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AA103
Air and Space Propulsion

Lecture 6 - Multistage Rockets

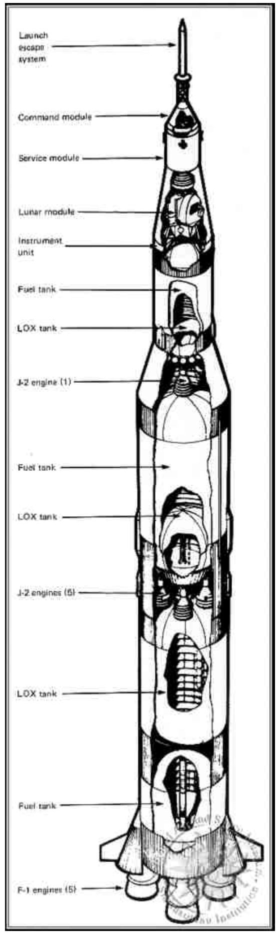
Recommended reading – AA283 course reader chapter 8

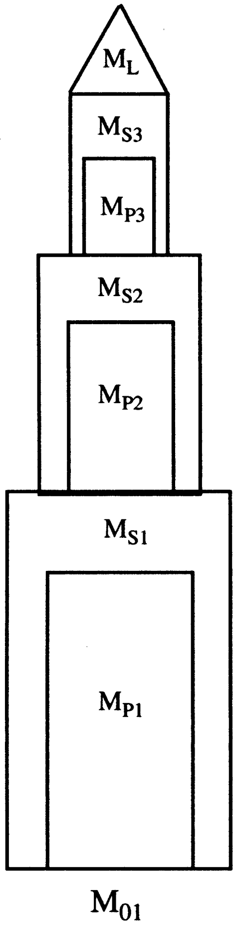
1) With current technology and fuels and without greatly increasing I_{sp} by airbreathing, a single stage rocket to orbit is still not possible.

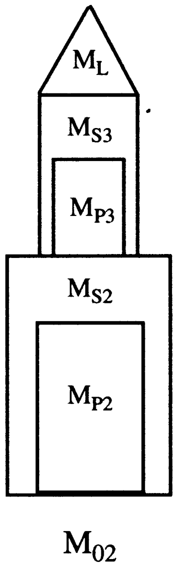
2) The final velocity of an n stage rocket is the sum of the velocity gains from each stage.

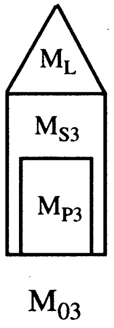
$$V_n = \Delta v_1 + \Delta v_2 + \Delta v_3 + \dots\dots\dots + \Delta v_n \tag{8.1}$$

7.1 Notation









The index i refers to the ith stage

M_{0i} - The total initial mass of the ith stage prior to firing including payload, ie, the mass of i, i+1, i+2, i+3, ..., n stages.

M_{pi} - The mass of propellant in the ith stage.

M_{si} - Structural mass of the ith stage alone including the mass of its engine, controllers and instrumentation as well as any residual propellant which is not expended by the end of the burn.

M_L - The payload

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7.2 Analysis

Payload ratio

$$\lambda_i = \frac{M_{0(i+1)}}{M_{0i} - M_{0(i+1)}}$$

$$\lambda_n = \frac{M_{0(n+1)}}{M_{0n} - M_{0(n+1)}} = \frac{M_L}{M_{0n} - M_L} \qquad (8.2)$$

Structural coefficient

$$\varepsilon_i = \frac{M_{Si}}{M_{0i} - M_{0(i+1)}} = \frac{M_{Si}}{M_{Si} + M_{Pi}} \qquad (8.3)$$

Mass ratio

$$R_i = \frac{M_{0i}}{M_{0i} - M_{Pi}} = \frac{1 + \lambda_i}{\varepsilon_i + \lambda_i} \qquad (8.4)$$

Ideal velocity increment

$$V_n = \sum_{i=1}^{n} C_i \ln R_i = \sum_{i=1}^{n} C_i \ln\left(\frac{1+\lambda_i}{\varepsilon_i+\lambda_i}\right) \tag{8.5}$$

Payload fraction

$$\Gamma = \frac{M_L}{M_{01}} = \left(\frac{M_{02}}{M_{01}}\right)\left(\frac{M_{03}}{M_{02}}\right)\left(\frac{M_{04}}{M_{03}}\right)\cdots\cdots\left(\frac{M_L}{M_{0n}}\right)$$

$$= \left(\frac{\lambda_1}{1+\lambda_1}\right)\left(\frac{\lambda_2}{1+\lambda_2}\right)\left(\frac{\lambda_3}{1+\lambda_3}\right)\cdots\cdots\left(\frac{\lambda_n}{1+\lambda_n}\right) \tag{8.6}$$

or

$$\ln\Gamma = \sum_{i=1}^{n} \ln\left(\frac{\lambda_i}{1+\lambda_i}\right) \tag{8.7}$$

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Apollo Saturn V

TABLE 10.3 *Saturn V Apollo* flight configuration

Mass and thrust features	Stage 1	Stage 2	Stage 3
Engine	F-1	J-2	J-2
Fuel	RP1 (hydrocarbon)	LH_2	LH_2
Oxidant	LO_2	LO_2	LO_2
Number of engines	5	5	1
Total thrust			
lb_f	7,500,000	1,000,000	200,000
kN	33,400	4,450	890
Total initial mass			
lb	6,115,000	1,488,000	473,000
kg	2,780,000	677,000	215,000
Mass of propellant			
lb	4,393,000	943,000	239,000
kg	1,997,000	429,000	109,000
Mass of structure and engines			
lb	234,000	71,600	56,500
kg	106,000	32,600	25,700
ϵ_i	0.050	0.071	0.191
Payload			
lb			178,000
kg			81,100
λ_i	0.321	0.466	0.603

$$C_1 = 2500 \qquad C_2 = 4250 \quad C_3 = 4250$$

$$V_3 = C_1 Ln\left(\frac{1+\lambda_1}{\varepsilon_1+\lambda_1}\right) + C_2 Ln\left(\frac{1+\lambda_2}{\varepsilon_2+\lambda_2}\right) + C_3 Ln\left(\frac{1+\lambda_3}{\varepsilon_3+\lambda_3}\right)$$

$$V_3 = 2500 Ln\left(\frac{1+0.321}{0.05+0.321}\right) + 4250 Ln\left(\frac{1+0.466}{0.071+0.466}\right) + 4250 Ln\left(\frac{1+0.603}{0.191+0.603}\right) = 10429 M/\sec$$

8.4 Example – Effective nozzle exit velocity and structural coefficient the same for all stages.

Let $C = C_i$ and $\varepsilon = \varepsilon_i$ be the same for all stages. In this case V_n is

$$V_n = \sum_{i=1}^{n} C \ln\left(\frac{1+\lambda}{\epsilon+\lambda}\right) = \ln\left(\frac{1+\lambda}{\varepsilon+\lambda}\right)^{nC}$$

In this case the payload ratio λ is independent of the payload fraction Γ.

$$\lambda = \frac{1-\varepsilon e^{\left(\frac{V_n}{nC}\right)}}{e^{\left(\frac{V_n}{nC}\right)}-1}$$

