and kinetic energy of a fluid particle of mass m depend on the frame of reference. The momentum transforms linearly like the velocity.

$$m\bar{U}' = m\bar{U} - md\bar{X}/dt \tag{1.57}$$

momentum in moving coordinates   momentum in fixed coordinates

The transformation of kinetic energy is a little more complex. In the two coordinate systems the definition of kinetic energy is the same.

$$\textit{kinetic energy in moving coordinates} = \frac{1}{2}m(U'^2 + V'^2 + W'^2) \tag{1.58}$$

and

$$\textit{kinetic energy in fixed coordinates} = \frac{1}{2}m(U^2 + V^2 + W^2) \tag{1.59}$$

The transformation of kinetic energy between frames is

$$\frac{1}{2}m(U'^2 + V'^2 + W'^2) = \frac{1}{2}m((U - \dot{X})^2 + (V - \dot{Y})^2 + (W - \dot{Z})^2). \tag{1.60}$$

which when expanded to show the kinetic energies explicitly becomes.

$$\begin{gathered} \frac{1}{2}m(U'^2 + V'^2 + W'^2) = \frac{1}{2}m(U^2 + V^2 + W^2) + \\ \frac{1}{2}m\dot{X}(\dot{X} - 2U) + \frac{1}{2}m\dot{Y}(\dot{Y} - 2V) + \frac{1}{2}m\dot{Z}(\dot{Z} - 2W) \end{gathered} \tag{1.61}$$

or

$$k' = k + \frac{1}{2}m\dot{X}(\dot{X} - 2U) + \frac{1}{2}m\dot{Y}(\dot{Y} - 2V) + \frac{1}{2}m\dot{Z}(\dot{Z} - 2W). \tag{1.62}$$

The transformation of kinetic energy depends nonlinearly on the velocity of the moving coordinate system.

In contrast, thermodynamic properties such as density, temperature and pressure are intrinsic properties of a given fluid element and so do not depend on the frame of reference

1.6 MOMENTUM TRANSPORT DUE TO CONVECTION

The law of conservation of momentum is stated as

$$\left\{\begin{matrix} \textit{Rate of} \\ \textit{momentum} \\ \textit{accumulation} \\ \textit{inside the} \\ \textit{control volume} \end{matrix}\right\} = \left\{\begin{matrix} \textit{Rate of} \\ \textit{momentum flow} \\ \textit{into the} \\ \textit{control volume} \end{matrix}\right\} - \left\{\begin{matrix} \textit{Rate of} \\ \textit{momentum flow} \\ \textit{out of the} \\ \textit{control volume} \end{matrix}\right\} + \left\{\begin{matrix} \textit{Sum of} \\ \textit{forces acting} \\ \textit{on the} \\ \textit{control volume} \end{matrix}\right\} \tag{1.63}$$

As a fluid moves it carries its momentum with it. This is called *convective momentum transport*. To study this kind of momentum transfer we use the same stationary control volume element $\Delta x \Delta y \Delta z$ that we used to develop the continuity equation. As before, the fluid velocity vector has components (U, V, W) in the (x, y, z) directions and the fluid density is ρ.

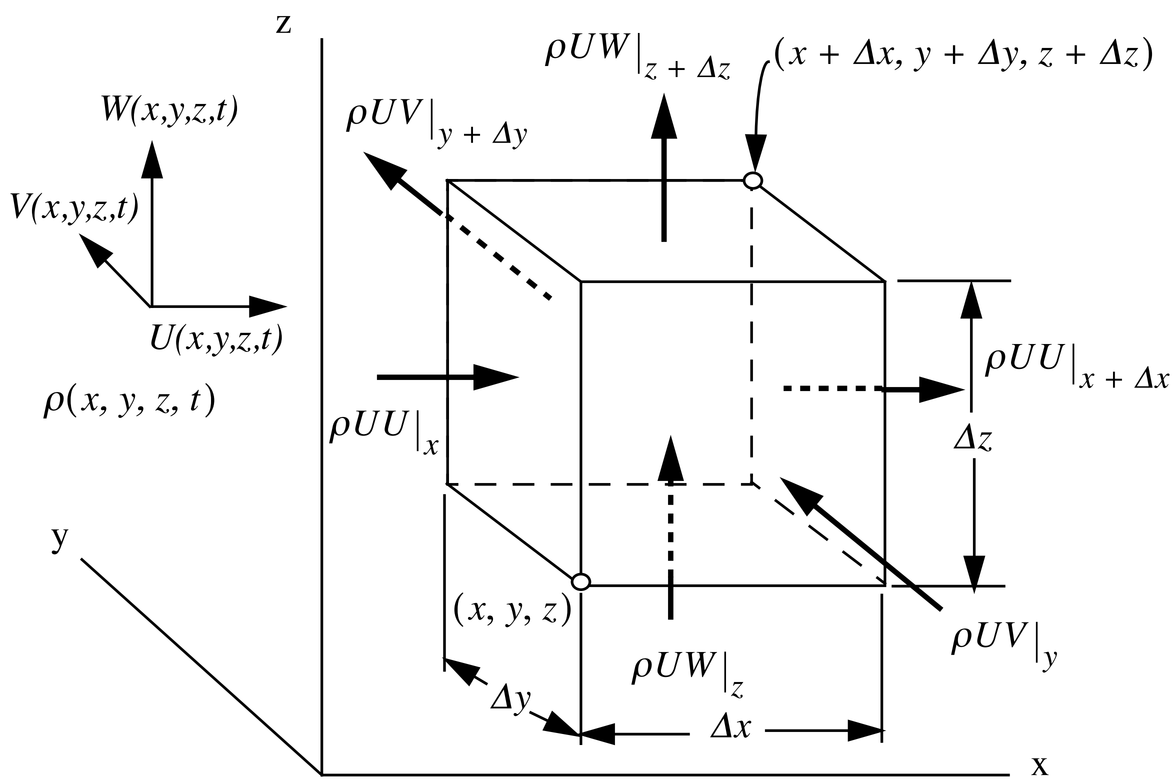


Figure 1.8 Fluxes of x-momentum through a fixed control volume. Arrows denote the velocity component carrying momentum into or out of the control volume.

Figure 1.8 shows the contribution to the x-momentum inside the control volume from the movement of fluid through all six faces of the control volume. Consider a pair of faces perpendicular to the x-axis. The flux of x-momentum is ρUU with units *momentum/area-time*. The rate at which the x-component of momentum enters the face at x is $\rho UU|_x \Delta y \Delta z$ and the rate at which it leaves through the face at $x + \Delta x$ is $\rho UU|_{x + \Delta x} \Delta y \Delta z$.

