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AA103
Air and Space Propulsion

Topic 4 – Introduction to gasdynamics – PART 2

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Recall this suggested viewing

National Science Foundation
Fluid Mechanics Films

http://web.mit.edu/fluids/www/Shapiro/ncfmf.html

Fluid Dynamics of Drag, Parts I to IV

Fundamental Boundary Layers

Turbulence

Channel flow of a Compressible Fluid

Waves in Fluids

Pressure Fields and Fluid Acceleration

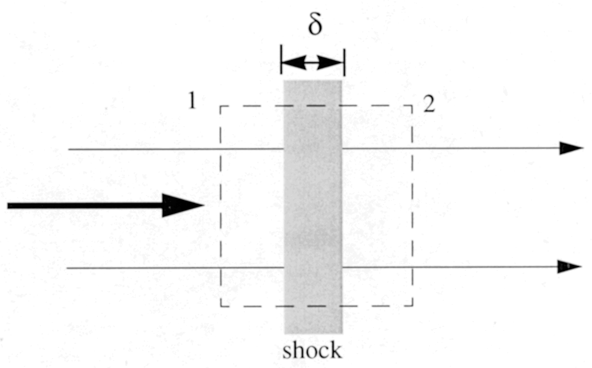
Finally the area averaged equations of motion take the concise form

$$d(\rho UA) = \delta\dot{m}$$

$$d(P - \tau_{xx}) + \rho U dU = -\frac{1}{2}\rho U^2\left(4C_f\frac{dx}{D}\right) + \frac{(U_{xm} - U)\delta\dot{m}}{A} - \frac{\delta F_x}{A}$$

$$d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) = \delta q - \delta w + \left(h_{tm} - \left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right)\right)\frac{\delta\dot{m}}{\rho UA}$$

Normal shock waves



The equations of motion reduce to a set of perfect differentials.

$$\left.\begin{aligned} d(\rho U) &= 0 \\ d(P - \tau_{xx} + \rho U^2) &= 0 \\ d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) &= 0 \end{aligned}\right\}$$

Each equation generates a conserved quantity.

$$\left.\begin{aligned} \rho U &= constant1 \\ P - \tau_{xx} + \rho U^2 &= constant2 \\ h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U} &= constant3 \end{aligned}\right\}$$

Equate conditions at states 1 and 2.

$$\left.\begin{aligned}\rho_1 U_1 &= \rho_2 U_2 \\ P_1 - \tau_{xx1} + \rho_1 U_1^{\,2} &= P_2 - \tau_{xx2} + \rho_2 U_2^{\,2} \\ h_{t1} - \frac{\tau_{xx1}}{\rho_1} + \frac{Q_{x1}}{\rho_1 U_1} &= h_{t2} - \frac{\tau_{xx2}}{\rho_2} + \frac{Q_{x2}}{\rho_2 U_2}\end{aligned}\right\}$$

For a Newtonian fluid

$$\tau_{xx} = \left(\frac{4}{3}\mu + \mu_v\right)\frac{\partial U}{\partial x}$$

and

$$Q_x = -k\frac{\partial T}{\partial x}.$$

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Now assume uniform flow at stations 1 and 2. That is assume that the velocity and temperature gradients are zero ahead of and behind the shock wave.

The classical shock jump conditions are:

$$\rho_1 U_1 = \rho_2 U_2$$

$$P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2$$

$$h_{t1} = h_{t2}$$

Shock waves in a calorically perfect gas

For a calorically perfect gas the third jump condition $h_{t1} = h_{t2}$ can be written

$$C_pT_1 + \frac{1}{2}{U_1}^2 = C_pT_2 + \frac{1}{2}{U_2}^2. \tag{9.46}$$

Consider a reference state where the flow speed equals the local speed of sound. The flow variables at this state will be denoted with a superscript *; Thus $U^* = a^*$, and $\rho = \rho^*$, $P = P^*$, $T = T^*$. Each side of (9.46) can be equated with the reference state.

$$\begin{aligned} C_pT_1 + \frac{1}{2}{U_1}^2 &= \frac{\gamma+1}{2(\gamma-1)}a^{*2} \\ C_pT_2 + \frac{1}{2}{U_2}^2 &= \frac{\gamma+1}{2(\gamma-1)}a^{*2} \end{aligned} \tag{9.47}$$

From the first two jump conditions

$$\rho_1 U_1 = \rho_2 U_2$$
$$P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2$$
$$P_1 + \rho_2 U_2 U_1 = P_2 + \rho_1 U_1 U_2$$
$$P_2 - P_1 = (\rho_2 - \rho_1) U_1 U_2$$

Use the ideal gas law to write (9.47) in the form

$$\begin{aligned} \frac{\gamma}{\gamma-1}P_1 + \frac{1}{2}\rho_2U_2U_1 &= \frac{\gamma+1}{2(\gamma-1)}\rho_1 a^{*2} \\ \frac{\gamma}{\gamma-1}P_2 + \frac{1}{2}\rho_1U_2U_1 &= \frac{\gamma+1}{2(\gamma-1)}\rho_2 a^{*2}. \end{aligned} \tag{9.48}$$

Subtract the relations in (9.48)

$$\frac{\gamma}{\gamma-1}(P_2 - P_1) - \frac{1}{2}(\rho_2 - \rho_1)U_1U_2 = \frac{\gamma+1}{2(\gamma-1)}a^{*2}(\rho_2 - \rho_1). \tag{9.49}$$

Now use (9.39) to replace $P_2 - P_1$ in (9.49) and gather terms. The result is the famous Prandtl relation

$$\boxed{U_1U_2 = a^{*2}.} \tag{9.50}$$

One of the implications of (9.50) is that for a normal shock the flow must be supersonic ahead of the shock and subsonic behind the shock. Note that a^* is essentially defined by (9.47) and can be determined entirely by values in either region 1 ahead of the shock or region 2 behind the shock. Prandtl's relation is the Rosetta stone for generating all of the shock jump relations in terms of the shock Mach number. For example, substitute (9.50) into the first relation in (9.47).

$$C_p T_1 + \frac{1}{2} {U_1}^2 = \frac{\gamma + 1}{2(\gamma - 1)} U_1 U_2 \tag{9.51}$$

Divide (9.51) by ${U_1}^2$. The result is

$$\frac{U_2}{U_1} = \frac{2}{\gamma + 1} \left(\frac{\gamma R T_1}{{U_1}^2} \right) + \left(\frac{\gamma - 1}{\gamma + 1} \right). \tag{9.52}$$

The Mach number ahead of the shock is

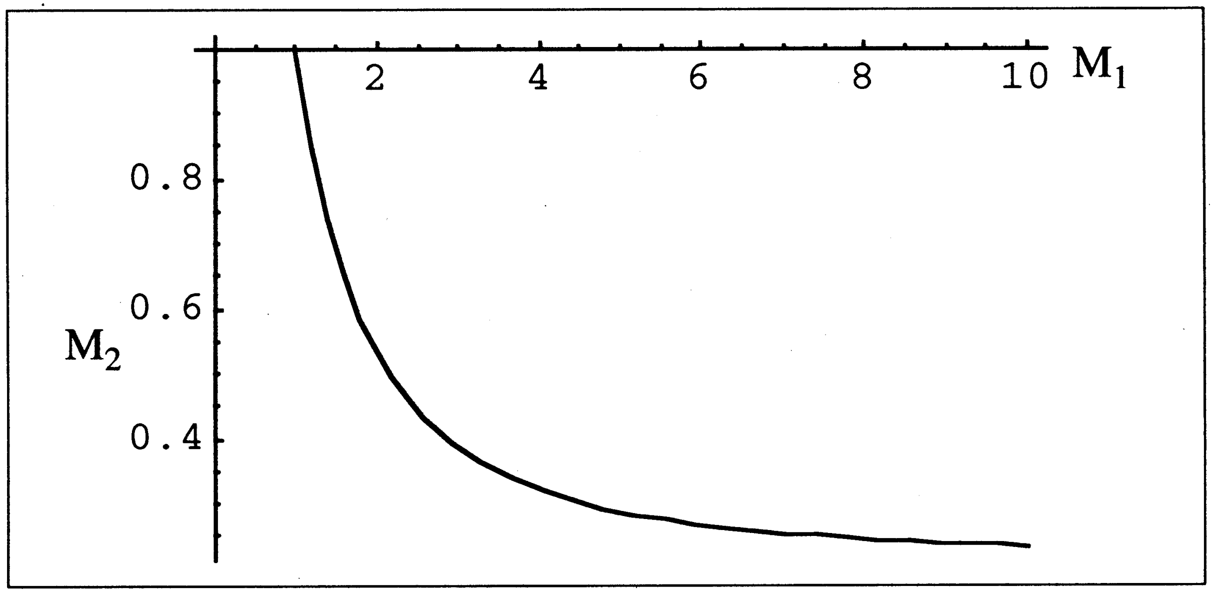
$$M_1 = \frac{U_1}{\sqrt{\gamma R T_1}} \tag{9.53}$$

and (9.52) becomes the basic relation for the velocity ratio (and density ratio) across the shock in terms of the upstream Mach number.