The mass ratio is

$$R = e^{\left(\frac{V_n}{nC}\right)}$$

and the payload fraction is

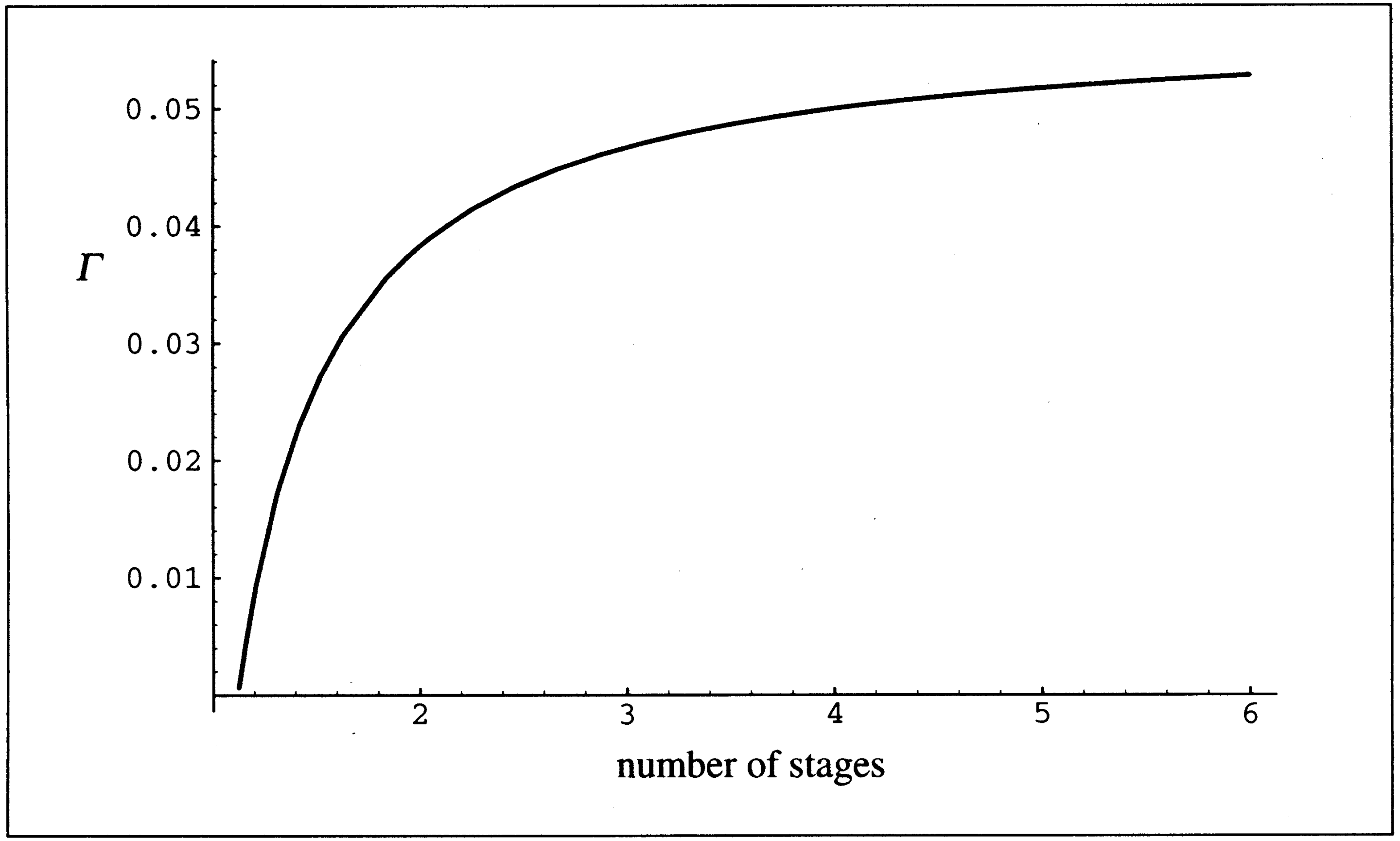
$$\ln\Gamma = \sum_{i=1}^{n} \ln\left(\frac{\lambda}{1+\lambda}\right) = \ln\left(\frac{\lambda}{1+\lambda}\right)^{n}$$

$$\Gamma = \left(\frac{1-\varepsilon e^{\left(\frac{V_n}{nC}\right)}}{(1-\varepsilon)e^{\left(\frac{V_n}{nC}\right)}}\right)^{n}$$

Consider a liquid oxygen, kerosene system. Take the specific impulse to be *360* sec implying $C = 3528$ M/sec. Let $V_n = 9077$ needed to reach orbital speed. The structural coefficient is $\varepsilon = 0.1$ and let the number of stages be $n = 3$. The stage design results are $\alpha = 2696$ M/sec, $\lambda = 0.563$, $R = 2.3575$ and the payload ratio is

$$\Gamma = 0.047 \quad (8.25)$$

Less than 5% of the overall mass of the vehicle is payload.



There is very little advantage to using more than about three stages.

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General case – Effective nozzle exit velocity and structural coefficient NOT the same for all stages.

The final velocity of a multistage system can be expressed as

$$V_n = \sum_{i=1}^{n} C_i \ln\left(\frac{M_{0i}/M_{Pi}}{M_{0i}/M_{Pi}-1}\right)$$

Consider a two stage design

$$V_2 = C_1 Ln\left(\frac{M_{01}/M_{P1}}{M_{01}/M_{P1}-1}\right) + C_2 Ln\left(\frac{M_{02}/M_{P2}}{M_{02}/M_{P2}-1}\right)$$

$$M_{01} = M_{S1} + M_{P1} + M_{S2} + M_{P2} + M_L$$

$$M_{02} = M_{S2} + M_{P2} + M_L$$

$$M_{s1} = \left(\frac{\varepsilon_1}{1-\varepsilon_1}\right) M_{P1} \qquad M_{s2} = \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right) M_{P2} \qquad \Gamma = \frac{M_L}{M_{01}}$$

Express payload mass in terms of propellant masses and payload fraction

$$M_L = \left(\frac{\Gamma}{1-\Gamma}\right)\left(\frac{1}{1-\varepsilon_1}\right) M_{P1} + \left(\frac{\Gamma}{1-\Gamma}\right)\left(\frac{1}{1-\varepsilon_2}\right) M_{P2}$$

Express stage mass ratios in terms of propellant mass ratios

$$\frac{M_{01}}{M_{P1}} = \left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{1}{1-\varepsilon_1}\right) + \left(\frac{1}{1-\varepsilon_2}\right)\frac{M_{P2}}{M_{P1}}\right)$$

$$\frac{M_{02}}{M_{P2}} = \left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{\Gamma}{1-\varepsilon_1}\right)\left(\frac{1}{M_{P2}/M_{P1}}\right) + \left(\frac{1}{1-\varepsilon_2}\right)\right)$$

$$V_2 = C_1 Ln\left(\frac{M_{01}/M_{P1}}{M_{01}/M_{P1} - 1}\right) + C_2 Ln\left(\frac{M_{02}/M_{P2}}{M_{02}/M_{P2} - 1}\right)$$

$$V_2 = C_1 Ln\left(\frac{\left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{1}{1-\varepsilon_1}\right)+\left(\frac{1}{1-\varepsilon_2}\right)\frac{M_{P2}}{M_{P1}}\right)}{\left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{1}{1-\varepsilon_1}\right)+\left(\frac{1}{1-\varepsilon_2}\right)\frac{M_{P2}}{M_{P1}}\right)-1} \right) + C_2 Ln\left(\frac{\left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{\Gamma}{1-\varepsilon_1}\right)\left(\frac{1}{M_{P2}/M_{P1}}\right)+\left(\frac{1}{1-\varepsilon_2}\right)\right)}{\left(\frac{1}{1-\Gamma}\right)\left(\left(\frac{\Gamma}{1-\varepsilon_1}\right)\left(\frac{1}{M_{P2}/M_{P1}}\right)+\left(\frac{1}{1-\varepsilon_2}\right)\right)-1} \right)$$

