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AA103
Air and Space Propulsion

Topic 4 – Introduction to gasdynamics PART 1

Recall this suggested viewing

National Science Foundation
Fluid Mechanics Films

http://web.mit.edu/fluids/www/Shapiro/ncfmf.html

Fluid Dynamics of Drag, Parts I to IV

Fundamental Boundary Layers

Turbulence

Channel flow of a Compressible Fluid

Waves in Fluids

Pressure Fields and Fluid Acceleration

Introduction to gasdynamics

Suggested reading – AA210 Course reader Chapters 1, 3, 5 and 9

Fluxes

$$"[\ ]" = units\ of$$

$$[\rho] = \frac{M}{L^3},\ [U] = \frac{L}{T},\ [e] = \frac{L^2}{T^2},\ [k] = \frac{L^2}{T^2}$$

Mass flux vector

$$\bar{U} = (U, V, W)$$

$$\rho\bar{U} = \frac{Mass}{Area - Time}$$

$$[\rho\bar{U}] = \frac{M}{L^2 - T}$$

Momentum flux tensor

$$\bar{U}\bar{U} = \begin{pmatrix} UU & UV & UW \\ UV & VV & VW \\ UW & VW & WW \end{pmatrix}$$

$$\rho\bar{U}\bar{U} = \frac{Momentum}{Area - Time}$$

$$[\rho\bar{U}\bar{U}] = \frac{M}{L - T^2}$$

Energy flux vector

$$k = \frac{1}{2}\bar{U}\cdot\bar{U} = \frac{1}{2}(U^2 + V^2 + W^2)$$

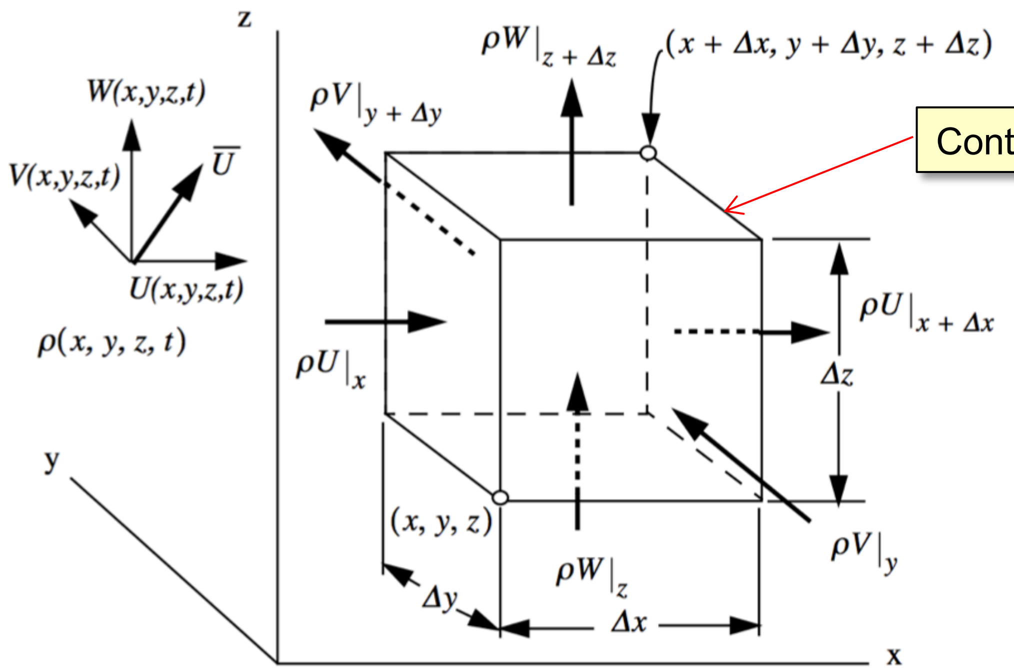
$$\rho(e+k)\bar{U} = \frac{Energy}{Area - Time}$$

$$[\rho(e+k)\bar{U}] = \frac{M}{T^3}$$

Conservation of mass

Add up mass in and mass out

$$\left\{\begin{array}{l} rate\ of\ mass \\ accumulation \\ inside\ the\ control \\ volume \end{array}\right\} = \left\{\begin{array}{l} rate\ of\ mass \\ flow \\ into\ the\ control \\ volume \end{array}\right\} - \left\{\begin{array}{l} rate\ of\ mass \\ flow \\ out\ of\ the\ control \\ volume \end{array}\right\}$$



Mass flux in the x-direction

$$[\rho U] = \frac{M\left(\frac{L}{T}\right)}{L^3} = \frac{M}{L^2 T}$$

Momentum per unit volume

Mass per unit area per second

$$\Delta x \Delta y \Delta z \left(\frac{\partial \rho}{\partial t}\right) = \Delta y \Delta z \rho\, U|_x - \Delta y \Delta z \rho\, U|_{x+\Delta x} + \Delta x \Delta z \rho\, V|_y - \Delta x \Delta z \rho\, V|_{y+\Delta y} + \Delta x \Delta y \rho\, W|_z - \Delta x \Delta y \rho\, W|_{z+\Delta z}$$

