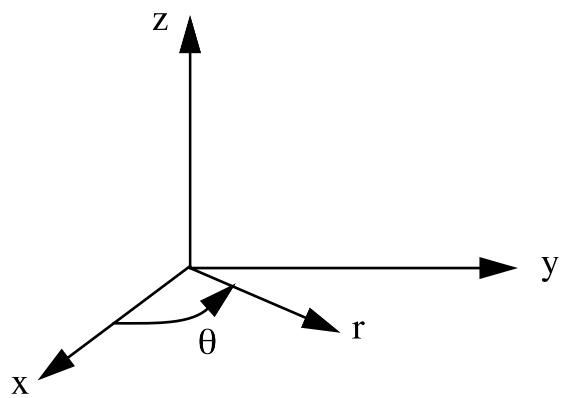
APPENDIX 1

EQUATIONS OF MOTION IN CYLINDRICAL AND SPHERICAL COORDINATES

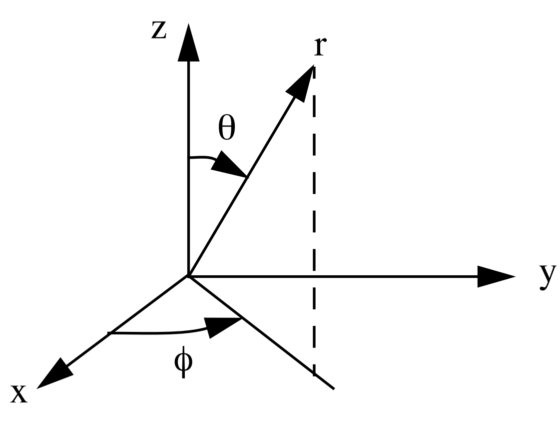
A1.1 COORDINATE SYSTEMS

A1.1.1 CYLINDRICAL COORDINATES



$$x = rCos\theta$$
$$y = rSin\theta$$
$$z = z$$
$$r = (x^2 + y^2)^{1/2}$$
$$\theta = Tan^{-1}(y/x)$$

A1.1.2 SPHERICAL POLAR COORDINATES



$$x = rSin\theta Cos\phi$$
$$y = rSin\theta Sin\phi$$
$$z = rCos\theta$$
$$r = (x^2 + y^2 + z^2)^{1/2}$$
$$\theta = Tan^{-1}\left((x^2 + y^2)^{1/2} / z\right)$$
$$\phi = Tan^{-1}(y/x)$$

A1.2 TRANSFORMATION OF VECTOR COMPONENTS

Basic trigonometry can be used to show that the Cartesian and curvilinear comnponents are related as follows.

A1.2.1 CYLINDRICAL COORDINATES

$$
\begin{aligned}
U_r &= U_x Cos\theta + U_y Sin\theta \\
U_\theta &= -U_x Sin\theta + U_y Cos\theta \\
U_z &= U_z
\end{aligned}
$$

(A1.1)

$$
\begin{aligned}
U_x &= U_r Cos\theta - U_\theta Sin\theta \\
U_y &= U_r Sin\theta + U_\theta Cos\theta \\
U_z &= U_z
\end{aligned}
$$

A1.2.2 SPHERICAL POLAR COORDINATES

$$
\begin{aligned}
U_r &= U_x Sin\theta Cos\phi + U_y Sin\theta Sin\phi + U_z Cos\theta \\
U_\theta &= U_x Cos\theta Cos\phi + U_y Cos\theta Sin\phi - U_z Sin\theta \\
U_\phi &= -U_x Sin\phi + U_y Cos\phi
\end{aligned}
$$

(A1.2)

$$
\begin{aligned}
U_x &= U_r Sin\theta Cos\phi + U_\theta Cos\theta Cos\phi - U_\phi Sin\phi \\
U_y &= U_r Sin\theta Sin\phi + U_\theta Cos\theta Sin\phi + U_\phi Cos\phi \\
U_z &= U_r Cos\theta - U_\theta Sin\theta
\end{aligned}
$$