$$\frac{U_2}{U_1} = \frac{1 + \left(\frac{\gamma - 1}{2} \right) {M_1}^2}{\left(\frac{\gamma + 1}{2} \right) {M_1}^2} = \frac{\rho_1}{\rho_2} \tag{9.54}$$

The downstream Mach number

$$M_2^2 = \frac{1 + \frac{\gamma - 1}{2} M_1^2}{\gamma M_1^2 - \left(\frac{\gamma - 1}{2}\right)}.$$



$$\lim_{M_1 \to \infty} M_2 = \sqrt{(\gamma - 1)/(2\gamma)}$$

Shock strength

$$\frac{P_2}{P_1} = \frac{\gamma M_1^2 - \frac{\gamma - 1}{2}}{\frac{\gamma + 1}{2}}.$$

Temperature jump

$$\frac{T_2}{T_1} = \frac{\left(\gamma M_1^2 - \frac{\gamma - 1}{2}\right)\left(1 + \frac{\gamma - 1}{2} M_1^2\right)}{\left(\frac{\gamma + 1}{2}\right)^2 M_1^2}$$

Density jump

$$\frac{\rho_2}{\rho_1} = \frac{\left(\frac{\gamma + 1}{2}\right) M_1^2}{1 + \frac{\gamma - 1}{2} M_1^2}.$$

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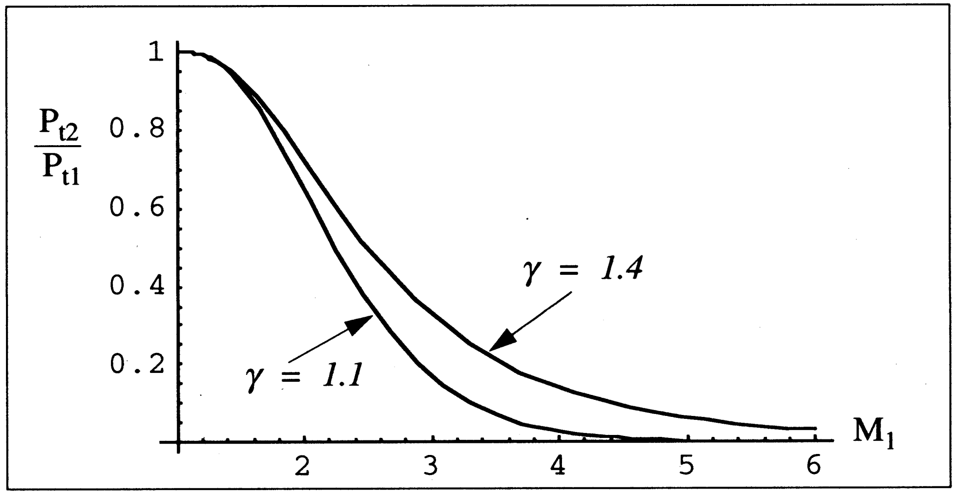
Stagnation pressure ratio

$$\frac{P_{t2}}{P_{t1}} = \left(\frac{\frac{\gamma+1}{\gamma-1}}{\frac{2\gamma}{\gamma-1}{M_1}^2 - 1} \right)^{\frac{1}{\gamma-1}} \left(\frac{\left(\frac{\gamma+1}{2}\right){M_1}^2}{1+\left(\frac{\gamma-1}{2}\right){M_1}^2} \right)^{\frac{\gamma}{\gamma-1}}$$

At high Mach numbers

$$\lim_{M_1 \to \infty} \frac{P_{t2}}{P_{t1}} = \left(\frac{\gamma+1}{2\gamma M_1^2}\right)^{\frac{1}{\gamma-1}} .$$



Entropy change

$$\frac{P_{t2}}{P_{t1}} = e^{-\left(\frac{s_2 - s_1}{R}\right)} .$$

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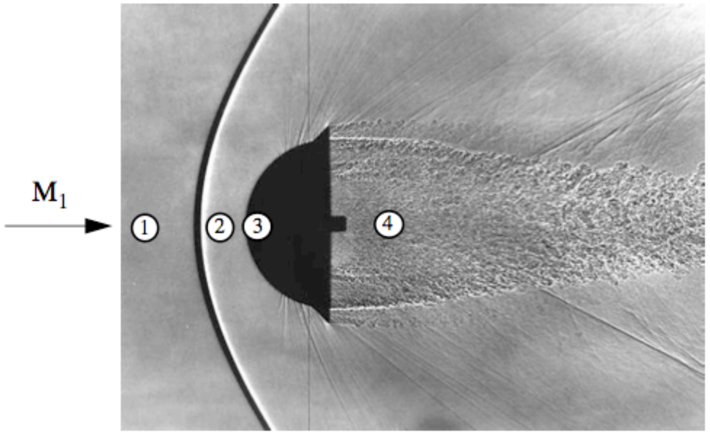
Examples

Problem 2 (15 points) - The figure below shows supersonic flow of air over a model of a re-entry body at a free stream Mach number, $M_1 = 2$.



The temperature of the free stream is $300°K$ and the pressure is one atmosphere.

1) Determine the stagnation temperature and pressure of a fluid element located at stations 1 (free stream), 2 (just behind the shock) and 3 (at the stagnation point on the body). State the assumptions used to solve the problem. Express your answer in °K and atmospheres.

Solution - The stagnation temperature of the free stream is determined from

$$\frac{T_{t1}}{T_1} = 1 + \left(\frac{\gamma - 1}{2}\right)M_1^2 = 1 + 0.2 \times 4 = 1.8 \ \textbf{(1 point)}$$

Therefore

$$T_{t1} = 1.8 \times 300 = 540°K \ \textbf{(1 point)}$$

If we assume the flow up to the stagnation point is adiabatic and the heat capacities are constant then the stagnation temperature is the same at stations 1, 2 and 3,

$$T_{t1} = T_{t2} = T_{t3}. \ \textbf{(1 point)}$$

The stagnation pressure is derived from

$$\frac{P_{t1}}{P_1} = \left(\frac{T_{t1}}{T_1}\right)^{\frac{\gamma}{\gamma - 1}} = 1.8^{3.5} = 7.824 \ \textbf{(1 point)}$$

The freestream stagnation pressure is $P_{t1} = 7.824\ atmospheres$.

Across the shock the stagnation pressure drops according to

$$\frac{P_{t2}}{P_{t1}} = \left(\frac{\frac{\gamma+1}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1}\right)^{\frac{1}{\gamma-1}} \left(\frac{\left(\frac{\gamma+1}{2}\right)M_1^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}} = 0.721 \quad \textbf{(1 point)}$$

The stagnation pressure behind the shock is

$$P_{t2} = 0.721 \times 7.824 = 5.64\ atmospheres. \ \textbf{(1 point)}$$

It is reasonable to assume that the flow from 2 to 3 is adiabatic and isentropic and so one may expect

$$P_{t3} = P_{t2} = 5.64\ atmospheres \ \textbf{(1 point)}$$

2) What can you say about the state of the gas at point 4?