The rate at which the x-component of momentum enters the face at y is $\rho UV|_y \Delta x \Delta z$ and the rate at which it leaves through the face at $y + \Delta y$ is $\rho UV|_{y+\Delta y} \Delta x \Delta z$. To understand how fluid motion in the y-direction contributes to the rate of x-momentum transfer into or out of the control volume one can make the following interpretation of the momentum transfer.

$$\rho U = \left\{ \begin{array}{c} \textit{x-momentum} \\ \textit{per unit volume} \end{array} \right\}$$

$$V|_y \Delta x \Delta z = \left\{ \begin{array}{c} \textit{volume of fluid per second} \\ \textit{passing into the control volume through} \\ \textit{the face normal to the V-velocity} \\ \textit{component at position y} \end{array} \right\} \tag{1.64}$$

$$V|_{y+\Delta y} \Delta x \Delta z = \left\{ \begin{array}{c} \textit{volume of fluid per second} \\ \textit{passing out of the control volume through} \\ \textit{the face normal to the V-velocity} \\ \textit{component at position } y+\Delta y \end{array} \right\}$$

Based on similar considerations, the rate at which the x-component of momentum enters the face at z is $\rho UW|_z \Delta x \Delta y$ and the rate at which it leaves through the face at $z + \Delta z$ is $\rho UW|_{z+\Delta z} \Delta x \Delta y$.

The rate of x-component of momentum accumulation inside the control volume is $\Delta x \Delta y \Delta z(\partial \rho U / \partial t)$. For the x-component of momentum, the balance (1.63) is expressed mathematically as

$$\begin{gathered} \Delta x \Delta y \Delta z\left(\frac{\partial \rho U}{\partial t}\right) = \Delta y \Delta z \rho UU|_x - \Delta y \Delta z \rho UU|_{x+\Delta x} + \\ \Delta x \Delta z \rho UV|_y - \Delta x \Delta z \rho UV|_{y+\Delta y} + \Delta x \Delta y \rho UW|_z - \Delta x \Delta y \rho UW|_{z+\Delta z} + \\ \{\textit{the sum of x-component forces acting on the system}\} \end{gathered} \tag{1.65}$$

which is rearranged to read

$$\begin{aligned} &\Delta x \Delta y \Delta z\left(\frac{\partial \rho U}{\partial t}\right) + \Delta y \Delta z(\rho UU|_{x+\Delta x} - \rho UU|_x) + \\ &\Delta x \Delta z(\rho UV|_{y+\Delta y} - \rho UV|_y) + \Delta x \Delta y(\rho UW|_{z+\Delta z} - \rho UW|_z) = \\ &\{\textit{the sum of x-component forces acting on the system}\} \end{aligned} \tag{1.66}$$

Divide (1.66) through by the volume of the control element $\Delta x \Delta y \Delta z$.

$$\begin{aligned} &\frac{\partial \rho U}{\partial t} + \frac{\rho UU|_{x+\Delta x} - \rho UU|_x}{\Delta x} + \frac{\rho UV|_{y+\Delta y} - \rho UV|_y}{\Delta y} + \frac{\rho UW|_{z+\Delta z} - \rho UW|_z}{\Delta z} = \\ &\left\{\begin{matrix} \textit{the sum of x-component forces acting on the system} \\ \textit{per unit volume} \end{matrix}\right\} \end{aligned} \tag{1.67}$$

Let $(\Delta x \to 0, \Delta y \to 0, \Delta z \to 0)$. In this limit (1.67) becomes

$$\frac{\partial \rho U}{\partial t} + \frac{\partial \rho UU}{\partial x} + \frac{\partial \rho UV}{\partial y} + \frac{\partial \rho UW}{\partial z} = \left\{\begin{matrix} \textit{The sum of} \\ \textit{x-component forces} \\ \textit{per unit volume acting} \\ \textit{on the control volume} \end{matrix}\right\} \tag{1.68}$$

Similar equations describe conservation of momentum in the y and z directions.

$$\begin{aligned} &\frac{\partial \rho V}{\partial t} + \frac{\partial \rho VU}{\partial x} + \frac{\partial \rho VV}{\partial y} + \frac{\partial \rho VW}{\partial z} = \left\{\begin{matrix} \textit{The sum of} \\ \textit{y-component forces} \\ \textit{per unit volume acting} \\ \textit{on the control volume} \end{matrix}\right\} \\ &\frac{\partial \rho W}{\partial t} + \frac{\partial \rho WU}{\partial x} + \frac{\partial \rho WV}{\partial y} + \frac{\partial \rho WW}{\partial z} = \left\{\begin{matrix} \textit{The sum of} \\ \textit{z-component forces} \\ \textit{per unit volume acting} \\ \textit{on the control volume} \end{matrix}\right\} \end{aligned} \tag{1.69}$$

Equations (1.68) and (1.69) are the components of the vector partial differential equation that describes conservation of momentum for any moving continuum.

In index notation, equations (1.68) and (1.69) are concisely written in the form

$$\frac{\partial \rho U_i}{\partial t} + \frac{\partial(\rho U_i U_j)}{\partial x_j} = \left\{ \begin{array}{c} Sum\ of\ the \\ ith\text{-}component\ forces \\ per\ unit\ volume\ acting \\ on\ the\ control\ volume \end{array} \right\} \quad ; \qquad i = 1,2,3 \tag{1.70}$$

where the subscript refers to the i-th vector component and the sum is over the repeated index j. Since the repeated index is always summed it makes no difference what symbol is used and so it is called a dummy index.

Rearrange (1.70) by carrying out the indicated differentiation.

$$\rho \frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial (U_i)}{\partial x_j} + U_i \left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U_j)}{\partial x_j} \right) = \left\{ \begin{array}{c} The\ sum\ of \\ ith\text{-}component\ forces \\ per\ unit\ volume\ acting \\ on\ the\ control\ volume \end{array} \right\} \tag{1.71}$$

The term in parentheses on the left hand side of (1.71) is zero by continuity and so the momentum equation becomes

$$\rho \frac{DU_i}{Dt} = \left\{ \begin{array}{c} The\ sum\ of \\ ith\text{-}component\ forces \\ per\ unit\ volume\ acting \\ on\ the\ control\ volume \end{array} \right\}. \tag{1.72}$$

Equation (1.72) can be stated in words as follows.

$$\left\{ \begin{array}{c} The\ rate\ of\ \ momentum\ change \\ of\ a\ fluid\ element \end{array} \right\} = \left\{ \begin{array}{c} The\ vector\ sum\ of \\ forces\ acting \\ on\ the\ fluid\ element \end{array} \right\} \tag{1.73}$$

Equation (1.72) clearly shows the connection between conservation of momentum for fluid motion and the classical Newton's law $m\boldsymbol{a} = \boldsymbol{F}$.