A1.3 SUMMARY OF DIFFERENTIAL OPERATIONS

A1.3.1 CYLINDRICAL COORDINATES

$$\nabla \bullet \overline{U} = \frac{1}{r}\frac{\partial}{\partial r}(rU_r) + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{\partial U_z}{\partial z} \quad \text{(A1.3)}$$

$$\nabla^2 \rho = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \rho}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \rho}{\partial \theta^2} + \frac{\partial^2 \rho}{\partial z^2} \tag{A1.4}$$

$$\begin{aligned} \bar{\bar{\tau}}:\nabla\bar{U} = {} & \tau_{rr}\left(\frac{\partial U_r}{\partial r}\right) + \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}\right) + \tau_{zz}\left(\frac{\partial U_z}{\partial z}\right) + \\ & \tau_{r\theta}\left(r\frac{\partial}{\partial r}\left(\frac{U_\theta}{r}\right) + \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right) + \tau_{\theta z}\left(\frac{1}{r}\frac{\partial U_z}{\partial \theta} + \frac{\partial U_\theta}{\partial z}\right) + \tau_{rz}\left(\frac{\partial U_z}{\partial r} + \frac{\partial U_r}{\partial z}\right) \end{aligned} \tag{A1.5}$$

$$\begin{aligned} \nabla\rho\big|_r &= \frac{\partial \rho}{\partial r} \\ \nabla\rho\big|_\theta &= \frac{1}{r}\frac{\partial \rho}{\partial \theta} \\ \nabla\rho\big|_z &= \frac{\partial \rho}{\partial z} \end{aligned} \tag{A1.6}$$

$$\begin{aligned} \nabla\times\bar{U}\big|_r &= \frac{1}{r}\frac{\partial U_z}{\partial \theta} - \frac{\partial U_\theta}{\partial z} \\ \nabla\times\bar{U}\big|_\theta &= \frac{\partial U_r}{\partial z} - \frac{\partial U_z}{\partial r} \\ \nabla\times\bar{U}\big|_z &= \frac{1}{r}\frac{\partial}{\partial r}(rU_\theta) - \frac{1}{r}\frac{\partial U_r}{\partial \theta} \end{aligned} \tag{A1.7}$$

$$\begin{aligned} \nabla\bullet\bar{\bar{\tau}}\big|_r &= \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z} \\ \nabla\bullet\bar{\bar{\tau}}\big|_\theta &= \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} \\ \nabla\bullet\bar{\bar{\tau}}\big|_z &= \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} \end{aligned} \tag{A1.8}$$

$$\nabla^2 \bar{U}\Big|_r = \nabla^2 U_r - \frac{2}{r^2}\frac{\partial U_\theta}{\partial \theta} - \frac{U_r}{r^2}$$

$$\nabla^2 \bar{U}\Big|_\theta = \nabla^2 U_\theta + \frac{2}{r^2}\frac{\partial U_r}{\partial \theta} - \frac{U_\theta}{r^2} \qquad \text{(A1.9)}$$

$$\nabla^2 \bar{U}\Big|_z = \nabla^2 U_z$$

where

$$\nabla^2(\ ) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(\ )}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2(\ )}{\partial \theta^2} + \frac{\partial^2(\ )}{\partial z^2} \qquad \text{(A1.10)}$$

$$\bar{U} \bullet \nabla \bar{U}\Big|_r = U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial \theta} - \frac{U_\theta^2}{r} + U_z\frac{\partial U_r}{\partial z}$$

$$\bar{U} \bullet \nabla \bar{U}\Big|_\theta = U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r U_\theta}{r} + U_z\frac{\partial U_\theta}{\partial z} \qquad \text{(A1.11)}$$

$$\bar{U} \bullet \nabla \bar{U}\Big|_z = U_r\frac{\partial U_z}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_z}{\partial \theta} + U_z\frac{\partial U_z}{\partial z}$$

A1.3.2 Spherical polar coordinates

$$\nabla \bullet \bar{U} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_r) + \frac{1}{rSin\theta}\frac{\partial(U_\theta Sin\theta)}{\partial \theta} + \frac{1}{rSin\theta}\frac{\partial U_\phi}{\partial \phi} \qquad \text{(A1.12)}$$

$$\nabla^2 \rho = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \rho}{\partial r}\right) + \frac{1}{r^2 Sin\theta}\frac{\partial}{\partial \theta}\left(Sin\theta\frac{\partial \rho}{\partial \theta}\right) + \frac{1}{r^2 Sin^2\theta}\frac{\partial^2 \rho}{\partial \phi^2} \qquad \text{(A1.13)}$$