For given values of

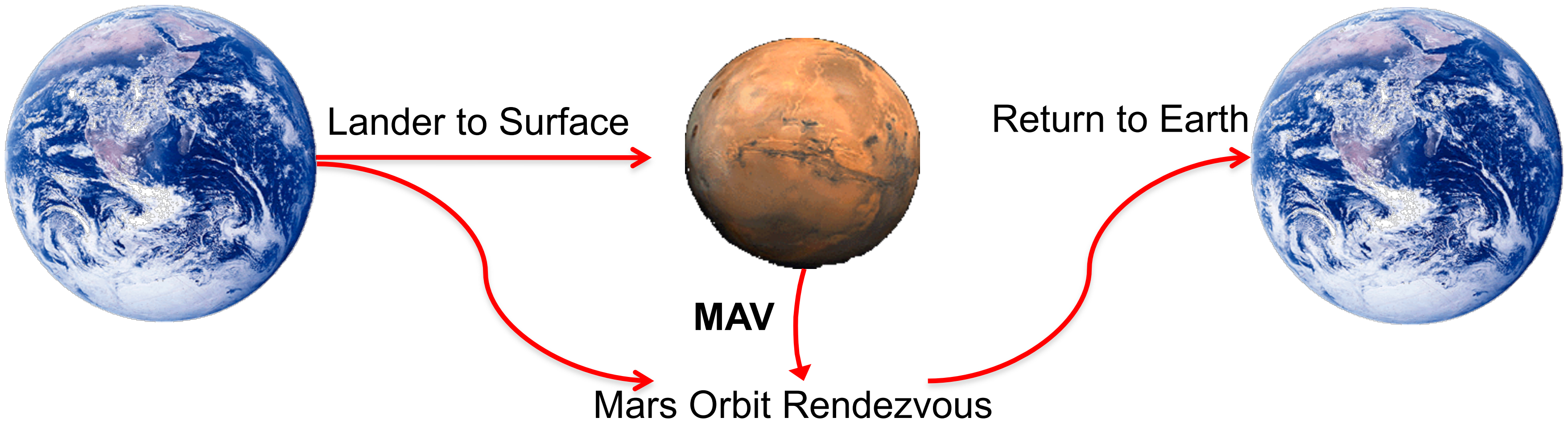
$$\Gamma, C_1, C_2, \varepsilon_1, \varepsilon_2$$

The final velocity is a function of the propellant ratio.

$$V_2 = F\left(\frac{M_{P2}}{M_{P1}}\right)$$

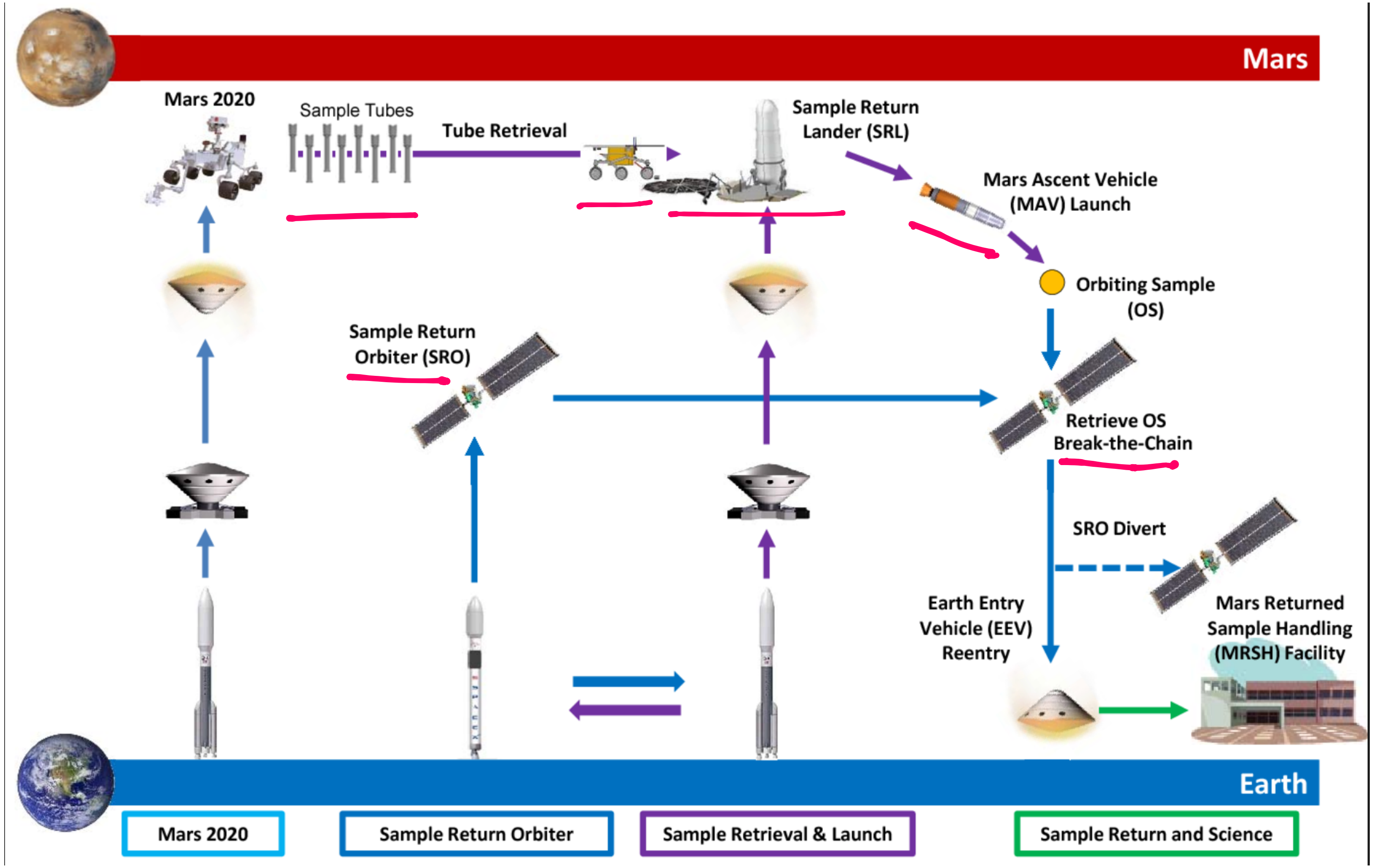
It is now just a matter of differentiating with respect to the propellant ratio to identify a maximum.

Application - Mars Sample Return Campaign 2020-2030(ish)



- Next major step in Mars Science
- Requires international collaboration
- Multiple new developments
  - **Mars Ascent Vehicle (MAV)**
  - Sample acquisition and handling
  - Precision entry descent and landing

Mars Sample Return Mission Architecture



A rover would drill rock samples and place them in individual containers.

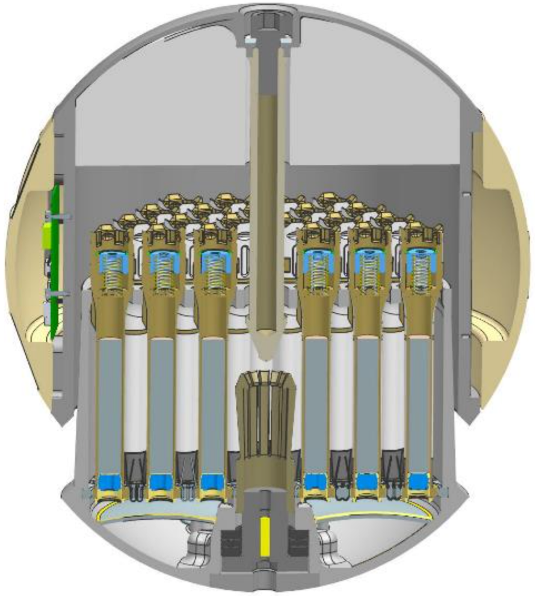
The samples would be left on the surface of Mars for later pick up by a second rover.

The second rover would place the samples in the payload bay of the MAV which would then launch to Mars orbit.