Solution - The flow near the back of the re-entry body is at a low pressure and nearly zero velocity. The path from state 1 to state 4 involves large thermal and velocity gradients leading to an entropy increase and loss of stagnation pressure and very likely a drop in stagnation enthalpy. So we would expect both the pressure and temperature at station 4 to be lower than the free stream stagnation values. **(2 points)**

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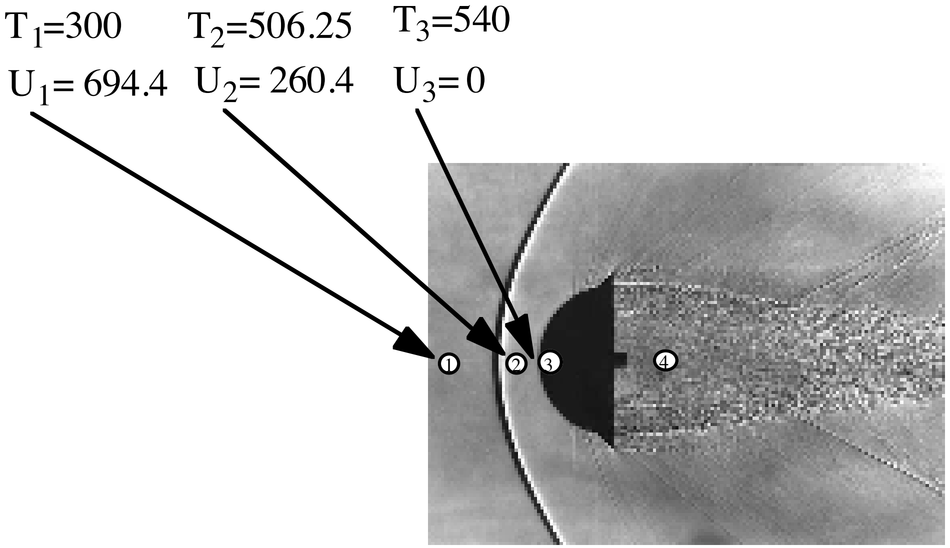
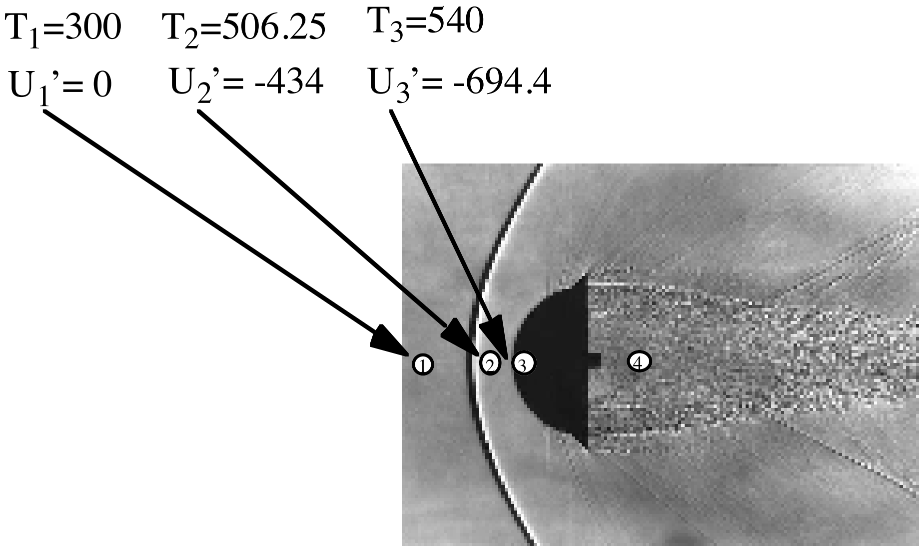


Observer at rest with respect to the body



Observer at rest with respect to the upstream gas

9.3.4 Example - stagnation at a leading edge in supersonic flow

The figure below shows a supersonic flow of Helium (atomic weight equals 4) over the leading edge of a thick flat plate at a free stream Mach number $M_1 = 2.0$.

The temperature of the free stream is $300\,K$ and the pressure is one atmosphere.

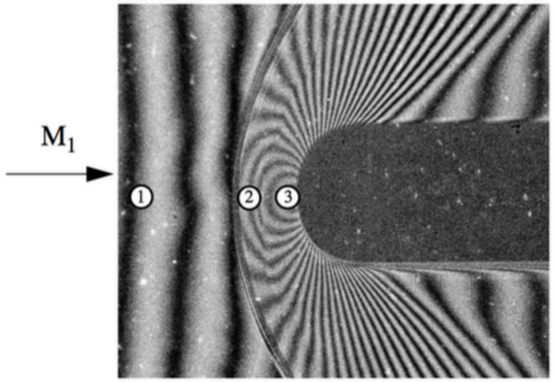


Figure 9.7: *Supersonic flow of helium over a leading edge.*

1) Determine the *energy* per unit mass of a fluid element located at points 1 (free stream), 2 (just behind the shock) and 3 (at the stagnation point on the body). State the assumptions used to solve the problem. Express your answer in Joules/kg.

Solution

The energy per unit mass of a flowing gas is the sum of internal energy and kinetic energy per unit mass, e + k .

a) Assume the gas is calorically perfect - constant heat capacities.

b) Assume the flow is adiabatic from station 1 to station 3.

c) Assume the body is adiabatic.

For Helium the number of degrees of freedom equals 3 and at the conditions of the free stream we have the following values.

$$
\begin{aligned}
R &= \frac{8314.472}{4} = 2078.62\, m^2/\mathrm{sec}^2 - K \\
C_v &= \frac{3}{2} R = 3117.93\, m^2/\mathrm{sec}^2 - K \\
C_p &= \frac{3+2}{2} R = 5196.55\, m^2/\mathrm{sec}^2 - K \\
\gamma &= 5/3 \\
a &= \sqrt{\gamma R T} = \sqrt{\frac{5}{3} 2078.62\,(300)} = 1019.46\, m^2/\mathrm{sec}^2 \\
U_1 &= 2\,(1019.46) = 2038.92\, m/\mathrm{sec} \\
e_1 + k_1 &= C_v T_1 + \frac{1}{2} {U_1}^2 = 3117.93\,(300) + 0.5(2038.92)^2 = 935379 + 2078597 \\
e_1 + k_1 &= 3013976\, J/kg
\end{aligned}
\tag{9.70}
$$

The stagnation temperature of the free stream is determined from

$$
\frac{T_t}{T} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2. \tag{9.71}
$$

Thus

$$
T_{t1} = 300 \left(1 + \frac{4}{3}\right) = 700\, K \tag{9.72}
$$

Across a normal shock at Mach 2 the temperature ratio is

$$
\frac{T_2}{T_1} = \frac{\left(1 + \left(\frac{\gamma-1}{2}\right) {M_1}^2\right)\left(\gamma {M_1}^2 - \left(\frac{\gamma-1}{2}\right)\right)}{\left(\frac{\gamma+1}{2}\right)^2 {M_1}^2} \tag{9.73}
$$

which gives

$$\frac{T_2}{T_1} = \frac{\left(1+\frac{4}{3}\right)\left(\frac{5}{3}(4)-\left(\frac{1}{3}\right)\right)}{\left(\frac{4}{3}\right)^2(4)} = \frac{\left(\frac{7}{3}\right)\left(\frac{19}{3}\right)}{\left(\frac{4}{3}\right)\left(\frac{16}{3}\right)} = \frac{7}{4}\left(\frac{19}{16}\right) = 2.078. \tag{9.74}$$

Assume the flow is adiabatic from the free stream to the stagnation point.

$$h_1 + \frac{1}{2}{U_1}^2 = h_2 + \frac{1}{2}{U_2}^2 = h_3 + \frac{1}{2}{U_3}^2 \tag{9.75}$$

We can rewrite this equation as follows.

$$RT_1 + (e_1 + k_1) = RT_2 + (e_2 + k_2) = RT_3 + (e_3 + k_3) \tag{9.76}$$

The temperatures at stations 1, 2 and 3 are respectively

$$\begin{aligned} T_1 &= 300\,K \\ T_2 &= 2.078\,(300) = 623.44\,K \\ T_3 &= T_{t1} = 700\,K. \end{aligned} \tag{9.77}$$