$$\bar{\bar{\tau}}:\nabla\bar{U} = \tau_{rr}\left(\frac{\partial U_r}{\partial r}\right) + \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial U_\theta}{\partial\theta} + \frac{U_r}{r}\right) +$$
$$\tau_{\phi\phi}\left(\frac{1}{rSin\theta}\frac{\partial U_\phi}{\partial\phi} + \frac{U_r}{r} + \frac{U_\theta Cot\theta}{r}\right) + \tau_{r\theta}\left(r\frac{\partial}{\partial r}\left(\frac{U_\theta}{r}\right) + \frac{1}{r}\frac{\partial U_r}{\partial\theta}\right) +$$
$$\tau_{r\phi}\left(\frac{\partial U_\phi}{\partial r} + \frac{1}{rSin\theta}\frac{\partial U_r}{\partial\phi} - \frac{U_\phi}{r}\right) + \tau_{\theta\phi}\left(\frac{1}{r}\frac{\partial U_\phi}{\partial\theta} + \frac{1}{rSin\theta}\frac{\partial U_\theta}{\partial\phi} - \frac{U_\phi Cot\theta}{r}\right) \tag{A1.14}$$

$$\nabla\rho\big|_r = \frac{\partial\rho}{\partial r}$$
$$\nabla\rho\big|_\theta = \frac{1}{r}\frac{\partial\rho}{\partial\theta} \tag{A1.15}$$
$$\nabla\rho\big|_\phi = \frac{1}{rSin\theta}\frac{\partial\rho}{\partial\phi}$$

$$\nabla\times\bar{U}\big|_r = \frac{1}{rSin\theta}\frac{\partial(U_\phi Sin\theta)}{\partial\theta} - \frac{1}{rSin\theta}\frac{\partial U_\theta}{\partial\phi}$$
$$\nabla\times\bar{U}\big|_\theta = \frac{1}{rSin\theta}\frac{\partial U_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rU_\phi) \tag{A1.16}$$
$$\nabla\times\bar{U}\big|_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rU_\theta) - \frac{1}{r}\frac{\partial U_r}{\partial\theta}$$

$$\nabla\bullet\bar{\bar{\tau}}\big|_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{rSin\theta}\frac{\partial(\tau_{r\theta}Sin\theta)}{\partial\theta} + \frac{1}{rSin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi} - \left(\frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right)$$
$$\nabla\bullet\bar{\bar{\tau}}\big|_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{rSin\theta}\frac{\partial(\tau_{\theta\theta}Sin\theta)}{\partial\theta} + \frac{1}{rSin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{\tau_{r\theta}}{r} - \frac{Cot\theta}{r}\tau_{\phi\phi} \tag{A1.17}$$
$$\nabla\bullet\bar{\bar{\tau}}\big|_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi}) + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{rSin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{\tau_{r\phi}}{r} + \frac{2Cot\theta}{r}\tau_{\theta\phi}$$

$$\nabla^2\bar{U}\big|_r = \nabla^2 U_r - \frac{2U_r}{r^2} - \frac{2}{r^2}\frac{\partial U_\theta}{\partial\theta} - \frac{2U_\theta Cot\theta}{r^2} - \frac{2}{r^2 Sin\theta}\frac{\partial U_\phi}{\partial\phi}$$

$$\nabla^2\bar{U}\big|_\theta = \nabla^2 U_\theta + \frac{2}{r^2}\frac{\partial U_r}{\partial\theta} - \frac{U_\theta}{r^2 Sin^2\theta} - \frac{2Cos\theta}{r^2 Sin^2\theta}\frac{\partial U_\phi}{\partial\phi} \quad \text{(A1.18)}$$

$$\nabla^2\bar{U}\big|_\phi = \nabla^2 U_\phi - \frac{U_\phi}{r^2 Sin^2\theta} + \frac{2}{r^2 Sin\theta}\frac{\partial U_r}{\partial\phi} + \frac{2Cos\theta}{r^2 Sin^2\theta}\frac{\partial U_\theta}{\partial\phi}$$