Current OS Reference Design

Mars 2020 Sample Tube Assembly

Engineering OS ready for impact testing



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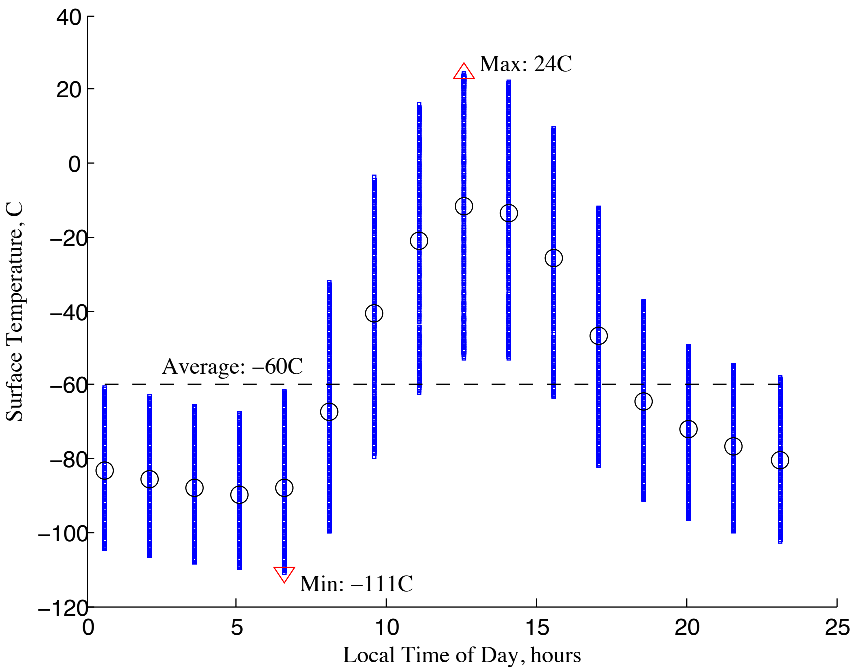
The Mars Ascent Vehicle

The MAV takes a container with Mars rock samples into orbit around Mars. There the container is transferred to another spacecraft for the return journey to Earth.

Critical Challenge: Mars Environmental Conditions

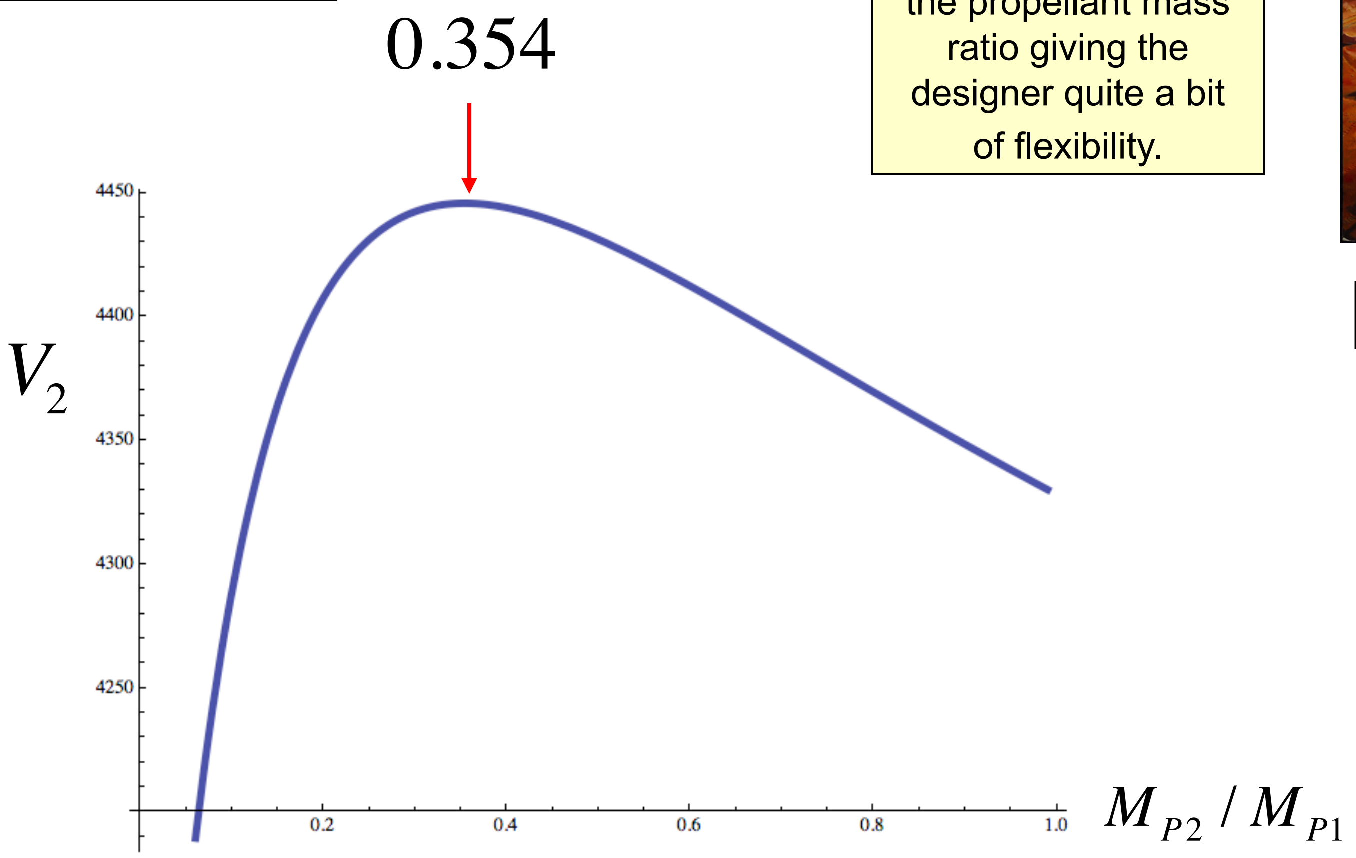
- Diurnal/seasonal minima and maxima (-111C to 24C)



Data from the NASA Ames Research Center Mars Global Climate Model for Holden Crater.

Use a two stage design for the Mars ascent vehicle

One part of Ashley Karp's PhD project



Note that the final velocity is actually quite insensitive to the propellant mass ratio giving the designer quite a bit of flexibility.

Notional values.

$$\varepsilon_1 = 0.13$$

$$\varepsilon_2 = 0.155$$

$$C_1 = 2883$$

$$C_2 = 3026$$

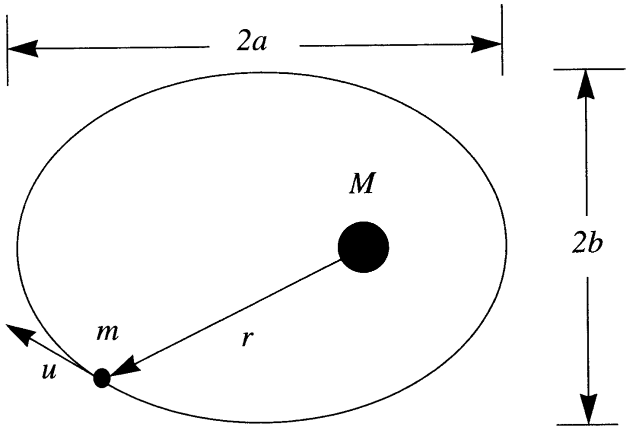
$$\Gamma = 0.147$$

In order to confirm the design it is necessary to fly it to orbit.

Kepler's Equations

Kepler's equations govern the motion of objects near gravitating bodies. This is called the two body problem.

$$\overline{F} = -G\frac{Mm}{r^2}\left(\frac{\overline{r}}{r}\right)$$



Universal gravitational constant

$$G = 6.67 \times 10^{-11} \text{ m}^3/\text{kg-sec}^2$$

$$\ddot{x}(t) + M_{Planet}G\frac{x(t)}{r(t)^3} = 0 \qquad \ddot{y}(t) + M_{Planet}G\frac{y(t)}{r(t)^3} = 0 \qquad \ddot{z}(t) + M_{Planet}G\frac{z(t)}{r(t)^3} = 0$$

$$r(t) = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$$

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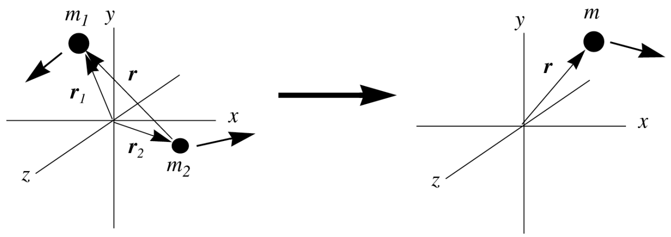
Constants of the motion – two body problem



Reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

Energy

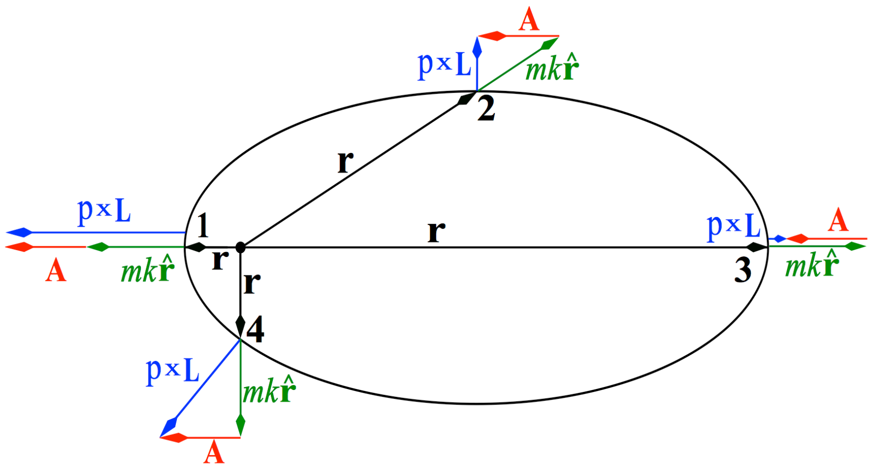
$$E = \frac{1}{2}mv^2 - \frac{k}{r}$$

Angular momentum

$$\bar{L} = \bar{r} \times \bar{p}$$

$$k = Gm_1 m_2$$

The Laplace vector



$$\bar{A} = \bar{p} \times \bar{L} - mk\left(\frac{\bar{r}}{r}\right)$$

Orbital Period

$$\frac{GMT^2}{(r_{\text{mean}})^3} = F\left(\frac{m}{M}, e\right)$$

$$r_{\text{mean}} = \sqrt{ab}$$

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

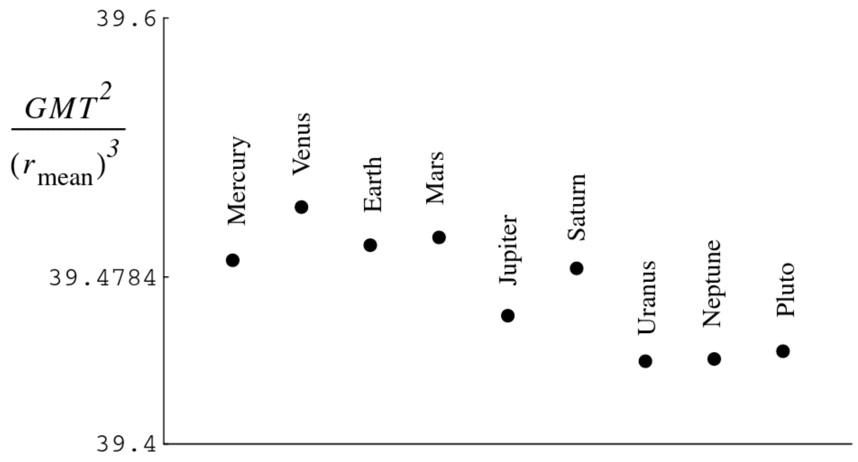
Kepler's theory gives

$$F\left(\frac{m}{M}, e\right) = 4\pi^2\left(\frac{1}{(1 + m/M)(1 - e^2)^{3/4}}\right)$$

Orbital Periods of the Planets about the Sun

Table 2.1. *The planets and their orbits.*

Heavenly body	Mass (Earth masses)	Diameter (Earth diameters)	Mean orbit Radius (10^6 km)	Eccentricity	Orbital period (years)
Sun	332,488.0	109.15	—	—	—
Mercury	0.0543	0.38	57.9	0.2056	0.241
Venus	0.8136	0.967	108.1	0.0068	0.615
Earth	1.0000	1.000	149.5	0.0167	1.000
Mars	0.1069	0.523	227.8	0.0934	1.881
Jupiter	318.35	10.97	777.8	0.0484	11.862
Saturn	95.3	9.03	1426.1	0.0557	29.458
Uranus	14.58	3.72	2869.1	0.0472	84.015
Neptune	17.26	3.38	4495.6	0.0086	164.788
Pluto	<0.1	0.45	5898.9	0.2485	247.697



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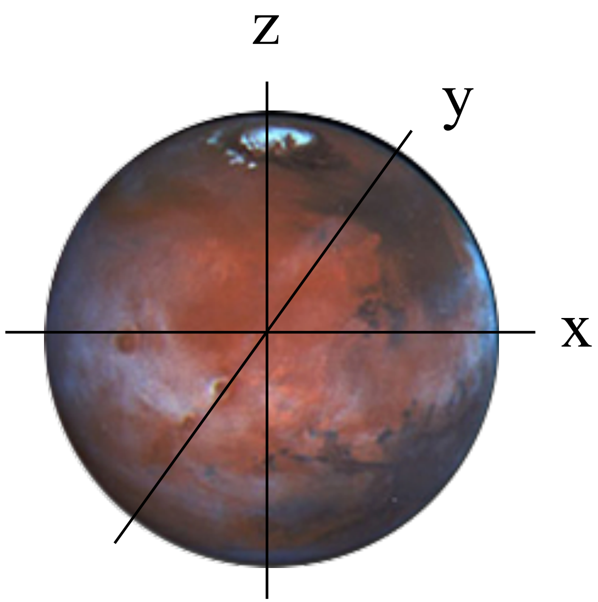
Mars Ascent Vehicle - launch to orbit

Equations of motion

$$\ddot{x}(t) + m_{MARS} G \frac{x(t)}{r(t)^3} + \frac{F_{x_{DRAG}}(t)}{m(t)} - \frac{F_{x_{THRUST}}(t)}{m(t)} = 0$$

$$\ddot{y}(t) + m_{MARS} G \frac{y(t)}{r(t)^3} + \frac{F_{y_{DRAG}}(t)}{m(t)} - \frac{F_{y_{THRUST}}(t)}{m(t)} = 0$$

$$\ddot{z}(t) + m_{MARS} G \frac{z(t)}{r(t)^3} + \frac{F_{z_{DRAG}}(t)}{m(t)} - \frac{F_{z_{THRUST}}(t)}{m(t)} = 0$$