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Continuity equation

Equation numbers AA210 chapter 1

Rearrange

$$\Delta x \Delta y \Delta z \left(\frac{\partial \rho}{\partial t}\right) + \Delta y \Delta z \left(\rho\, U|_{x+\Delta x} - \rho\, U|_{x}\right) + \Delta x \Delta z \left(\rho\, V|_{y+\Delta y} - \rho\, V|_{y}\right) + \Delta x \Delta y \left(\rho\, W|_{z+\Delta z} - \rho\, W|_{z}\right) = 0 . \tag{1.3}$$

Divide (1.3) through by the infinitesimal volume $\Delta x \Delta y \Delta z$.

$$\frac{\partial \rho}{\partial t} + \frac{\rho\, U|_{x+\Delta x} - \rho\, U|_{x}}{\Delta x} + \frac{\rho\, V|_{y+\Delta y} - \rho\, V|_{y}}{\Delta y} + \frac{\rho\, W|_{z+\Delta z} - \rho\, W|_{z}}{\Delta z} = 0 \tag{1.4}$$

Let $\Delta x \to 0, \Delta y \to 0, \Delta z \to 0$. In this limit (1.4) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z} = 0. \tag{1.5}$$

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Index notation and the Einstein convention

Make the following replacements

$$(x, y, z) \rightarrow (x_1, x_2, x_3)$$

$$(U, V, W) \rightarrow (U_1, U_2, U_3)$$

Using index notation the continuity equation is

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial(\rho U_i)}{\partial x_i} = 0$$

Einstein recognized that such sums from vector calculus always involve a repeated index. For convenience he dropped the summation symbol.

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U_i)}{\partial x_i} = 0}$$

Coordinate independent form

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \bar{U}) = 0 \qquad \nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

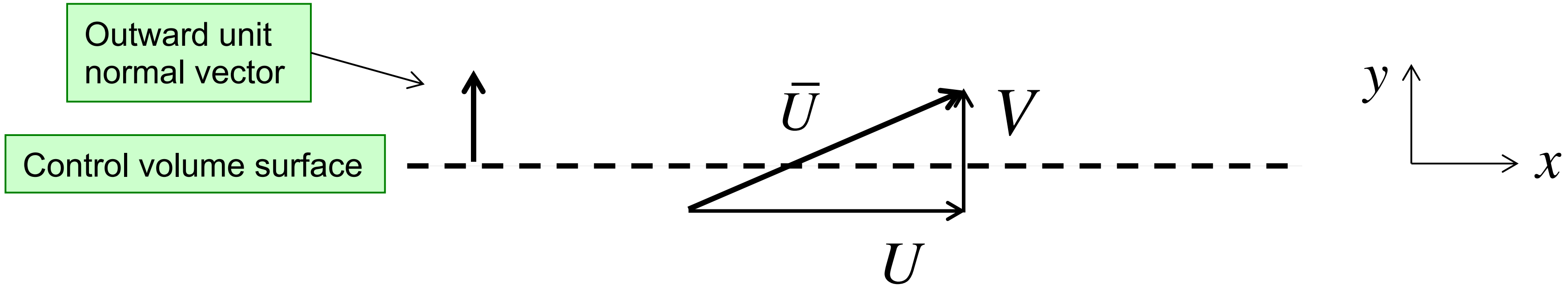
Momentum transport due to convection

Density

$$[\rho] = \frac{M}{L^3}$$

Volume flux in the y direction

$$[V] = \frac{L}{T} = \frac{L^3}{L^2 T} = \frac{Volume}{Area \cdot Sec}$$



Momentum flux

$$[\rho U V] = \frac{M\left(\frac{L}{T}\right)}{L^3}\left(\frac{L}{T}\right) = \frac{M\left(\frac{L}{T}\right)}{L^2 T}$$