Now

$$\begin{aligned} &e_1 + k_1 = 3.014 \times 10^6\,J/kg \\ &(e_2 + k_2) = (e_1 + k_1) - R\,(T_2 - T_1) = 3.014 \times 10^6 - 2078.62\,(623.44 - 300) \\ &= 3.014 \times 10^6 - 0.6723 \times 10^6 = 2.3417 \times 10^6\,J/kg \\ &(e_3 + k_3) = (e_1 + k_1) - R\,(T_3 - T_1) = 3.014 \times 10^6 - 2078.62\,(700 - 300) \\ &= 3.014 \times 10^6 - 0.8314 \times 10^6 = 2.1826 \times 10^6\,J/kg. \end{aligned} \tag{9.78}$$

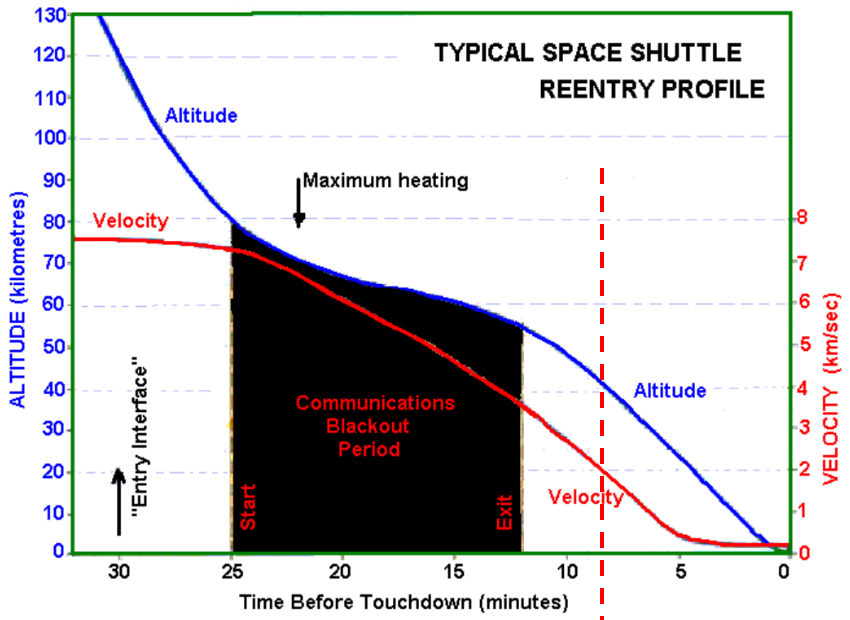
The energy of a fluid element decreases considerably across the shock and then decreases further to the stagnation point.

2) Describe the mechanism by which the energy of the fluid element changes as it moves from station 1 to station 3.

The work done by the pressure and viscous normal force field on the fluid element is the mechanism by which the energy decreases in moving from station 1 to station 3. The flow energy decreases across the shock wave through a combination of pressure and viscous normal stress forces of roughly equal magnitude that act to compress the fluid element increasing its internal energy while decelerating it and reducing its kinetic energy. The loss of kinetic energy dominates the increase in internal energy.

Between stations 2 and 3 the flow further decelerates as the pressure increases toward the stagnation point. Viscous normal forces also act in region 2 to 3 but because the streamwise velocity gradients are small (compared to the shock) viscous forces are generally much smaller than the pressure forces.

Space shuttle re-entry
ignore heat capacity changes and real gas effects



8 minutes to touchdown
Altitude 40km
Speed 2000m/sec

$$P_\infty = 277.522\ Pa$$

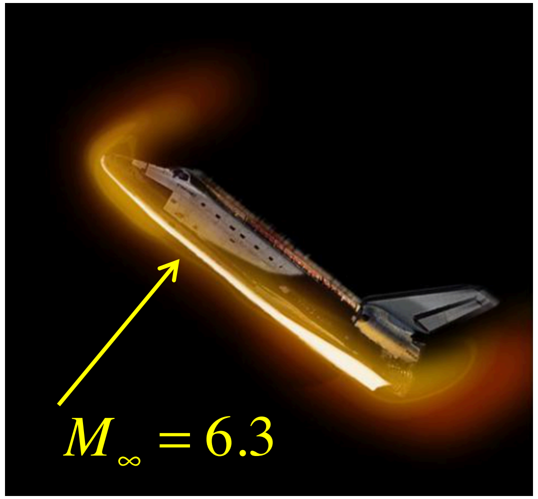
$$\rho_\infty = 0.003851\ kg/m^3$$

$$T_\infty = 251.050\ K$$

$$a_\infty = 317.633\ m/\sec$$

$$M_\infty = 6.297$$

$$Assume\ \gamma = 1.4$$



$$M_2 = 0.402$$

$$\frac{T_{t\infty}}{T_\infty} = 1 + \frac{\gamma - 1}{2} M_\infty^2 = 8.930$$

$$\frac{P_{t\infty}}{P_\infty} = \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma - 1}} = 8.930^{3.5} = 2128.41$$

Across the normal shock

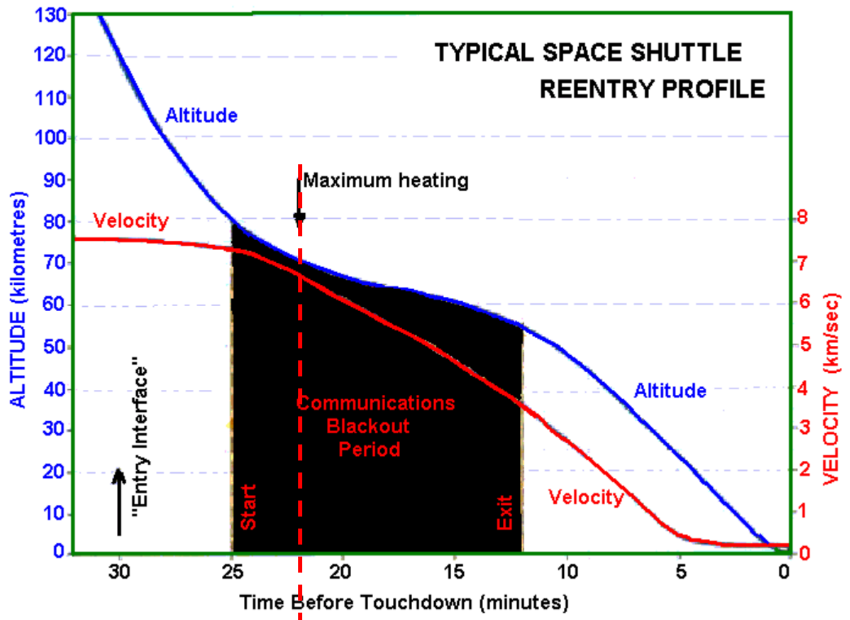
$$\frac{P_{t2}}{P_{t\infty}} = \left(\frac{\frac{\gamma+1}{\gamma-1}}{\frac{2\gamma}{\gamma-1} M_\infty^2 - 1}\right)^{\frac{1}{\gamma-1}} \left(\frac{\frac{\gamma+1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M_\infty^2}\right)^{\frac{\gamma}{\gamma-1}} = 0.02416$$

$$P_{t\infty} = 590{,}680\ Pa$$

$$P_{t2} = 14{,}270\ Pa$$

$$T_{t\infty} = 2241.87\ K$$

Space shuttle re-entry
ignore heat capacity changes and real gas effects



22 minutes to touchdown
Altitude 70km
Speed 7000m/sec

$P_\infty = 4.63422\ Pa$

$\rho_\infty = 0.000074243\ kg/m^3$

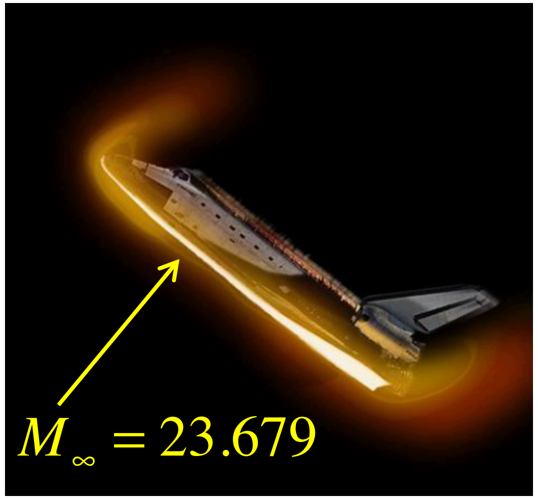
$T_\infty = 217.45\ K$

$a_\infty = 295.614\ m/\sec$

$M_\infty = 23.679$

$Assume\ \gamma = 1.4$

At such a high Mach number the flow is in fact totally dominated by real gas effects including dissociation. The temperatures reached are much lower, and the pressure behind the shock tends to be higher that predicted here.