Mars radius=3.376×10^6 m

Mars mass=6.418×10^{23} kg

$$g_{MARS} = m_{MARS} G / r^2 = 3.756 \ \text{m/sec}^2$$

Mars time scale: $\tau_{MARS} = \sqrt{\frac{r^3}{mG}} = 948.03$ sec

Mars velocity scale: $U_{MARS} = \sqrt{\frac{mG}{r}} = 3561$ m/sec

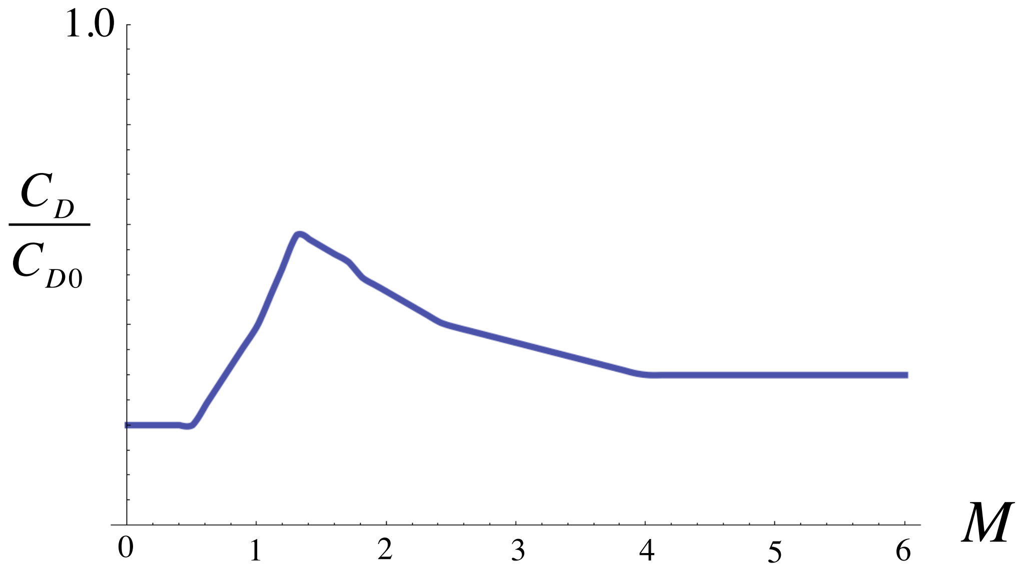
Universal gravitational constant

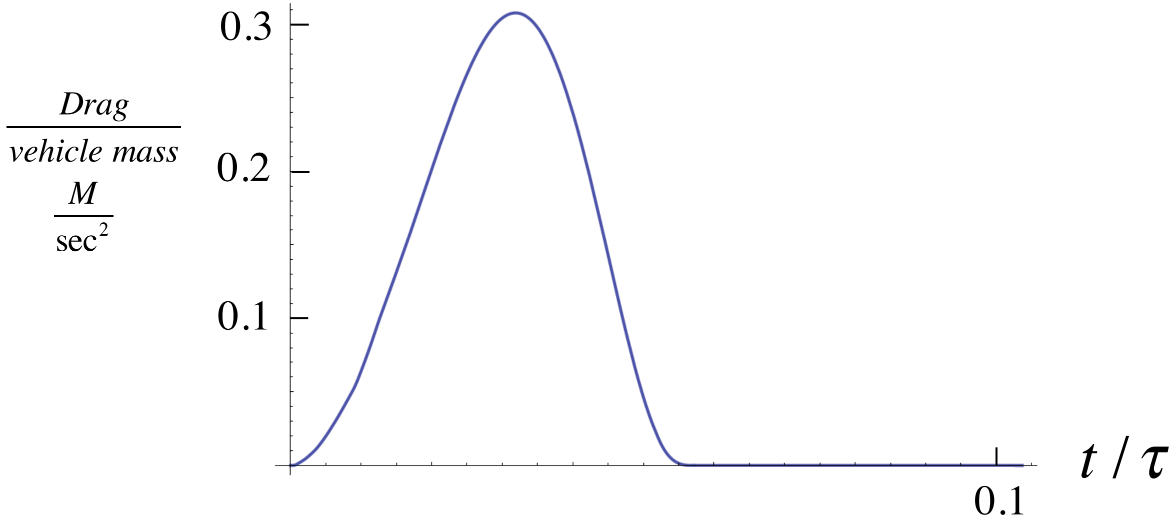
$$G = 6.67 \times 10^{-11} \ \text{m}^3/\text{kg-sec}^2$$

Vehicle mass

$$m(t)$$

Aerodynamic drag





Gravity turn equations

$$\ddot{x}(t)+m_{MARS}G\frac{x(t)}{r(t)^3}+\frac{F_{x_{DRAG}}(t)}{m(t)}-\frac{F_{x_{THRUST}}(t)}{m(t)}=0$$

$$\ddot{y}(t)+m_{MARS}G\frac{y(t)}{r(t)^3}+\frac{F_{y_{DRAG}}(t)}{m(t)}-\frac{F_{y_{THRUST}}(t)}{m(t)}=0$$

$$\ddot{z}(t)+m_{MARS}G\frac{z(t)}{r(t)^3}+\frac{F_{z_{DRAG}}(t)}{m(t)}-\frac{F_{z_{THRUST}}(t)}{m(t)}=0$$

Gravity turn - no drag

$$\bar{F}_{THRUST}\times\bar{V}=0$$

$$\bar{V}=(\dot{x},\dot{y},\dot{z})$$

See the paper *Universal Gravity Turn Trajectories* on my website.

Assume there is no lift on the rocket. Combine thrust and drag.

$$\left(\bar{F}_{THRUST}-\bar{F}_{DRAG}\right)\times\bar{V}=0$$

$$\bar{V}=(\dot{x},\dot{y},\dot{z})$$

$\longrightarrow$

$$\dot{y}(t)F_z(t)-\dot{z}(t)F_y(t)=0$$

$$\dot{z}(t)F_x(t)-\dot{x}(t)F_z(t)=0$$

$$\dot{x}(t)F_y(t)-\dot{y}(t)F_x(t)=0$$

$\longrightarrow$

$$F_y(t)=\frac{\dot{y}(t)}{\dot{x}(t)}F_x(t)$$

$$F_z(t)=\frac{\dot{z}(t)}{\dot{x}(t)}F_x(t)$$

Vehicle acceleration

$$a(t)=\frac{\left(F_x(t)^2+F_y(t)^2+F_z(t)^2\right)^{1/2}}{m(t)}=\frac{F_x(t)}{\dot{x}(t)m(t)}\left(\dot{x}(t)^2+\dot{y}(t)^2+\dot{z}(t)\right)^{1/2}$$

$$\frac{F_x(t)}{m(t)}=\frac{\dot{x}(t)}{\dot{r}(t)}a(t)$$

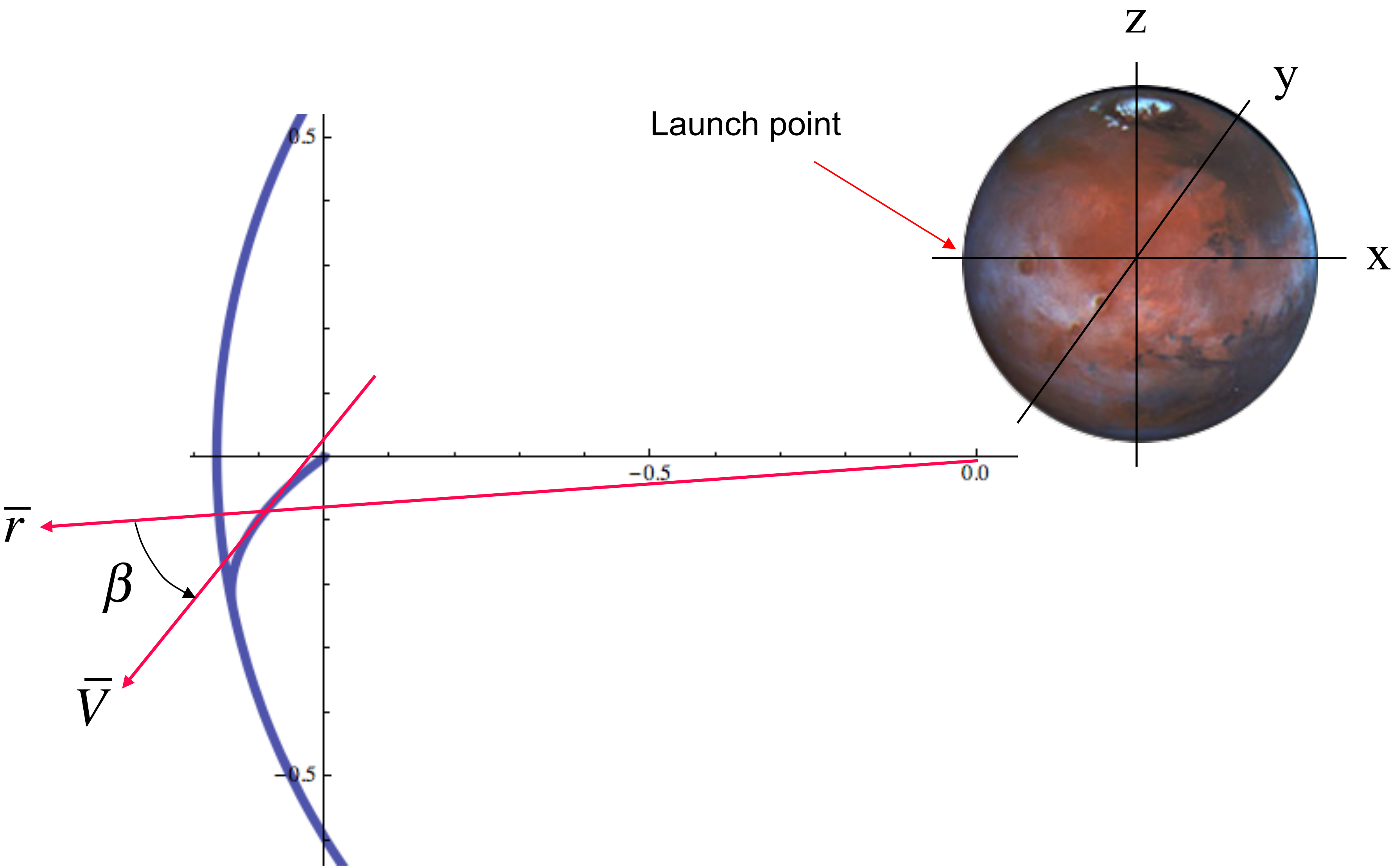
$$\ddot{x}(t)+m_{MARS}G\frac{x(t)}{r(t)^3}-a(t)\frac{\dot{x}(t)}{\dot{r}(t)}=0$$