1.7 Momentum transport due to molecular motion at microscopic scales

Appendix I describes the molecular origin of two of the most basic forces that act on a fluid element; pressure and viscous friction. Although the derivations are restricted to gases, both types of forces exist in all fluid motion and both arise from momentum transfer due to the motion and collisions of the molecules that make up the fluid. Molecules in the fluid move in random and chaotic ways and across any selected section of a surface within the fluid there are molecules moving in both directions through the surface. The pressure and viscosity are defined through an average of the momentum transfer due to the collective motion of a very large set of molecules over a time scale that is short compared to the unsteady macroscopic motion of the fluid.

1.7.1 Pressure

Pressure forces arise from collisions between the molecules within the fluid and with whatever form of containment is used to hold or bound the fluid. In a fluid at rest, in the absence of gravitational forces, the pressure is uniform everywhere. When the fluid is set into motion the pressure becomes a scalar function of space and time. For example, variations in the pressure over a wing give rise to the lift of the wing. Pressure variations in the atmosphere are the primary driving force behind wind patterns. The drag of a motor vehicle is primarily due to variations in the pressure field over the surface of the vehicle.

1.7.2 Viscous friction - Plane Couette Flow

All simple fluids exhibit resistance to flow due to the physical property of viscosity. The effects of viscosity are readily apparent in many everyday activities such as the spreading of honey on a piece of bread or the slow dripping of paint from a brush. Anyone who has purchased oil for their car knows that different grades of oil are characterized by different viscosities and that the viscosity depends on the oil temperature. The viscosity of air and other gases is less apparent but nonetheless just as real. The most important effect of viscosity on the motion of a fluid is that it causes the fluid to stick to the surface of a moving solid body imposing the so-called *no-slip condition* on the velocity field at the surface. Even though the viscosity of air may be very small, viscous forces profoundly affect the flow and are critical to the generation of lift and drag by moving bodies.

To illustrate viscous friction consider a fluid contained between two large parallel plates with area A as shown in Figure 1.9 below. The fluid is initially at rest. At $t = 0$ the upper plate is set into motion and, due to viscous forces, the adjacent fluid layers are dragged along with it. Eventually the velocity field reaches a steady state. If the speed of the upper plate is small compared to the speed of sound in the fluid the final velocity profile is a straight line as shown.

This is called plane Couette flow after Maurice Frederic Alfred Couette a Professor of Physics at the French University of Angers at the end of the nineteenth century. Couette actually studied the flow between inner and outer concentric rotating cylinders with the goal of using his device to measure viscosity.

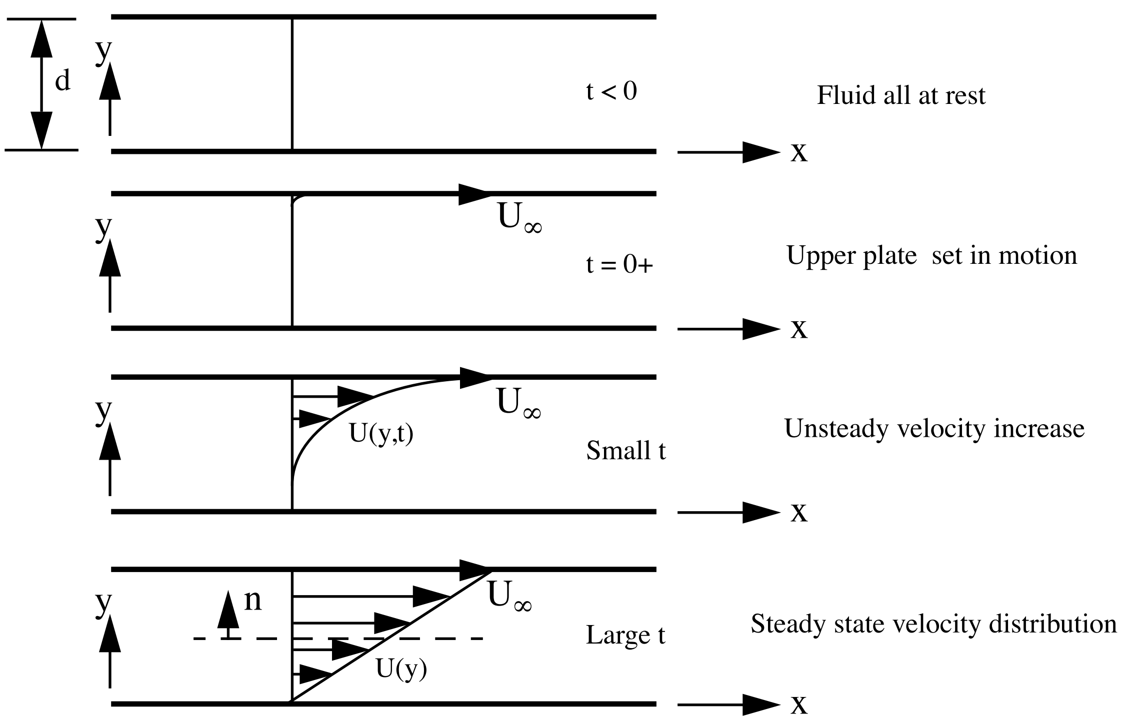


Figure 1.9 Build-up to a steady laminar velocity profile for a viscous fluid contained between two parallel plates. At t=0 the upper plate is set into motion at a constant speed U_∞.

Once the flow reaches steady state a constant force F is required to maintain the motion of the upper plate. Provided the flow is laminar and the plate speed is not too large the force may be expressed as

$$\frac{F}{A} = \mu \frac{U_{\infty}}{d} \tag{1.74}$$

The viscous stress on the plate (*force/area*) is proportional to the velocity gradient and the constant of proportionality is the viscosity of the fluid.

Later we will generalize the notion of viscous stress to flow in three dimensions and so it is useful to write (1.74) in a somewhat more general form. The shear stress acting in the x direction exerted on a layer of fluid with an outward unit normal vector n in the positive y direction (as shown) is designated τ_{xy}. In terms of the velocity derivative (1.74) is written,

$$\tau_{xy} = \mu \frac{dU}{dy} . \tag{1.75}$$

Equation (1.75) is called Newton's law of viscosity after Sir Isaac Newton who first proposed it.

In Chapter 5 we will consider the compressible case where the plate speed is not small and the steady-state velocity profile deviates from a straight line.

1.7.3 A QUESTION OF SIGNS

There is a subtle sign issue that comes up when one defines the viscous stress. In the discussion just above we have adopted the convention normally used in aeronautics that the stress is the force on a surface defined by its outward normal vector. In Figure 1.9 the outward normal on the lower surface is positive and the force is in the positive x-direction; on the upper surface the force is in the negative x-direction on a surface defined by an outward normal that points in the negative y-direction. Either way τ_{xy} is positive as indicated in (1.75).

But in certain fields such as chemical engineering a different interpretation of the stress is used. Here the stress is viewed as the transport of momentum in a certain direction. In this view the flow in Figure 1.9 would be interpreted as the transport of positive x-momentum in the negative y-direction and the expression in (1.75) would have a negative sign. If the general momentum conservation equation is developed with this interpretation of stress then another negative sign appears in front of the stress term. When the constitutive relation between stress and rate-of-strain is introduced both camps wind up solving the same system of equations.