$M_2 = 0.379$

$$\frac{T_{t\infty}}{T_\infty} = 1 + \frac{\gamma-1}{2} M_\infty^{\ 2} = 113.14$$

$$\frac{P_{t\infty}}{P_\infty} = \left(1 + \frac{\gamma-1}{2} M_\infty^{\ 2}\right)^{\frac{\gamma}{\gamma-1}} = 113.14^{3.5} = 1.54048 \times 10^7$$

Across the normal shock

$$\frac{P_{t2}}{P_{t\infty}} = \left(\frac{\dfrac{\gamma+1}{\gamma-1}}{\dfrac{2\gamma}{\gamma-1} M_\infty^{\ 2} - 1}\right)^{\frac{1}{\gamma-1}} \left(\frac{\dfrac{\gamma+1}{2} M_\infty^{\ 2}}{1 + \dfrac{\gamma-1}{2} M_\infty^{\ 2}}\right)^{\frac{\gamma}{\gamma-1}} = 4.68951 \times 10^{-5}$$

$P_{t\infty} = 7.13893 \times 10^7\ Pa$

$P_{t2} = 3347.81\ Pa$

$? \longrightarrow T_{t\infty} = 24602.3\ K$

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Gasdynamics of nozzle flow

10.1 The area-Mach number function

Quasi-1D equations of motion

$$d(\rho U) = \frac{\delta \dot{m}}{A} - \rho U \frac{dA}{A}$$

$$d(P - \tau_{xx}) + \rho U dU = -\frac{1}{2}\rho U^2\left(4C_f \frac{dx}{D}\right) + \frac{(U_{xm} - U)\delta \dot{m}}{A} - \frac{\delta F_x}{A}$$

$$d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) = \delta q - \delta w + \left(h_{tm} - \left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right)\right)\frac{\delta \dot{m}}{\rho U A}$$

Assume

$$\delta \dot{m} = C_f = \delta F_x = \delta q = \delta w = 0$$

$$d(\rho U A) = 0$$

$$dP + \rho U dU = 0$$

$$C_p dT + U dU = 0$$

$$P = \rho R T$$

Equations of motion in fractional differential form

$$\frac{d\rho}{\rho} + \frac{dU^2}{2U^2} + \frac{dA}{A} = 0$$

$$\frac{dP}{P} + \frac{\gamma M^2}{2}\frac{dU^2}{U^2} = 0$$

$$\frac{dT}{T} + \frac{(\gamma - 1)M^2}{2}\frac{dU^2}{U^2} = 0$$

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

Equation of state

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

$$-\frac{\gamma M^2}{2}\frac{dU^2}{U^2} = -\frac{dU^2}{2U^2} - \frac{dA}{A} - \frac{(\gamma - 1)M^2}{2}\frac{dU^2}{U^2}$$

Effect of area change on velocity

$$\frac{dU^2}{U^2} = \left(\frac{2}{M^2 - 1}\right)\frac{dA}{A}$$

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Effect of area change on other flow variables

$$\frac{d\rho}{\rho} = -\left(\frac{M^2}{M^2 - 1}\right)\frac{dA}{A}$$

$$\frac{dP}{P} = -\left(\frac{\gamma M^2}{M^2 - 1}\right)\frac{dA}{A}$$

$$\frac{dT}{T} = -\left(\frac{(\gamma - 1)M^2}{M^2 - 1}\right)\frac{dA}{A}$$

Effect of area change on Mach number $U^2 = \gamma R T M^2$

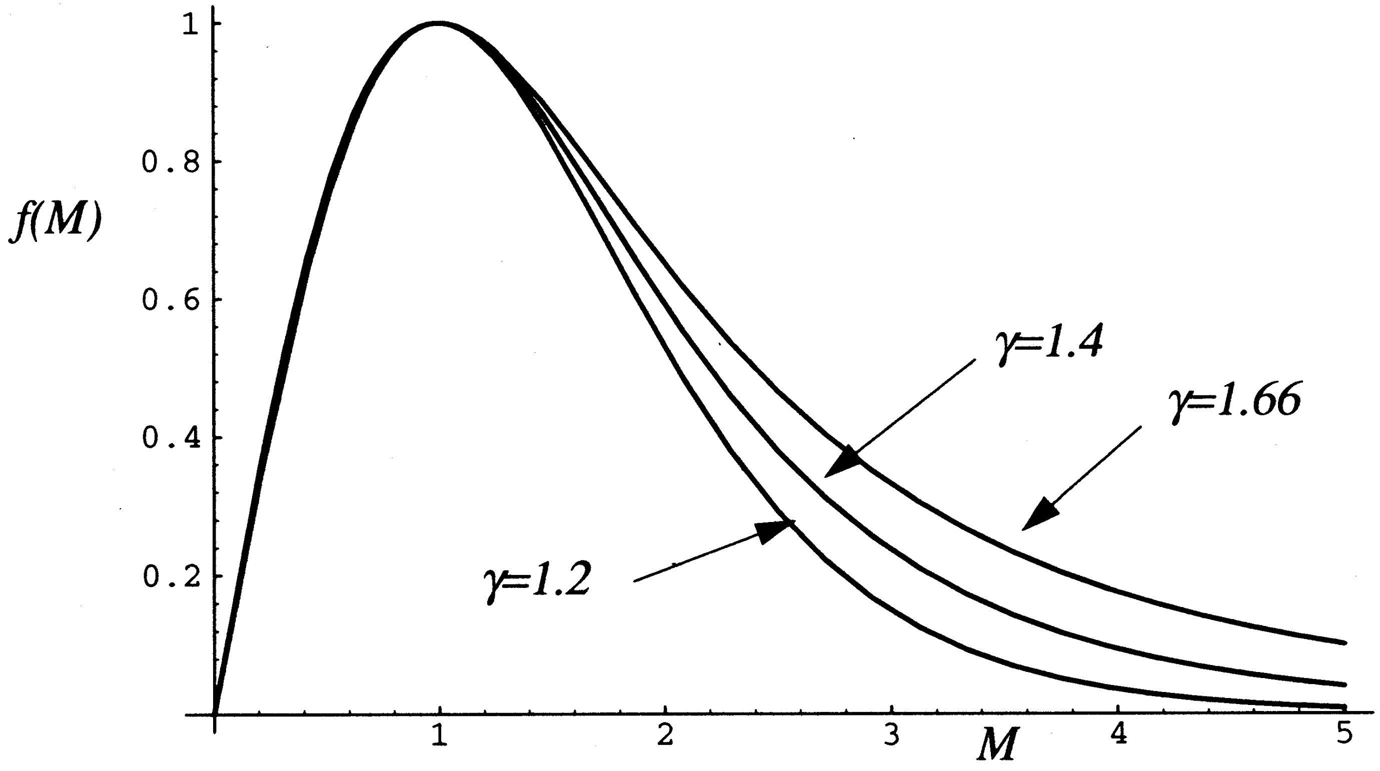
$$\frac{dU^2}{U^2} = \frac{dT}{T} + \frac{dM^2}{M^2}$$

$$\left(\frac{2}{M^2 - 1}\right)\frac{dA}{A} = -\frac{(\gamma - 1)M^2}{M^2 - 1}\frac{dA}{A} + \frac{dM^2}{M^2}$$

$$\frac{dA}{A} = \frac{M^2 - 1}{2\left(1 + \left(\frac{\gamma - 1}{2}\right)M^2\right)}\frac{dM^2}{M^2}$$

$$\int_{M^2}^{1} \frac{M^2 - 1}{2\left(1 + \left(\frac{\gamma - 1}{2}\right)M^2\right)}\frac{dM^2}{M^2} = \int_{A}^{A^*} \frac{dA}{A}$$

$$f(M) = \frac{A^*}{A} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{M}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}.$$