where

$$\nabla^2(\ ) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial(\ )}{\partial r}\right) + \frac{1}{r^2 Sin\theta}\frac{\partial}{\partial\theta}\left(Sin\theta\frac{\partial(\ )}{\partial\theta}\right) + \frac{1}{r^2 Sin^2\theta}\frac{\partial^2(\ )}{\partial\phi^2} \quad \text{(A1.19)}$$

$$\bar{U}\bullet\nabla\bar{U}\big|_r = U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial\theta} + \frac{U_\phi}{rSin\theta}\frac{\partial U_r}{\partial\phi} - \frac{(U_\theta^2 + U_\phi^2)}{r}$$

$$\bar{U}\bullet\nabla\bar{U}\big|_\theta = U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial\theta} + \frac{U_\phi}{rSin\theta}\frac{\partial U_\theta}{\partial\phi} + \frac{U_r U_\theta}{r} - \frac{U_\phi^2 Cot\theta}{r} \quad \text{(A1.20)}$$

$$\bar{U}\bullet\nabla\bar{U}\big|_\phi = U_r\frac{\partial U_\phi}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\phi}{\partial\theta} + \frac{U_\phi}{rSin\theta}\frac{\partial U_\phi}{\partial\phi} + \frac{U_r U_\phi}{r} + \frac{U_\theta U_\phi Cot\theta}{r}$$

A1.4 Continuity

A1.4.1 Cylindrical coordinates

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r U_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho U_\theta) + \frac{\partial}{\partial z}(\rho U_z) = 0 \quad \text{(A1.21)}$$

A1.4.2 Spherical polar coordinates

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 U_r) + \frac{1}{rSin\theta}\frac{\partial}{\partial\theta}(\rho U_\theta Sin\theta) + \frac{1}{rSin\theta}\frac{\partial}{\partial\phi}(\rho U_\phi) = 0 \quad \text{(A1.22)}$$

A1.5 MOMENTUM

A1.5.1 CYLINDRICAL COORDINATES

r - component

$$\rho\left(\frac{\partial U_r}{\partial t} + U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial \theta} - \frac{U_\theta^2}{r} + U_z\frac{\partial U_r}{\partial z}\right) + \frac{\partial P}{\partial r} =$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z} + \rho g_r$$

θ - component

$$\rho\left(\frac{\partial U_\theta}{\partial t} + U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r U_\theta}{r} + U_z\frac{\partial U_\theta}{\partial z}\right) + \frac{1}{r}\frac{\partial P}{\partial \theta} = \tag{A1.23}$$

$$\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \rho g_\theta$$

z - component

$$\rho\left(\frac{\partial U_z}{\partial t} + U_r\frac{\partial U_z}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_z}{\partial \theta} + U_z\frac{\partial U_z}{\partial z}\right) + \frac{\partial P}{\partial z} =$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \rho g_z$$

A1.5.2 SPHERICAL POLAR COORDINATES

r - component

$$\rho\left(\frac{\partial U_r}{\partial t} + U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial \theta} + \frac{U_\phi}{rSin\theta}\frac{\partial U_r}{\partial \phi} - \frac{(U_\theta^2 + U_\phi^2)}{r}\right) + \frac{\partial P}{\partial r} =$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{rSin\theta}\frac{\partial(\tau_{r\theta}Sin\theta)}{\partial \theta} + \frac{1}{rSin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} - \left(\frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right) + \rho g_r$$

θ - component

$$\rho\left(\frac{\partial U_\theta}{\partial t} + U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_\phi}{rSin\theta}\frac{\partial U_\theta}{\partial \phi} + \frac{U_r U_\theta}{r} - \frac{U_\phi^2 Cot\theta}{r}\right) + \frac{1}{r}\frac{\partial P}{\partial \theta} = \quad \text{(A1.24)}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{rSin\theta}\frac{\partial(\tau_{\theta\theta}Sin\theta)}{\partial \theta} + \frac{1}{rSin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} - \frac{Cot\theta}{r}\tau_{\phi\phi} + \rho g_\theta$$

ϕ - component

$$\rho\left(\frac{\partial U_\phi}{\partial t} + U_r\frac{\partial U_\phi}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\phi}{\partial \theta} + \frac{U_\phi}{rSin\theta}\frac{\partial U_\phi}{\partial \phi} + \frac{U_r U_\phi}{r} + \frac{U_\theta U_\phi Cot\theta}{r}\right) + \frac{1}{rSin\theta}\frac{\partial P}{\partial \phi} =$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi}) + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{rSin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\phi}}{r} + \frac{2Cot\theta}{r}\tau_{\theta\phi} + \rho g_\phi$$