1.7.4 NEWTONIAN FLUIDS

A fluid for which the viscous stress is linearly proportional to the velocity gradient is called Newtonian. Virtually all gases and most liquids are Newtonian. The few exceptions include polymers, pastes and slurries that exhibit non-Newtonian behavior where the stress depends nonlinearly on the rate of strain and may exhibit dependence on the total strain as in a solid. Some fluids are said to have memory where the stress can depend on the past history of the rate-of-strain. We will introduce the Newtonian stress-rate-of-strain relation in Chapter 5. For the present the stress is left in general terms.

A truly exceptional case is liquid Helium that, when cooled to below 2.3°K, undergoes a transition to a strange macroscopic quantum state called Helium II. A given sample of Helium below this temperature consists of two fluids that appear to co-exist. The normal fluid component exhibits viscosity whereas the superfluid component appears to flow without any viscous friction. In fact Helium II is an extraordinarily difficult fluid to contain since the superfluid component can pass through ultrafine leaks in the dewar used to hold it.

Exceptions to the no-slip condition occur in the flow of highly rarefied gases such as in the upper reaches of the atmosphere during re-entry of a space craft. When the density of the gas is extremely low the fluid can no longer be treated as a continuous medium. In this case the flow must be treated by modeling the motion and collisions of individual molecules. Near a wall molecular collisions may not be perfectly specular and may on average exhibit a small difference between the velocity of the surface and the mean velocity of fluid particles near the surface. This is called the slip velocity and can become quite large as the density of the gas decreases to extremely low values.

1.7.5 Forces acting on a fluid element

At macroscopic scales the transport of momentum due to the net molecular motion of a very large number of molecules is equivalent to continuous pressure and viscous forces acting within the fluid. Figure 1.10 below shows the contribution to the x-momentum inside our small control volume from the pressure and viscous stresses acting on the six faces of the control volume.

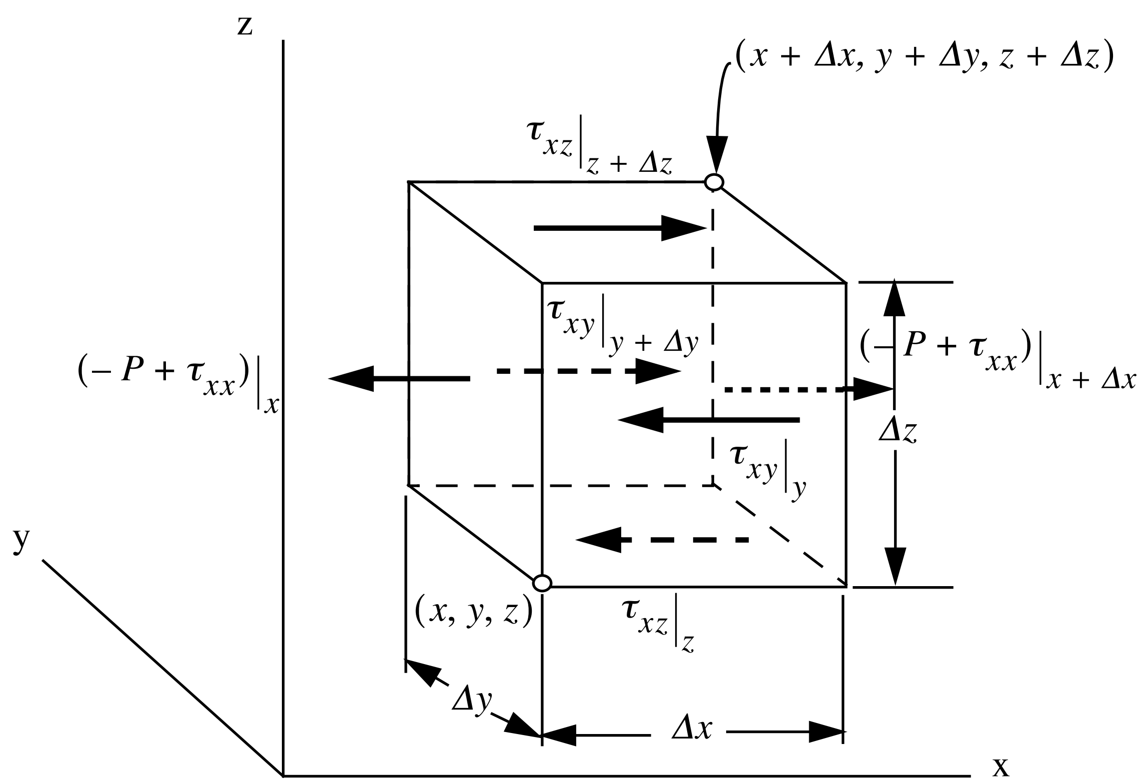


Figure 1.10 Pressure and viscous stresses acting in the x-direction

Consider a pair of faces perpendicular to the x-axis in Figure 1.10. The force on the face the face at x with outward normal in the negative x-direction is $-(-P + \tau_{xx})\big|_x \Delta y \Delta z$ and the force on the face at $x + \Delta x$ with outward normal in the positive x-direction is $(-P + \tau_{xx})\big|_{x + \Delta x} \Delta y \Delta z$. The x-component of force on the face at y due to viscous stress is $-\tau_{xy}\big|_y \Delta x \Delta z$ and the x-component of force on the face at $y + \Delta y$ is $\tau_{xy}\big|_{y + \Delta y} \Delta x \Delta z$. Finally, the x-component of force on the face at z is $-\tau_{xz}\big|_{z} \Delta y \Delta z$ and the x-component of force on the face at $z + \Delta z$ is $\tau_{xz}\big|_{z + \Delta z} \Delta y \Delta z$. Note that the pressure is a scalar that acts normal to any surface in the fluid and therefore there is no component of the pressure acting in the x-direction on the faces perpendicular to the y and z directions. The pressure-viscous stress forces acting in all three directions on the control volume are

$$
\begin{aligned}
F_x &= \Delta y \Delta z\left((-P + \tau_{xx})\big|_{x + \Delta x} - (-P + \tau_{xx})\big|_{x}\right) + \Delta x \Delta z\left(\tau_{xy}\big|_{y + \Delta y} - \tau_{xy}\big|_{y}\right) + \\
&\qquad \Delta x \Delta y\left(\tau_{xz}\big|_{z + \Delta z} - \tau_{xz}\big|_{z}\right) \\
F_y &= \Delta y \Delta z\left(\tau_{xy}\big|_{x + \Delta x} - \tau_{xy}\big|_{x}\right) + \Delta x \Delta z\left((-P + \tau_{yy})\big|_{y + \Delta y} - (-P + \tau_{yy})\big|_{y}\right) + \\
&\qquad \Delta x \Delta y\left(\tau_{yz}\big|_{z + \Delta z} - \tau_{yz}\big|_{z}\right) \\
F_z &= \Delta y \Delta z\left(\tau_{xz}\big|_{x + \Delta x} - \tau_{xz}\big|_{x}\right) + \Delta x \Delta z\left(\tau_{yz}\big|_{y + \Delta y} - \tau_{yz}\big|_{y}\right) + \\
&\qquad \Delta x \Delta y\left((-P + \tau_{zz})\big|_{z + \Delta z} - (-P + \tau_{zz})\big|_{z}\right)
\end{aligned}
\tag{1.76}
$$