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Mass Conservation

$$\dot{m} = \rho U A$$

$$\dot{m} = \rho U A = \frac{P}{RT}(\gamma RT)^{1/2} M A$$

$$\frac{T_t}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

$$\frac{P_t}{P} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

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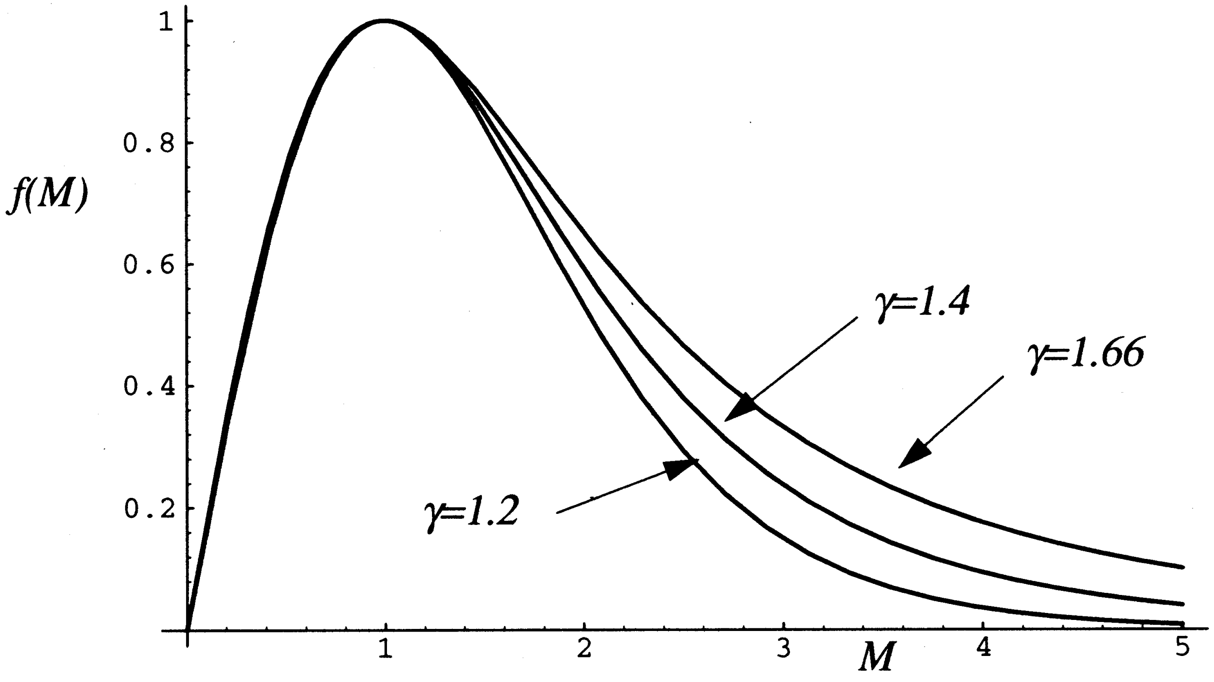
$$\dot{m} = \rho U A = \frac{\gamma}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \left(\frac{P_t A}{\sqrt{\gamma R T_t}}\right) f(M)$$



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Between any two points in a channel with zero mass addition

$$\dot{m}_1 = \dot{m}_2$$

$$\frac{P_{t1}A_1}{\sqrt{T_{t1}}} f\left(M_1\right) = \frac{P_{t2}A_2}{\sqrt{T_{t2}}} f\left(M_2\right)$$

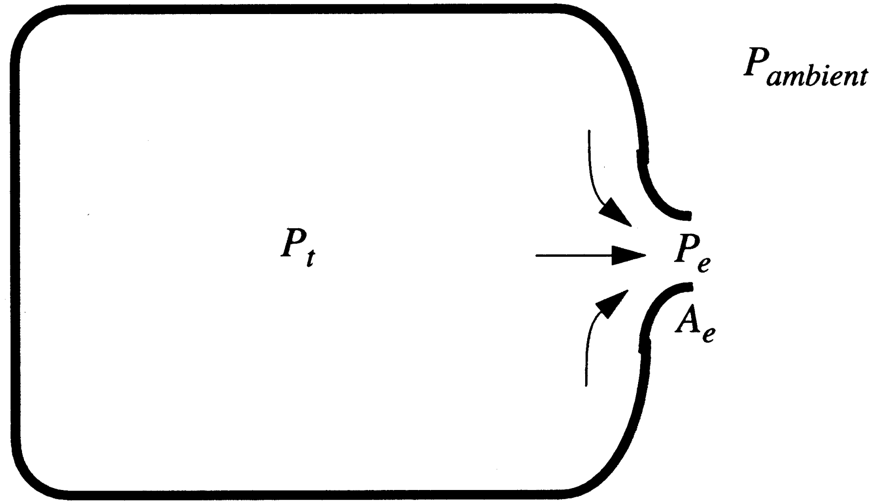
If the flow is adiabatic

$$P_{t1}A_1 f\left(M_1\right) = P_{t2}A_2 f\left(M_2\right)$$

If the flow is adiabatic and isentropic

$$A_1 f\left(M_1\right) = A_2 f\left(M_2\right)$$

10.2 A Simple convergent nozzle



If the flow is subsonic

$$P_e = P_{ambient}$$

For subsonic flow the exit Mach number can be determined from

$$\frac{P_t}{P_e} = \left(1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\right)^{\frac{\gamma}{\gamma - 1}}$$

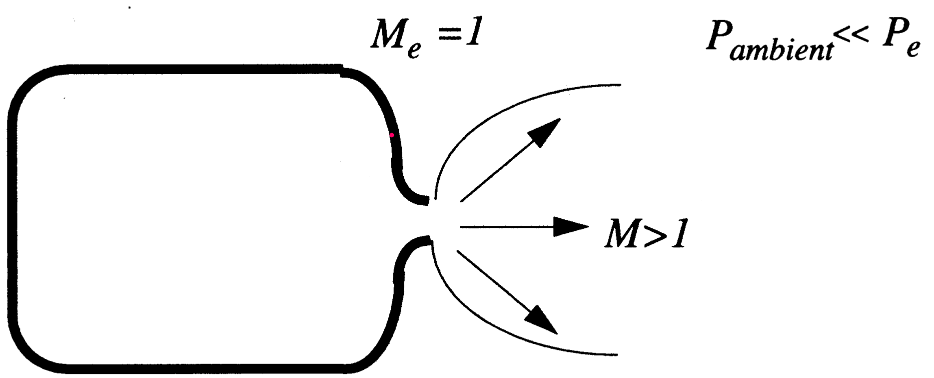
Thus

$$M_e = \left(\frac{2}{\gamma - 1}\right)^{\frac{1}{2}}\left(\left(\frac{P_t}{P_{ambient}}\right)^{\frac{\gamma - 1}{\gamma}} - 1\right)^{\frac{1}{2}}$$