A1.6 ENERGY EQUATION

Recall that the stagnation enthalpy is

$$h_t = e + \frac{P}{\rho} + k \tag{A1.25}$$

A1.6.1 CYLINDRICAL COORDINATES

$$\begin{aligned}
&\frac{\partial\rho(e+k)}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho h_t U_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho h_t U_\theta) + \frac{\partial}{\partial z}(\rho h_t U_z) + \\
&\left(\frac{1}{r}\frac{\partial}{\partial r}(rQ_r) + \frac{1}{r}\frac{\partial Q_\theta}{\partial\theta} + \frac{\partial Q_z}{\partial z}\right) - \left(\tau_{rr}\frac{\partial U_r}{\partial r} + \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial U_\theta}{\partial\theta} + \frac{U_r}{r}\right) + \tau_{zz}\frac{\partial U_z}{\partial z}\right) - \\
&\left(\tau_{r\theta}\left(r\frac{\partial}{\partial r}\left(\frac{U_\theta}{r}\right) + \frac{1}{r}\frac{\partial U_r}{\partial\theta}\right) + \tau_{rz}\left(\frac{\partial U_z}{\partial r} + \frac{\partial U_r}{\partial z}\right) + \tau_{\theta z}\left(\frac{1}{r}\frac{\partial U_z}{\partial\theta} + \frac{\partial U_\theta}{\partial z}\right)\right) - \\
&U_r\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial\tau_{rz}}{\partial z}\right) - \\
&U_\theta\left(\frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{\partial\tau_{\theta z}}{\partial z}\right) - \\
&U_z\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z}\right) = \{\textit{power sources}\}
\end{aligned} \tag{A1.26}$$

A1.6.2 Spherical polar coordinates

$$
\begin{aligned}
&\frac{\partial\rho(e+k)}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho h_t U_r) + \frac{1}{rSin\theta}\frac{\partial(\rho h_t U_\theta Sin\theta)}{\partial\theta} + \frac{1}{rSin\theta}\frac{\partial(\rho h_t U_\phi)}{\partial\phi} + \\
&\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 Q_r) + \frac{1}{rSin\theta}\frac{\partial(Q_\theta Sin\theta)}{\partial\theta} + \frac{1}{rSin\theta}\frac{\partial Q_\phi}{\partial\phi}\right) - \\
&\tau_{rr}\left(\frac{\partial U_r}{\partial r}\right) - \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial U_\theta}{\partial\theta} + \frac{U_r}{r}\right) - \\
&\tau_{\phi\phi}\left(\frac{1}{rSin\theta}\frac{\partial U_\phi}{\partial\phi} + \frac{U_r}{r} + \frac{U_\theta Cot\theta}{r}\right) - \tau_{r\theta}\left(r\frac{\partial}{\partial r}\left(\frac{U_\theta}{r}\right) + \frac{1}{r}\frac{\partial U_r}{\partial\theta}\right) - \\
&\tau_{r\phi}\left(\frac{\partial U_\phi}{\partial r} + \frac{1}{rSin\theta}\frac{\partial U_r}{\partial\phi} - \frac{U_\phi}{r}\right) - \tau_{\theta\phi}\left(\frac{1}{r}\frac{\partial U_\phi}{\partial\theta} + \frac{1}{rSin\theta}\frac{\partial U_\theta}{\partial\phi} - \frac{U_\phi Cot\theta}{r}\right) - \\
&U_r\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{rSin\theta}\frac{\partial(\tau_{r\theta}Sin\theta)}{\partial\theta} + \frac{1}{rSin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi} - \left(\frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right)\right) - \\
&U_\theta\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{rSin\theta}\frac{\partial(\tau_{\theta\theta}Sin\theta)}{\partial\theta} + \frac{1}{rSin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{\tau_{r\theta}}{r} - \frac{Cot\theta}{r}\tau_{\phi\phi}\right) - \\
&U_\phi\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi}) + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{rSin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{\tau_{r\phi}}{r} + \frac{2Cot\theta}{r}\tau_{\theta\phi}\right) = \{power\ sources\}
\end{aligned}
\tag{A1.27}
$$