With convection, pressure and viscous forces included, the *x*-component momentum balance over the control volume is, from (1.65)

$$\Delta x\Delta y\Delta z\left(\frac{\partial\rho U}{\partial t}\right) = \Delta y\Delta z(\rho UU|_x - \rho UU|_{x+\Delta x}) +$$
$$\Delta x\Delta z(\rho UV|_y - \rho UV|_{y+\Delta y}) + \Delta x\Delta y(\rho UW|_z - \rho UW|_{z+\Delta z}) +$$
$$\Delta y\Delta z\left((-P+\tau_{xx})\Big|_{x+\Delta x} - (-P+\tau_{xx})\Big|_x\right) + \qquad . \tag{1.77}$$
$$\Delta x\Delta z\left(\tau_{xy}\Big|_{y+\Delta y} - \tau_{xy}\Big|_y\right) + \Delta x\Delta y\left(\tau_{xz}\Big|_{z+\Delta z} - \tau_{xz}\Big|_z\right)$$

Divide (1.77) through by $\Delta x\Delta y\Delta z$ and move the pressure and viscous stress terms to the left hand side. The result is

$$\frac{\partial\rho U}{\partial t} + \frac{\rho UU|_{x+\Delta x} - \rho UU|_x + (P-\tau_{xx})\Big|_{x+\Delta x} - (P-\tau_{xx})\Big|_x}{\Delta x} +$$
$$\frac{\rho UV|_{y+\Delta y} - \rho UV|_y - \left(\tau_{xy}\Big|_{y+\Delta y} - \tau_{xy}\Big|_y\right)}{\Delta y} + \tag{1.78}$$
$$\frac{\rho UW|_{z+\Delta z} - \rho UW|_z - \left(\tau_{xz}\Big|_{z+\Delta z} - \tau_{xz}\Big|_z\right)}{\Delta z} = 0$$

Let $(\Delta x \to 0, \Delta y \to 0, \Delta z \to 0)$. In this limit (1.78) becomes

$$\frac{\partial\rho U}{\partial t} + \frac{\partial(\rho UU + P - \tau_{xx})}{\partial x} + \frac{\partial(\rho UV - \tau_{xy})}{\partial y} + \frac{\partial(\rho UW - \tau_{xz})}{\partial z} = 0 \tag{1.79}$$

The equations for conservation of momentum in the y and z directions are derived in a similar way using the expressions in (1.69) and (1.76).

$$\frac{\partial\rho V}{\partial t} + \frac{\partial(\rho VU - \tau_{xy})}{\partial x} + \frac{\partial(\rho VV + P - \tau_{yy})}{\partial y} + \frac{\partial(\rho VW - \tau_{yz})}{\partial z} = 0$$
$$\frac{\partial\rho W}{\partial t} + \frac{\partial(\rho WU - \tau_{xz})}{\partial x} + \frac{\partial(\rho WV - \tau_{yz})}{\partial y} + \frac{\partial(\rho WW + P - \tau_{zz})}{\partial z} = 0 \tag{1.80}$$

In index form, using the Einstein convention, the momentum conservation equation is

$$\boxed{\frac{\partial \rho U_i}{\partial t} + \frac{\partial(\rho U_i U_j)}{\partial x_j} + \frac{\partial P}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} = 0 \quad ; \qquad i = 1,2,3}\,. \tag{1.81}$$

The coordinate independent form of (1.81) is

$$\frac{\partial \rho \overline{U}}{\partial t} + \nabla \bullet (\rho \overline{U}\,\overline{U}) + \nabla P - \nabla \bullet \bar{\bar{\tau}} = 0\,. \tag{1.82}$$

The tensor product of vectors that appears in (1.82), $\overline{U}\,\overline{U} = U_i U_j$, also called a dyadic product, is sometimes written $\overline{U} \otimes \overline{U}$. Fully written out, this product is

$$\overline{U}\,\overline{U} = \begin{bmatrix} UU & UV & UW \\ VU & VV & VW \\ WU & WV & WW \end{bmatrix}. \tag{1.83}$$

Note that the matrix (1.83) is symmetric and as we shall see in Chapter 5, so is the stress tensor τ_{ij}.

1.8 CONSERVATION OF ENERGY

The law of conservation of energy is stated as

$$\begin{Bmatrix} \textit{Rate of} \\ \textit{energy} \\ \textit{accumulation} \\ \textit{inside the} \\ \textit{control volume} \end{Bmatrix} = \begin{Bmatrix} \textit{Rate of} \\ \textit{energy flow} \\ \textit{into the} \\ \textit{control volume} \\ \textit{by convection} \end{Bmatrix} - \begin{Bmatrix} \textit{Rate of} \\ \textit{energy flow} \\ \textit{out of the} \\ \textit{control volume} \\ \textit{by convection} \end{Bmatrix} + \begin{Bmatrix} \textit{Work done on the} \\ \textit{control volume} \\ \textit{by pressure and} \\ \textit{viscous forces} \end{Bmatrix} + \begin{Bmatrix} \textit{Rate of energy} \\ \textit{addition due to} \\ \textit{heat conduction} \end{Bmatrix} + \begin{Bmatrix} \textit{Energy generation} \\ \textit{due to sources} \\ \textit{inside the} \\ \textit{control volume} \end{Bmatrix} \tag{1.84}$$

The energy per unit mass of a moving continuum is $e + k$ where e is the internal energy per unit mass and

$$k = (1/2)(U^2 + V^2 + W^2) \tag{1.85}$$

is the kinetic energy per unit mass. The convection of energy into and out of the control volume is depicted in Figure 1.11.