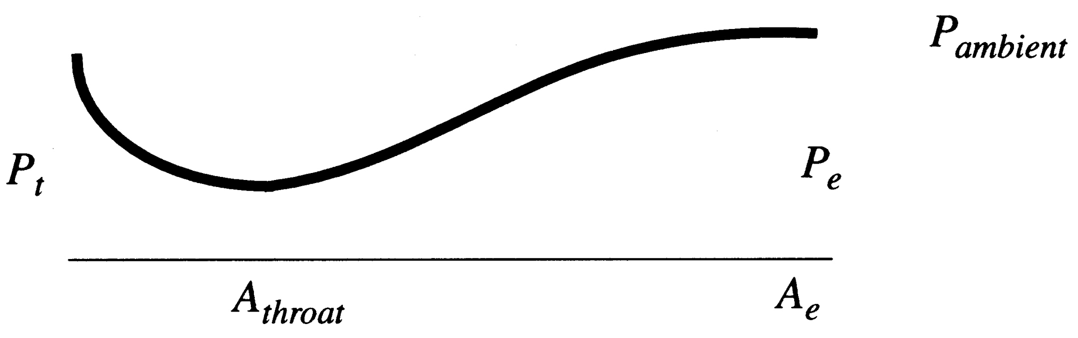
The exit Mach number reaches one when

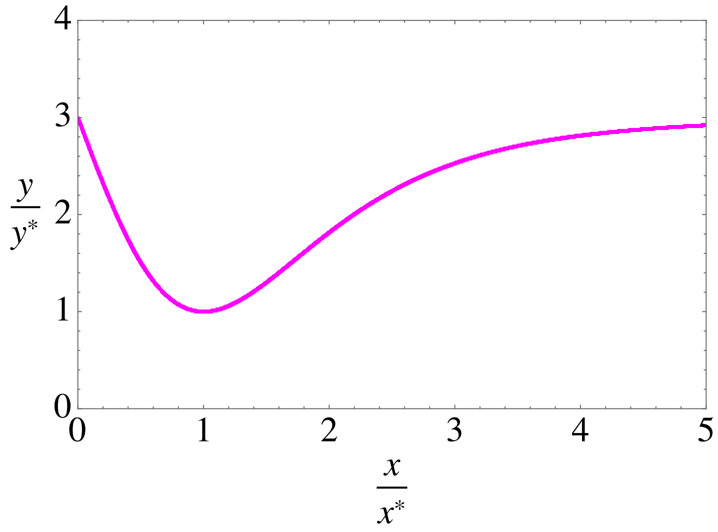
$$\frac{P_t}{P_{ambient}} = \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma}{\gamma - 1}}$$

If the pressure ratio is very high the flow from the nozzle will spread rapidly.

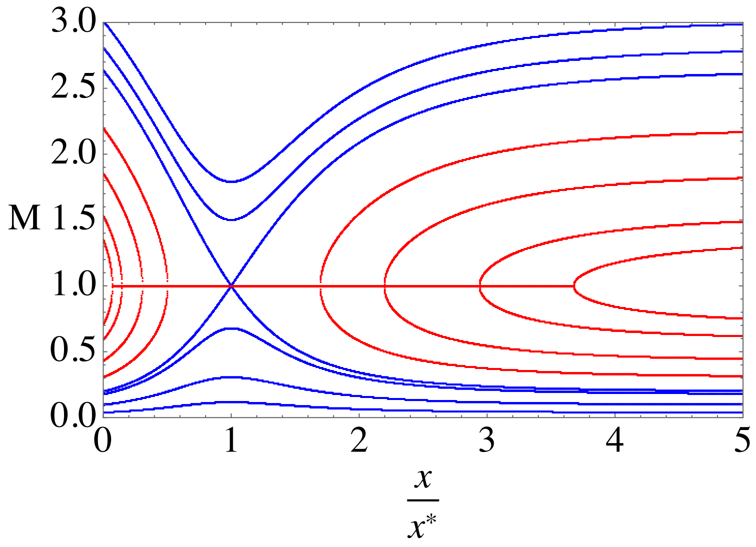


10.3 Converging-diverging nozzle





Determine two critical exit Mach numbers from



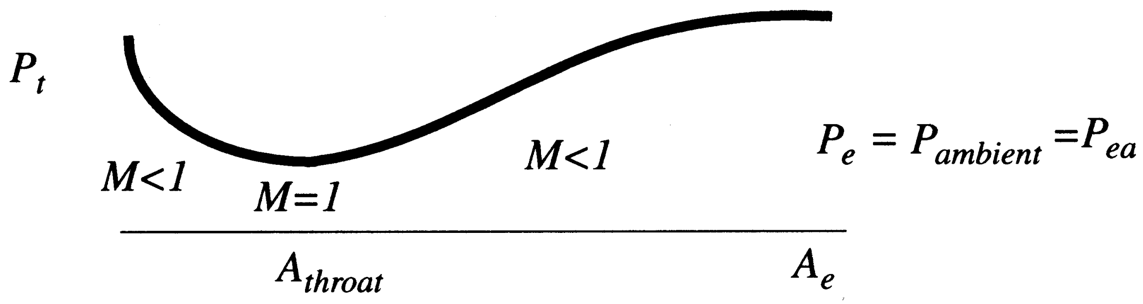
$$\frac{A_t}{A_e} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{M_e}{\left(1+\frac{\gamma-1}{2}M_e^{\,2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}.$$

The corresponding critical exit pressures are determined from

$$\frac{P_t}{P_{ea}} = \left(1 + \left(\frac{\gamma - 1}{2}\right)M_{ea}^2\right)^{\frac{\gamma}{\gamma - 1}}$$

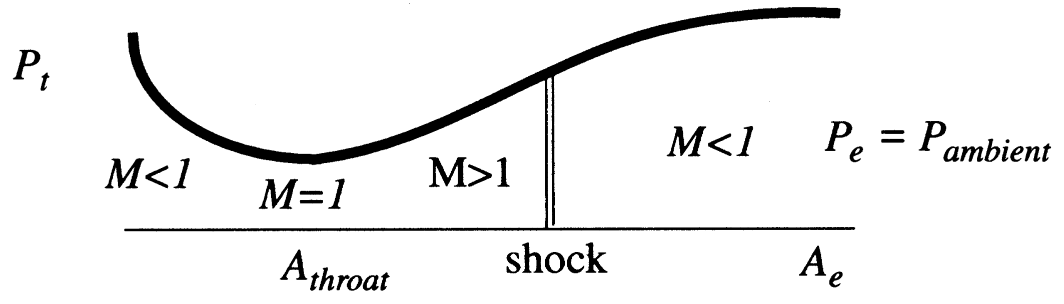
$$\frac{P_t}{P_{eb}} = \left(1 + \left(\frac{\gamma - 1}{2}\right)M_{eb}^2\right)^{\frac{\gamma}{\gamma - 1}}$$

10.3.1 Case 1 - Isentropic, subsonic flow in the nozzle



$$1 < P_t/P_{ambient} < P_t/P_{ea}$$

10.3.2 Case 2 - Non-isentropic flow - shock in the nozzle



$$\dot{m}_{throat} = \dot{m}_{exit}$$

$$P_t A_{throat} = P_{te} A_e f(M_e)$$

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The exit flow is subsonic and so the exit pressure matches the ambient pressure.

$$P_{te} = P_e\left(1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\right)^{\frac{\gamma}{\gamma - 1}} = P_{ambient}\left(1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\right)^{\frac{\gamma}{\gamma - 1}}.$$

Solve for the exit mach number

$$\left(\frac{P_t}{P_{ambient}}\right)\left(\frac{A_{throat}}{A_e}\right) = \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma + 1}{2(\gamma - 1)}} M_e\left(1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\right)^{\frac{1}{2}}$$

Now determine the stagnation pressure ratio across the nozzle.

$$\frac{P_{te}}{P_t} = \frac{A_{throat}}{A_e}\frac{1}{f(M_e)} < 1$$

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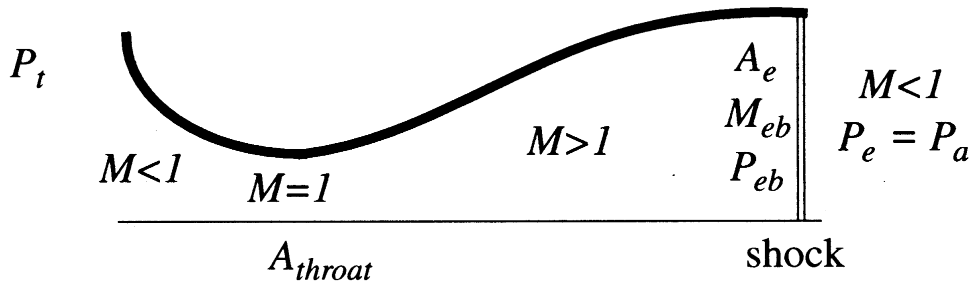
The shock Mach number is now determined from

$$\frac{P_{te}}{P_t} = \left(\frac{\left(\frac{\gamma+1}{2}\right)M_{shock}^2}{1+\frac{\gamma-1}{2}M_{shock}^2}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{\frac{\gamma+1}{2}}{\gamma M_{shock}^2-\left(\frac{\gamma-1}{2}\right)}\right)^{\frac{1}{\gamma-1}}$$

As the nozzle pressure ratio is increased the shock moves downstream until it sits at the nozzle exit.