$$\ddot{y}(t)+m_{MARS}G\frac{y(t)}{r(t)^3}-a(t)\frac{\dot{y}(t)}{\dot{r}(t)}=0$$

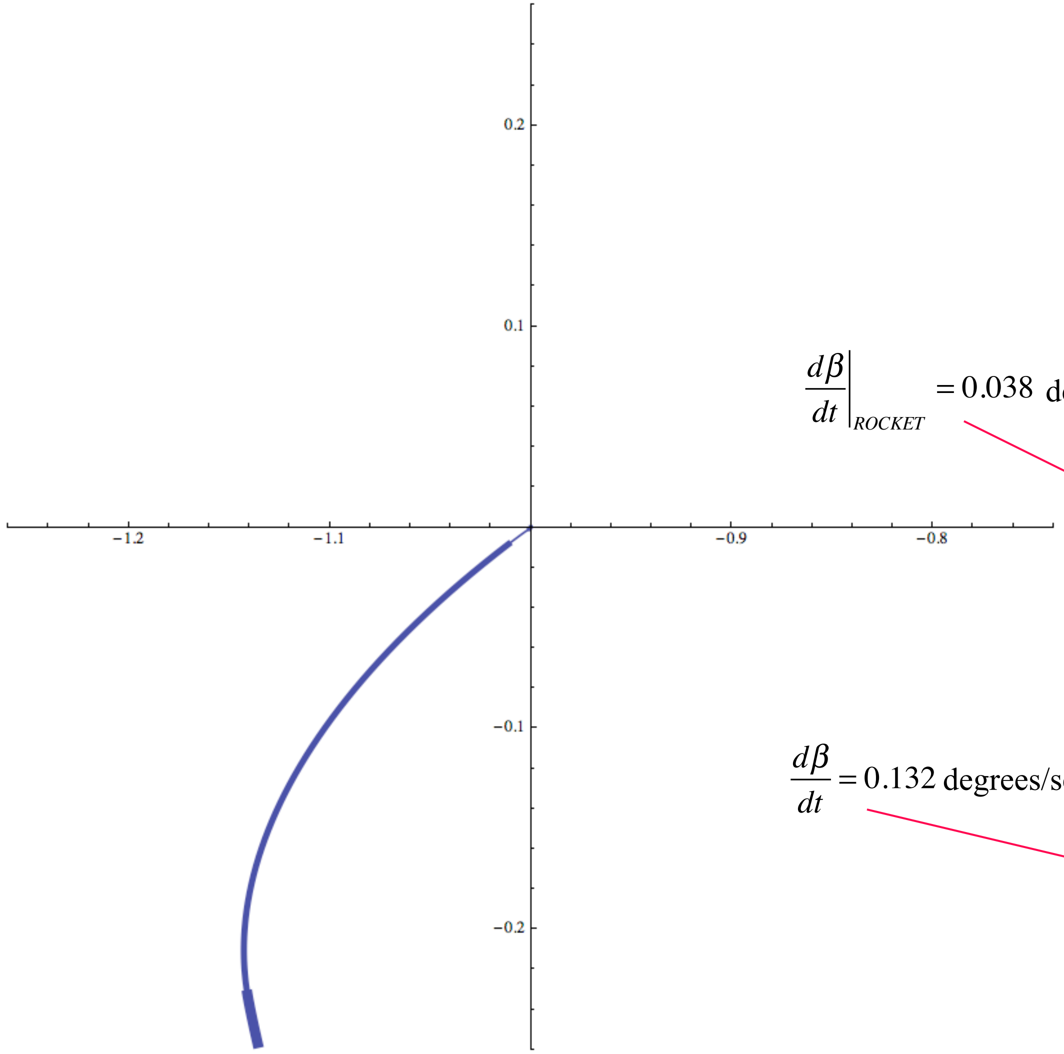
$$\ddot{z}(t)+m_{MARS}G\frac{z(t)}{r(t)^3}-a(t)\frac{\dot{z}(t)}{\dot{r}(t)}=0$$

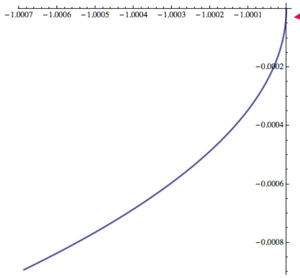
$$g_{MARS}=m_{MARS}G/r^2=3.756\quad \text{m/sec}^2$$

Angle between velocity vector and planet radius



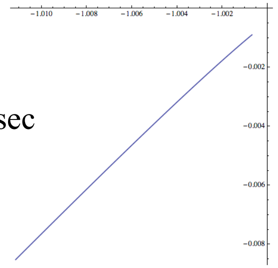
Launch trajectory





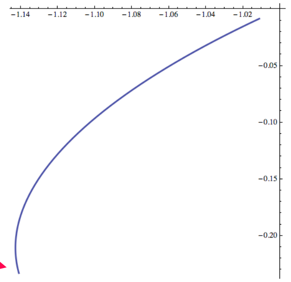
Surface speed at the equator=241.17 m/sec

Vertical launch
12.5 sec



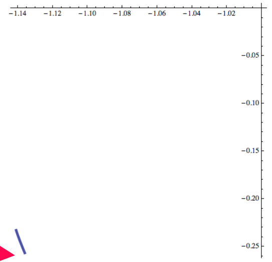
$$\left.\frac{d\beta}{dt}\right|_{ROCKET} = 0.038 \text{ degrees/sec}$$