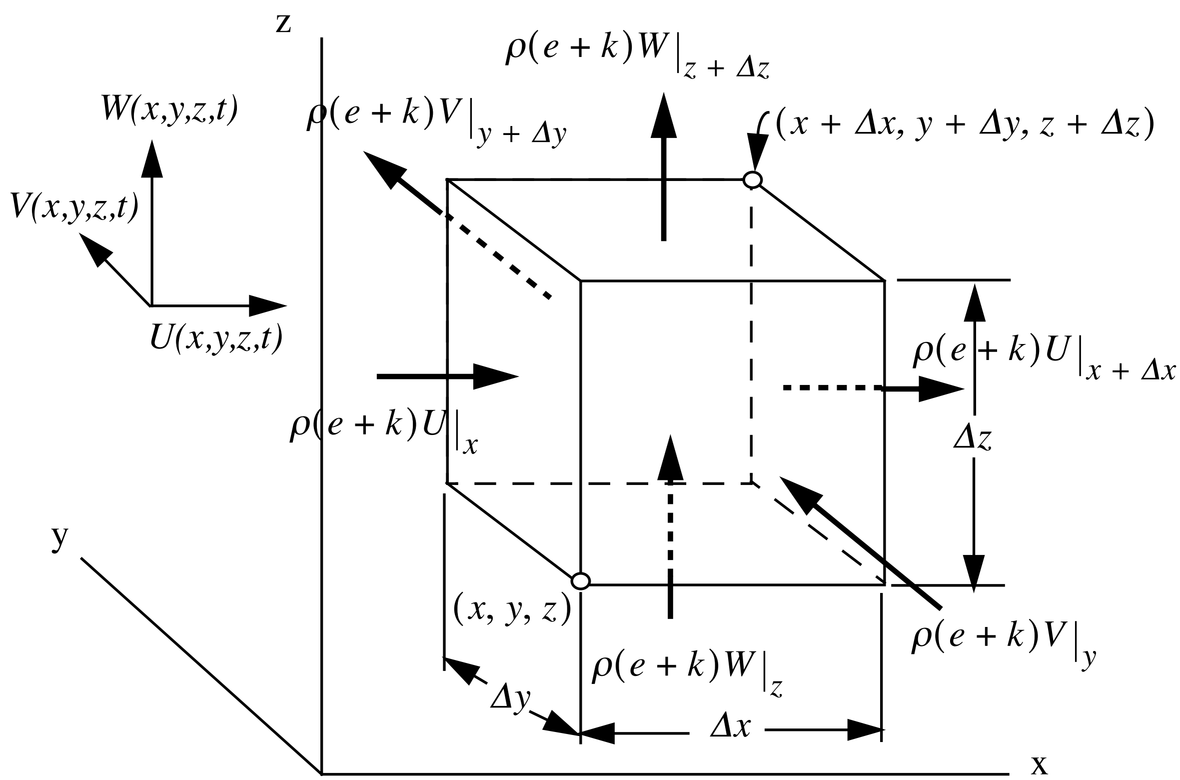


Figure 1.11 Convection of energy into and out of a control volume.

1.8.1 Pressure and viscous stress work

The forces that act on the surface of a control volume do work on the fluid in the control volume and therefore contribute to the energy balance. The power (energy per second) put into the fluid within the control volume is given by the classical relation from mechanics

$$\textit{Power input to the control volume} = \overline{F} \bullet \overline{U} \tag{1.86}$$

Where $\overline{U}$ is the flow velocity and $\overline{F}$ is the net vector force on the control volume by pressure and viscous forces with components (1.76). Fully written out (1.86) is

$$
\begin{aligned}
&\textit{Power input to the control volume} = \\
&\Delta y\Delta z\left\{\left((-P+\tau_{xx})\big|_{x+\Delta x}-(-P+\tau_{xx})\big|_{x}\right)U+\left(\tau_{xy}\big|_{x+\Delta x}-\tau_{xy}\big|_{x}\right)V+\right. \\
&\left.\left(\tau_{xz}\big|_{x+\Delta x}-\tau_{xz}\big|_{x}\right)W\right\}+ \\
&\Delta x\Delta z\left\{\left(\tau_{xy}\big|_{y+\Delta y}-\tau_{xy}\big|_{y}\right)U+\left((-P+\tau_{yy})\big|_{y+\Delta y}-(-P+\tau_{yy})\big|_{y}\right)V+\right. \\
&\left.\left(\tau_{yz}\big|_{y+\Delta y}-\tau_{yz}\big|_{y}\right)W\right\}+ \\
&\Delta x\Delta y\left\{\left(\tau_{xz}\big|_{z+\Delta z}-\tau_{xz}\big|_{z}\right)U+\left(\tau_{yz}\big|_{z+\Delta z}-\tau_{yz}\big|_{z}\right)V+\right. \\
&\left.\left((-P+\tau_{zz})\big|_{z+\Delta z}-(-P+\tau_{zz})\big|_{z}\right)W\right\}
\end{aligned}
\tag{1.87}
$$

The right-hand-side of (1.87) can be parsed into a series of energy fluxes into and out of the various faces of the control volume.

$$
\begin{aligned}
&\textit{Power input to the control volume} = \\
&\Delta y\Delta z\left\{(-PU+\tau_{xx}U+\tau_{xy}V+\tau_{xz}W)\big|_{x+\Delta x}-(-PU+\tau_{xx}U+\tau_{xy}V+\tau_{xz}W)\big|_{x}\right\}+ \\
&\Delta x\Delta z\left\{(-PV+\tau_{xy}U+\tau_{yy}V+\tau_{yz}W)\big|_{y+\Delta y}-(-PV+\tau_{xy}U+\tau_{yy}V+\tau_{yz}W)\big|_{y}\right\}+ \\
&\Delta x\Delta y\left\{(-PW+\tau_{zx}U+\tau_{zy}V+\tau_{zz}W)\big|_{z+\Delta z}-(-PW+\tau_{zx}U+\tau_{zy}V+\tau_{zz}W)\big|_{z}\right\}
\end{aligned}
\tag{1.88}
$$

The contribution of these fluxes to the energy balance over the control volume is illustrated in Figure 1.12.