A1.7 COMPONENTS OF THE STRESS TENSOR

A1.7.1 CYLINDRICAL COORDINATES

$$\begin{aligned}
\tau_{rr} &= 2\mu\frac{\partial U_r}{\partial r} - \left(\frac{2}{3}\mu - \mu_v\right)\nabla \bullet \bar{U} \\
\tau_{\theta\theta} &= 2\mu\left(\frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}\right) - \left(\frac{2}{3}\mu - \mu_v\right)\nabla \bullet \bar{U} \\
\tau_{zz} &= 2\mu\frac{\partial U_z}{\partial z} - \left(\frac{2}{3}\mu - \mu_v\right)\nabla \bullet \bar{U} \\
\tau_{r\theta} &= \mu\left(r\frac{\partial}{\partial r}\left(\frac{U_\theta}{r}\right) + \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right) \\
\tau_{rz} &= \mu\left(\frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}\right) \\
\tau_{\theta z} &= \mu\left(\frac{\partial U_\theta}{\partial z} + \frac{1}{r}\frac{\partial U_z}{\partial \theta}\right)
\end{aligned} \qquad (A1.28)$$

A1.7.2 SPHERICAL POLAR COORDINATES

$$\begin{aligned}
\tau_{rr} &= 2\mu\frac{\partial U_r}{\partial r} - \left(\frac{2}{3}\mu - \mu_v\right)\nabla \bullet \bar{U} \\
\tau_{\theta\theta} &= 2\mu\left(\frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}\right) - \left(\frac{2}{3}\mu - \mu_v\right)\nabla \bullet \bar{U} \\
\tau_{\phi\phi} &= 2\mu\left(\frac{1}{rSin\theta}\frac{\partial U_\phi}{\partial \phi} + \frac{U_r}{r} + \frac{U_\theta Cot\theta}{r}\right) - \left(\frac{2}{3}\mu - \mu_v\right)\nabla \bullet \bar{U} \\
\tau_{r\theta} &= \mu\left(r\frac{\partial}{\partial r}\left(\frac{U_\theta}{r}\right) + \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right) \\
\tau_{r\phi} &= \mu\left(\frac{1}{rSin\theta}\frac{\partial U_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{U_\phi}{r}\right)\right) \\
\tau_{\theta\phi} &= \mu\left(\frac{Sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{U_\phi}{Sin\theta}\right) + \frac{1}{rSin\theta}\frac{\partial U_\theta}{\partial \phi}\right)
\end{aligned} \qquad (A1.29)$$

A1.8 KINETIC ENERGY DISSIPATION FUNCTION $\varepsilon = \bar{\bar{\tau}} : \nabla \bar{U}$

Cylindrical coordinates

$$\varepsilon = 2\mu\left(\left(\frac{\partial U_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}\right)^2 + \left(\frac{\partial U_z}{\partial z}\right)^2\right) + \mu\left(r\frac{\partial}{\partial r}\left(\frac{U_\theta}{r}\right) + \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right)^2 + \mu\left(\frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}\right)^2 + \mu\left(\frac{\partial U_\theta}{\partial z} + \frac{1}{r}\frac{\partial U_z}{\partial \theta}\right)^2 - \left(\frac{2}{3}\mu - \mu_v\right)(\nabla \bullet \bar{U})^2 \tag{A1.30}$$

Spherical coordinates

$$\varepsilon = 2\mu\left(\left(\frac{\partial U_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}\right)^2 + \left(\frac{1}{rSin\theta}\frac{\partial U_\phi}{\partial \phi} + \frac{U_r}{r} + \frac{U_\theta Cot\theta}{r}\right)^2\right) + \mu\left(r\frac{\partial}{\partial r}\left(\frac{U_\theta}{r}\right) + \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right)^2 + \mu\left(\frac{1}{rSin\theta}\frac{\partial U_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{U_\phi}{r}\right)\right)^2 + \mu\left(\frac{Sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{U_\phi}{Sin\theta}\right) + \frac{1}{rSin\theta}\frac{\partial U_\theta}{\partial \phi}\right)^2 - \left(\frac{2}{3}\mu - \mu_v\right)(\nabla \bullet \bar{U})^2 \tag{A1.31}$$