First stage
gravity turn
36.6 sec



$$\frac{d\beta}{dt} = 0.132 \text{ degrees/sec}$$

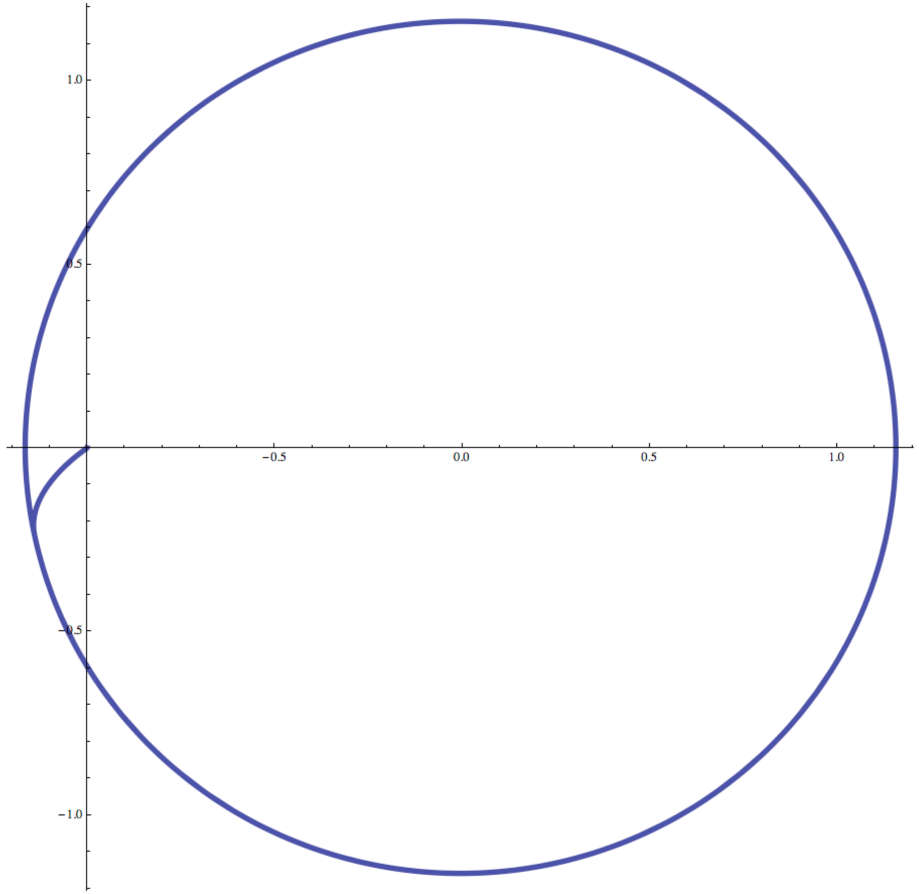
Coast
620 sec



$$\frac{d\beta}{dt} = 0.0007 \text{ degrees/sec}$$

Second stage
gravity turn to orbit
39.4 sec

Launch and orbit trajectory



Maximum altitude

557.9 km

Minimum altitude

527.5 km

Three stage design

$$V_3 = C_1 \ln\left(\frac{M_{01}/M_{P1}}{M_{01}/M_{P1}-1}\right) + C_2 \ln\left(\frac{M_{02}/M_{P2}}{M_{02}/M_{P2}-1}\right) + C_3 \ln\left(\frac{M_{03}/M_{P3}}{M_{03}/M_{P3}-1}\right)$$

The mass ratios can be written in terms of the payload fraction as follows.

$$M_{01}/M_{P1} = \left(\frac{1}{1-\Gamma}\right)\left(\frac{1}{1-\varepsilon_1} + \frac{1}{1-\varepsilon_2}\left(M_{P2}/M_{P1}\right) + \frac{1}{1-\varepsilon_3}\left(M_{P3}/M_{P1}\right)\right)$$

$$M_{02}/M_{P2} = \left(\frac{1}{1-\Gamma}\right)\left(\frac{1}{1-\varepsilon_2} + \frac{\Gamma}{1-\varepsilon_1}\left(\frac{1}{M_{P2}/M_{P1}}\right) + \frac{1}{1-\varepsilon_3}\left(\frac{M_{P3}/M_{P1}}{M_{P2}/M_{P1}}\right)\right)$$

$$M_{03}/M_{P3} = \left(\frac{1}{1-\Gamma}\right)\left(\frac{1}{1-\varepsilon_3} + \frac{\Gamma}{1-\varepsilon_1}\left(\frac{1}{M_{P3}/M_{P1}}\right) + \frac{\Gamma}{1-\varepsilon_2}\left(\frac{M_{P2}/M_{P1}}{M_{P3}/M_{P1}}\right)\right)$$

For given values of

$$\Gamma, C_1, C_2, C_3, \varepsilon_1, \varepsilon_2, \varepsilon_3$$

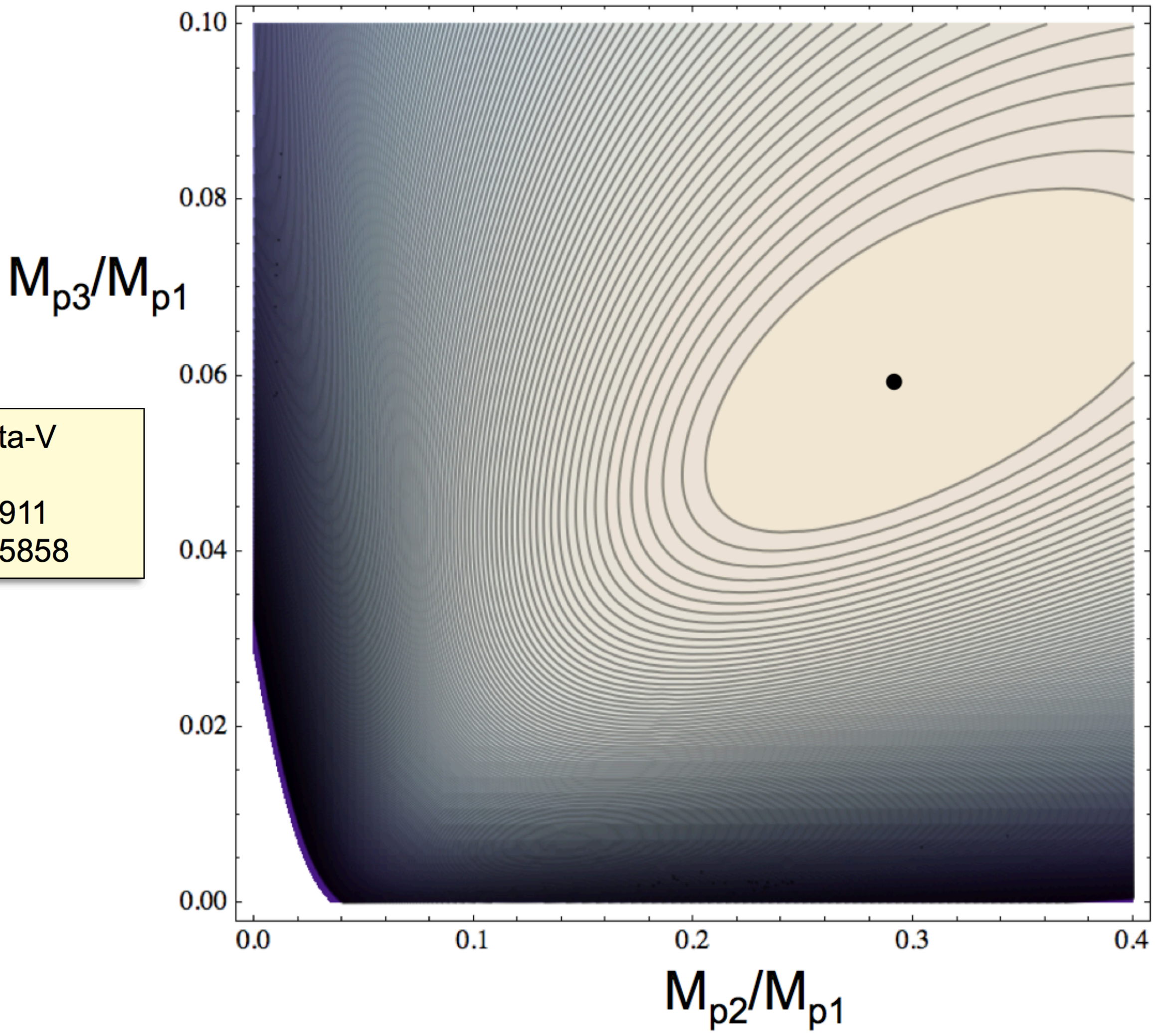
The final velocity is a function of of the propellant ratios.

$$V_3 = F\left(\frac{M_{P2}}{M_{P1}}, \frac{M_{P3}}{M_{P1}}\right)$$

Differentiate with respect to the two variables to identify a maximum.

STANFORD AERONAUTICS & ASTRONAUTICS

Three stage launch vehicle for small satellites



Maximum delta-V occurs at
$M_{p2}/M_{p1} = 0.2911$
$M_{p3}/M_{p1} = 0.05858$

$\varepsilon_1 = 0.116$

$\varepsilon_2 = 0.117$

$\varepsilon_3 = 0.148$

$C_1 = 3040\ m/\sec$

$C_2 = 3210\ m/\sec$

$C_3 = 3350\ m/\sec$

$\Gamma = 0.01125$

$\Delta V = 10{,}800\ m/\sec$

$Payload\ mass\ =\ 85\ kg$

$M_{01} = 7557\ kg$