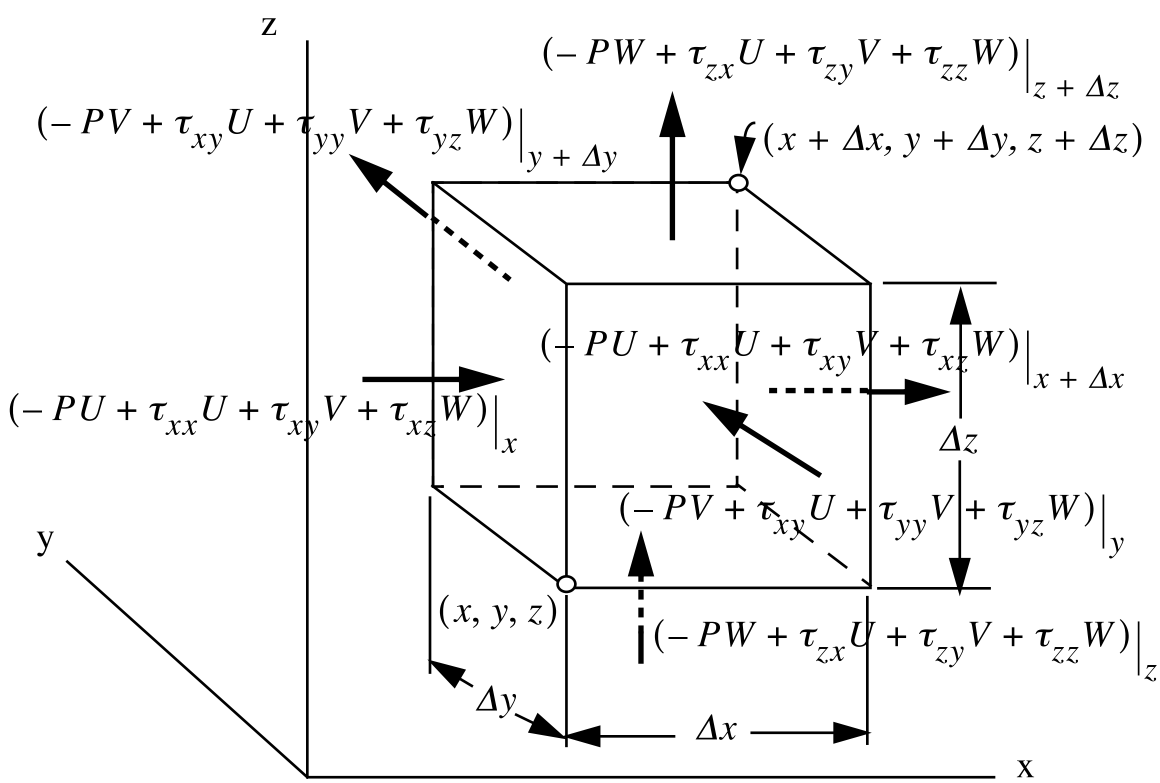


Figure 1.12 Energy fluxes due to the work done on the control volume by pressure and viscous forces.

In addition to the energy input due to work done by pressure and viscous forces one must also include the flux of heat $\overline{Q}$ into and out of the control volume due to thermal conduction.

The rate of internal and kinetic energy accumulation inside the control volume is $\Delta x \Delta y \Delta z(\partial\rho(e+k)/\partial t)$. With all the various sources of energy taken into account, the energy balance (1.84) is expressed mathematically as

$$
\begin{aligned}
\Delta x \Delta y \Delta z\left(\frac{\partial \rho(e+k)}{\partial t}\right) = {} & \Delta y \Delta z(\rho(e+k)U|_x - \rho(e+k)U|_{x+\Delta x}) + \\
& \Delta x \Delta z(\rho(e+k)V|_y - \rho(e+k)V|_{y+\Delta y}) + \\
& \Delta x \Delta y(\rho(e+k)W|_z - \rho(e+k)W|_{z+\Delta z}) + \\
& \Delta y \Delta z\left((-PU + \tau_{xx}U + \tau_{xy}V + \tau_{xz}W)|_{x+\Delta x} - (-PU + \tau_{xx}U + \tau_{xy}V + \tau_{xz}W)|_x\right) \\
& \Delta x \Delta z\left((-PV + \tau_{xy}U + \tau_{yy}V + \tau_{yz}W)|_{y+\Delta y} - (-PV + \tau_{xy}U + \tau_{yy}V + \tau_{yz}W)|_y\right) \\
& \Delta x \Delta y\left((-PW + \tau_{zx}U + \tau_{zy}V + \tau_{zz}W)|_{z+\Delta z} - (-PW + \tau_{zx}U + \tau_{zy}V + \tau_{zz}W)|_z\right) + \\
& \Delta y \Delta z\left(Q_x|_x - Q_x|_{x+\Delta x}\right) + \Delta x \Delta z\left(Q_y|_y - Q_y|_{y+\Delta y}\right) + \Delta x \Delta y\left(Q_z|_z - Q_z|_{z+\Delta z}\right) + \\
& \{\textit{Power generation due to sources inside the control volume}\}
\end{aligned}
\tag{1.89}
$$

Divide (1.89) through by the infinitesimal volume $\Delta x \Delta y \Delta z$ and take the limit $(\Delta x \to 0, \Delta y \to 0, \Delta z \to 0)$. The conservation equation for the energy becomes

$$
\begin{aligned}
& \frac{\partial \rho(e+k)}{\partial t} + \frac{\partial(\rho(e+k)U)}{\partial x} + \frac{\partial(\rho(e+k)V)}{\partial y} + \frac{\partial(\rho(e+k)W)}{\partial z} + \\
& \frac{\partial(PU - \tau_{xx}U - \tau_{xy}V - \tau_{xz}W)}{\partial x} + \frac{\partial(PV - \tau_{xy}U - \tau_{yy}V - \tau_{yz}W)}{\partial y} + \\
& \frac{\partial(PW - \tau_{zx}U - \tau_{zy}V - \tau_{zz}W)}{\partial z} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} = \\
& \{\textit{Power generation due to sources inside the control volume}\}
\end{aligned}
\tag{1.90}
$$

In index form the energy conservation equation is

$$
\frac{\partial \rho(e+k)}{\partial t} + \frac{\partial(\rho(e+k)U_j)}{\partial x_j} + \frac{\partial PU_j}{\partial x_j} - \frac{\partial(U_i \tau_{ij})}{\partial x_j} + \frac{\partial Q_j}{\partial x_j} = \{\textit{ Power sources}\}\,. \tag{1.91}
$$

and the coordinate independent form of (1.91) is

$$
\frac{\partial \rho(e+k)}{\partial t} + \nabla \bullet (\rho(e+k)\overline{U} + P\overline{U} - \bar{\bar{\tau}} \bullet \overline{U} + \overline{Q}) = \{\textit{Power sources}\}\,. \tag{1.92}
$$

Power sources within the control volume can take on a wide variety of forms including chemical reactions, phase change, light absorption from an external source, nuclear reactions, etc.

1.9 SUMMARY - THE EQUATIONS OF MOTION

The combined conservation equations in Cartesian coordinates are

Conservation of mass

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho U}{\partial x} + \frac{\partial\rho V}{\partial y} + \frac{\partial\rho W}{\partial z} = 0$$

Conservation of momentum

$$\begin{aligned}
&\frac{\partial\rho U}{\partial t} + \frac{\partial(\rho UU + P - \tau_{xx})}{\partial x} + \frac{\partial(\rho UV - \tau_{xy})}{\partial y} + \frac{\partial(\rho UW - \tau_{xz})}{\partial z} = 0 \\
&\frac{\partial\rho V}{\partial t} + \frac{\partial(\rho VU - \tau_{xy})}{\partial x} + \frac{\partial(\rho VV + P - \tau_{yy})}{\partial y} + \frac{\partial(\rho VW - \tau_{yz})}{\partial z} = 0 \\
&\frac{\partial\rho W}{\partial t} + \frac{\partial(\rho WU - \tau_{xz})}{\partial x} + \frac{\partial(\rho WV - \tau_{yz})}{\partial y} + \frac{\partial(\rho WW + P - \tau_{zz})}{\partial z} = 0
\end{aligned} \tag{1.93}$$

Conservation of energy

$$\begin{aligned}
&\frac{\partial\rho(e+k)}{\partial t} + \frac{\partial(\rho(e+k)U)}{\partial x} + \frac{\partial(\rho(e+k)V)}{\partial y} + \frac{\partial(\rho(e+k)W)}{\partial z} + \\
&\frac{\partial(PU - \tau_{xx}U - \tau_{xy}V - \tau_{xz}W)}{\partial x} + \frac{\partial(PV - \tau_{xy}U - \tau_{yy}V - \tau_{yz}W)}{\partial y} + \\
&\frac{\partial(PW - \tau_{zx}U - \tau_{zy}V - \tau_{zz}W)}{\partial z} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} = \{Power\ sources\}
\end{aligned}$$

See Appendix 2 for the equations of motion in cylindrical and spherical polar coordinates.