The Mach number behind the shock is

$$M^2_{e(behind\ shock)} = \frac{1 + \frac{\gamma - 1}{2} M^2_{eb}}{\gamma M^2_{eb} - \left(\frac{\gamma - 1}{2}\right)}$$

This condition is reached when

$$\left(\frac{P_t}{P_{ambient}}\right)\Bigg|_{exit\ \perp\ shock} =$$

$$\left(\frac{A_e}{A_{throat}}\right)\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} M_{e(behind\ shock)}\left(1+\left(\frac{\gamma-1}{2}\right)M^2_{e(behind\ shock)}\right)^{\frac{1}{2}}$$

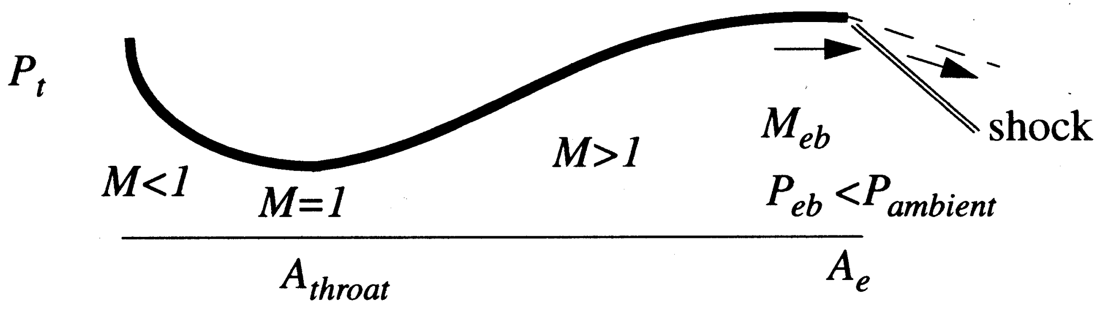
In summary, the shock-in-nozzle case occurs over the range

$$\frac{P_t}{P_{ea}} < \frac{P_t}{P_{ambient}} < \left(\frac{P_t}{P_{ambient}}\right)\Bigg|_{exit\ \perp\ shock}$$

10.3.3 Case 3 - Isentropic supersonic flow in the nozzle

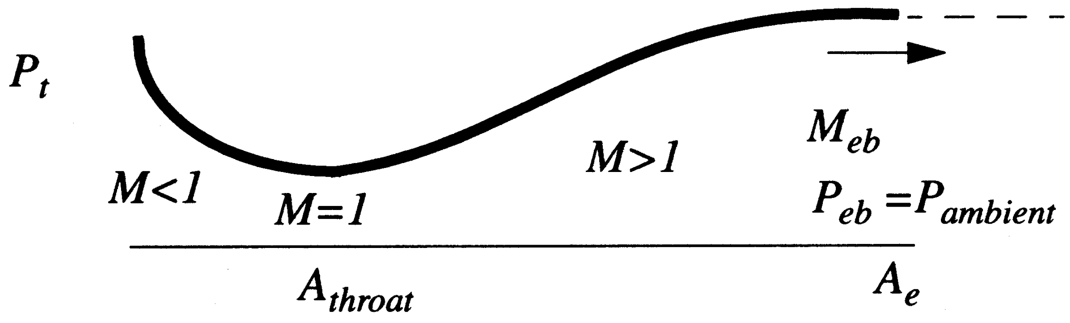
i) Over expanded flow

$$\left.\left(\frac{P_t}{P_{ambient}}\right)\right|_{exit \perp shock} < P_t/P_{ambient} < P_t/P_{eb}$$



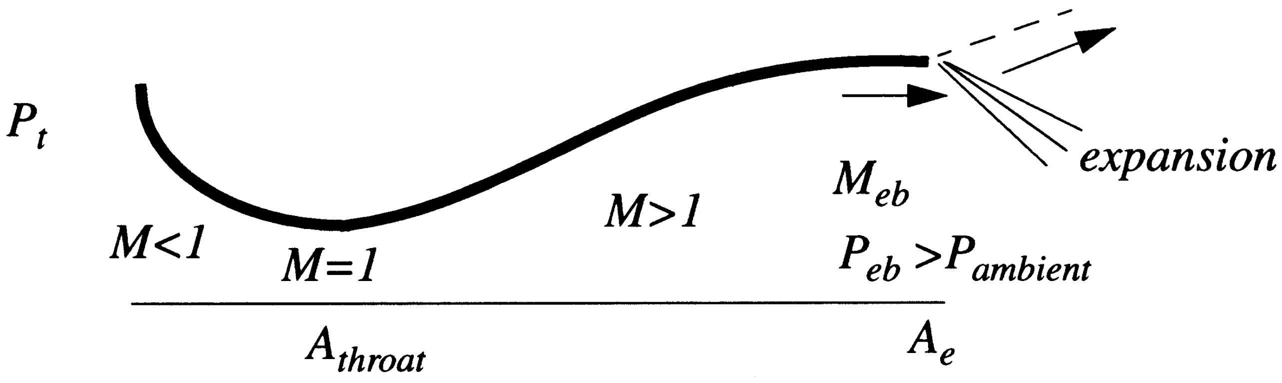
ii) Fully expanded flow

$$P_t/P_{ambient} = P_t/P_{eb}$$



iii) Under expanded flow

$$P_t/P_{ambient} > P_t/P_{eb}$$



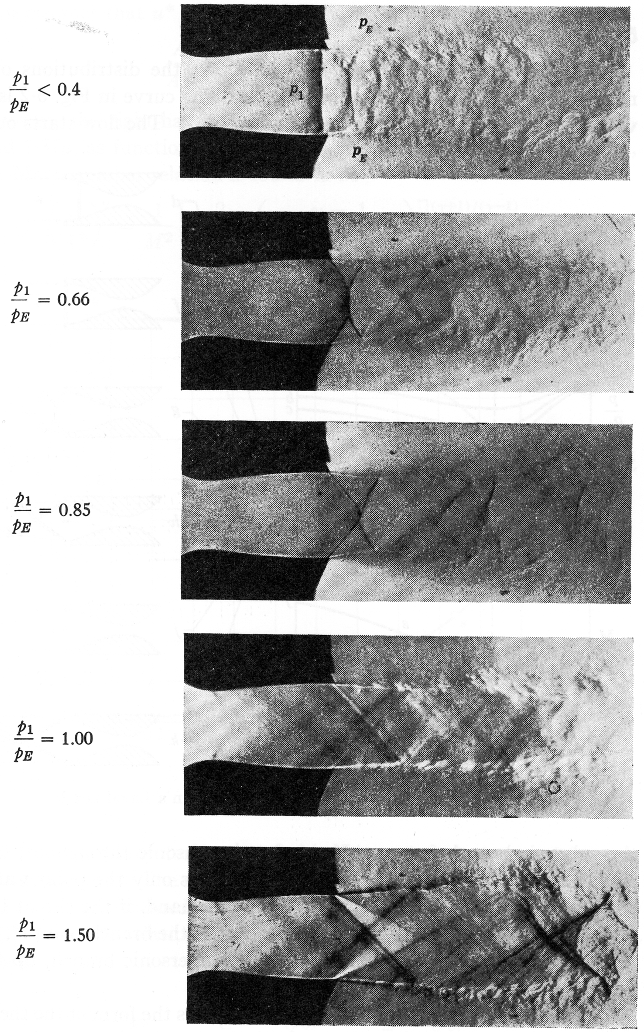


FIG. 5·4 Schlieren photographs of flow from a supersonic nozzle at different back pressures. The photographs, from top to bottom, may be compared with Fig. 5·3, sketches *d*, *g*, *h*, *j*, *k*, respectively. Reproduced from: L. Howarth (ed.), *Modern Developments in Fluid Dynamics, High Speed Flow*, Oxford, 1953.

Figure 5.4 from Liepmann and Roshko

Space Shuttle Main Engine

Approximately adiabatic isentropic flow!