In their current form (1.93) the equations cannot be solved until there is an additional equation that relates the viscous stresses to the velocity field. In Chapter 5 the general form of the Newtonian constitutive law (1.75) will be defined.

Also required is a way of relating the density and pressure that appear in the momentum equation to the internal energy that appears in the energy equation. This will come from thermodynamics and the definition of an equation of state. The heat flux in the energy equation will be related to the fluid temperature using Fick's law for linear heat conduction.

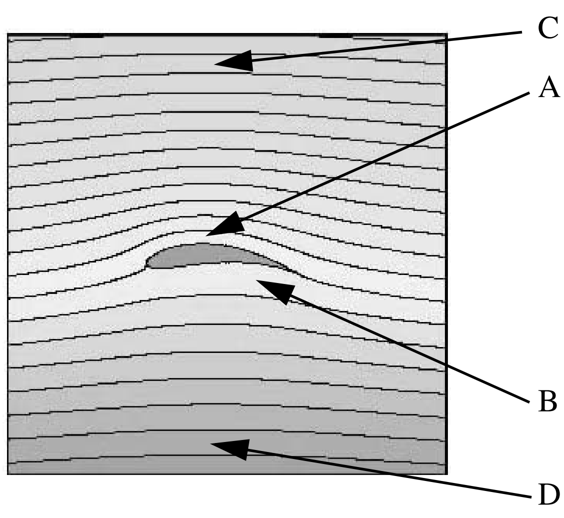
Inclusion of these relations leads to a closed system of governing equations for viscous compressible flow where the number of equations is equal to the number of unknowns. Under certain flow assumptions that are of considerable aeronautical interest it will be possible to directly relate the density and pressure without having to know the temperature. In this case the momentum and mass conservation equations become a closed system that is sufficient to fully describe the flow. The two most common cases where this occurs are incompressible flow where the density is constant and isentropic compressible flow where the pressure is related to the density by a power law.

1.10 Problems

Problem 1 - Show that the continuity equation can be expressed as

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial U_j}{\partial x_j} = 0 \tag{1.94}$$

Problem 2 - Use direct measurements from the streamline figure below to estimate the percent change from the free stream velocity at points A, B, C and D.



Problem 3 - The general, first order, linear ODE

$$\frac{dy}{dx} = -g(x)y + f(x) \tag{1.95}$$

can be written as the differential form

$$(g(x)y - f(x))dx + dy = 0\,. \tag{1.96}$$

Show that (1.96) can be converted to a perfect differential by multiplying by the integrating factor.

$$M = e^{\int g\,dx} \tag{1.97}$$

Work out the solution of (1.95) in terms of integrals. What is the solution for the case $g = \sin x$, $f = \cos x$? Sketch the corresponding streamline pattern.

Problem 4 - Solve

$$y\frac{\partial\Psi}{\partial x} - \frac{x}{3}\frac{\partial\Psi}{\partial y} = 0 \tag{1.98}$$

Sketch the resulting streamline pattern.

Problem 5 - Show that the following expression is a perfect differential.

$$-(\sin x \sin y)dx + (\cos x \cos y)dy = 0 \tag{1.99}$$

Integrate (1.99) to determine the stream function and sketch the corresponding flow pattern. Work out the substantial derivatives of the velocity components and sketch the acceleration vector field.

Problem 6 - Determine the acceleration of a particle in the 1-D velocity field

$$\boldsymbol{u} = \left(k\frac{x}{t}, 0, 0\right) \tag{1.100}$$

where k is constant.

Problem 7 - In a fixed frame of reference a fluid element has the velocity components

$$(U, V, W) = (100, 60, 175)\ meters/sec \tag{1.101}$$

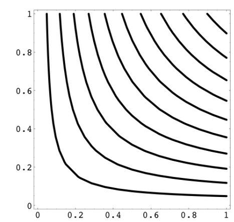
Suppose the same fluid element is observed in a frame of reference moving at

$$\dot{\bar{X}} = (25, 110, 90)\ meters/sec \tag{1.102}$$

with respect to the fixed frame. Determine the velocity components measured by the observer in the moving frame. Determine the kinetic energy per unit mass in each frame.

Problem 8 - The stream function of a steady, 2-D compressible flow in a corner is shown below.

$$\psi = \frac{xy}{1 + x + y}$$



Determine plausible expressions for the velocity components and density field. Does a pressure field exist for this flow if it is assumed to be inviscid?

Problem 9 - The expansion into vacuum of a spherical cloud of a monatomic gas such as helium has a well-known exact solution of the equations for compressible isentropic flow. The velocity field is

$$U = \frac{xt}{t_0^2 + t^2} \qquad V = \frac{yt}{t_0^2 + t^2} \qquad W = \frac{zt}{t_0^2 + t^2}. \tag{1.103}$$

The density and pressure are

$$\frac{\rho}{\rho_0} = \frac{t_0^3}{(t_0^2 + t^2)^{3/2}}\left(1 - \left(\frac{t_0^2}{R_{initial}^2}\right)\frac{x^2 + y^2 + z^2}{t_0^2 + t^2}\right)^{3/2} \tag{1.104}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{5/3}$$

where $R_{initial}$ is the initial radius of the cloud. This problem has served as a model of the expanding gas nebula from an exploding star.

1) Determine the particle paths $(x(t), y(t), z(t))$.

2) Work out the substantial derivative of the density $D\rho/Dt$.

Problem 10 - A moving fluid contains a passive non-diffusing scalar contaminant Smoke in a wind tunnel would be a pretty good example of such a contaminant. Let the concentration of the contaminant be $C(x, y, z, t)$ The units of C are

$$\textit{mass of contaminant/unit mass of fluid} \tag{1.105}$$

Derive a conservation equation for C.

Problem 11 - Include the effects of gravity in the equations of motion (1.93). You can check your answer with the equations derived in Chapter 5.