x-momentum per unit volume

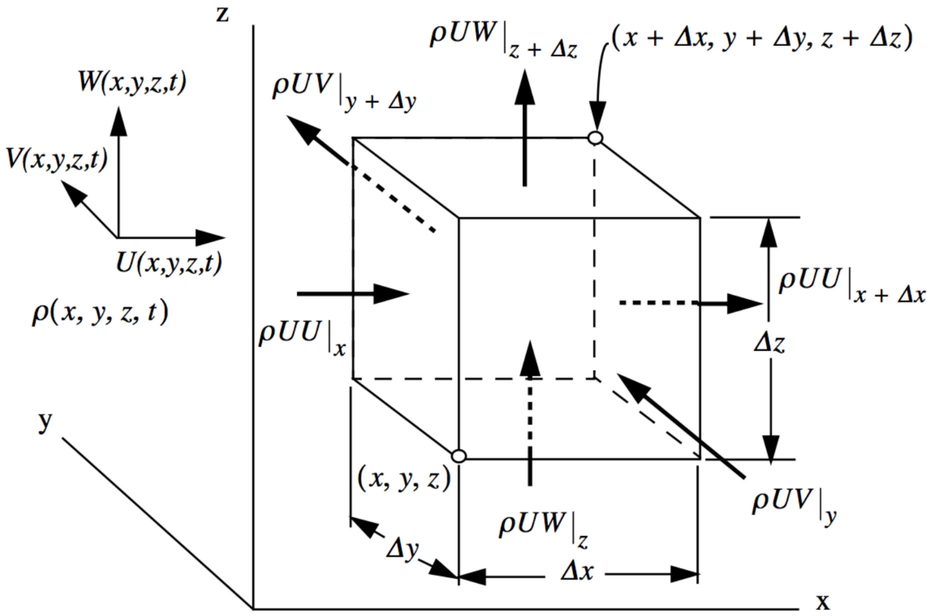
Volume per unit area per second

x-momentum convected in the y-direction per unit area per second

Conservation of momentum

$$\left\{\begin{matrix} \textit{rate of momentum} \\ \textit{accumulation inside} \\ \textit{the control volume} \end{matrix}\right\} =$$

$$\left\{\begin{matrix} \textit{rate of} \\ \textit{momentum} \\ \textit{flow into} \\ \textit{the control volume} \end{matrix}\right\} - \left\{\begin{matrix} \textit{rate of} \\ \textit{momentum} \\ \textit{flow out of} \\ \textit{the control volume} \end{matrix}\right\} + \left\{\begin{matrix} \textit{sum of} \\ \textit{forces acting} \\ \textit{on the} \\ \textit{control volume} \end{matrix}\right\}$$



Add up momentum in and momentum out in the x-direction

$$\Delta x \Delta y \Delta z \left(\frac{\partial \rho U}{\partial t}\right) = \Delta y \Delta z \rho\, UU|_x - \Delta y \Delta z \rho\, UU|_{x+\Delta x} +$$

$$\Delta x \Delta z \rho U\; V|_y - \Delta x \Delta z \rho U\; V|_{y+\Delta y} + \Delta x \Delta y \rho\; UW|_z - \Delta x \Delta y \rho U\; W|_{z+\Delta z} +$$

$$\{\textit{the sum of } x - \textit{component forces acting on the system}\}$$

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Conservation of momentum cont'd

Rearrange

$$\Delta x \Delta y \Delta z \left(\frac{\partial \rho U}{\partial t}\right) + \Delta y \Delta z \left(\rho\, UU|_{x+\Delta x} - \rho\, UU|_{x}\right) +$$

$$\Delta x \Delta z \left(\rho U\; V|_{y+\Delta y} - \rho U\; V|_{y}\right) + \Delta x \Delta y \left(\rho\, UW|_{z+\Delta z} - \rho U\; W|_{z}\right) =$$

$$\{the\ sum\ of\ x - component\ forces\ acting\ on\ the\ system\}\,.$$

x equation

$$\frac{\partial \rho U}{\partial t} + \frac{\rho\, UU|_{x+\Delta x} - \rho\, UU|_{x}}{\Delta x} + \frac{\rho U\; V|_{y+\Delta y} - \rho U\; V|_{y}}{\Delta y} + \frac{\rho\, UW|_{z+\Delta z} - \rho U\; W|_{z}}{\Delta z} =$$

$$\{the\ sum\ of\ x - component\ forces\ acting\ on\ the\ system\ per\ unit\ volume\} \tag{1.67}$$

Let $(\Delta x \to 0, \Delta y \to 0, \Delta z \to 0)$. In this limit (1.67) becomes

$$\frac{\partial \rho U}{\partial t} + \frac{\partial \rho UU}{\partial x} + \frac{\partial \rho UV}{\partial y} + \frac{\partial \rho UW}{\partial z} = \left\{ \begin{array}{l} the\ sum\ of \\ x - component\ forces \\ per\ unit\ volume\ acting \\ on\ the\ control\ volume \end{array} \right\}. \tag{1.68}$$

Similar equations describe conservation of momentum in the y and z directions.

y and z equations

$$\begin{aligned} \frac{\partial \rho V}{\partial t} + \frac{\partial \rho VU}{\partial x} + \frac{\partial \rho VV}{\partial y} + \frac{\partial \rho VW}{\partial z} &= \left\{ \begin{array}{l} the\ sum\ of \\ y - component\ forces \\ per\ unit\ volume\ acting \\ on\ the\ control\ volume \end{array} \right\} \\ \frac{\partial \rho W}{\partial t} + \frac{\partial \rho WU}{\partial x} + \frac{\partial \rho WV}{\partial y} + \frac{\partial \rho WW}{\partial z} &= \left\{ \begin{array}{l} the\ sum\ of \\ z - component\ forces \\ per\ unit\ volume\ acting \\ on\ the\ control\ volume \end{array} \right\} \end{aligned} \tag{1.69}$$

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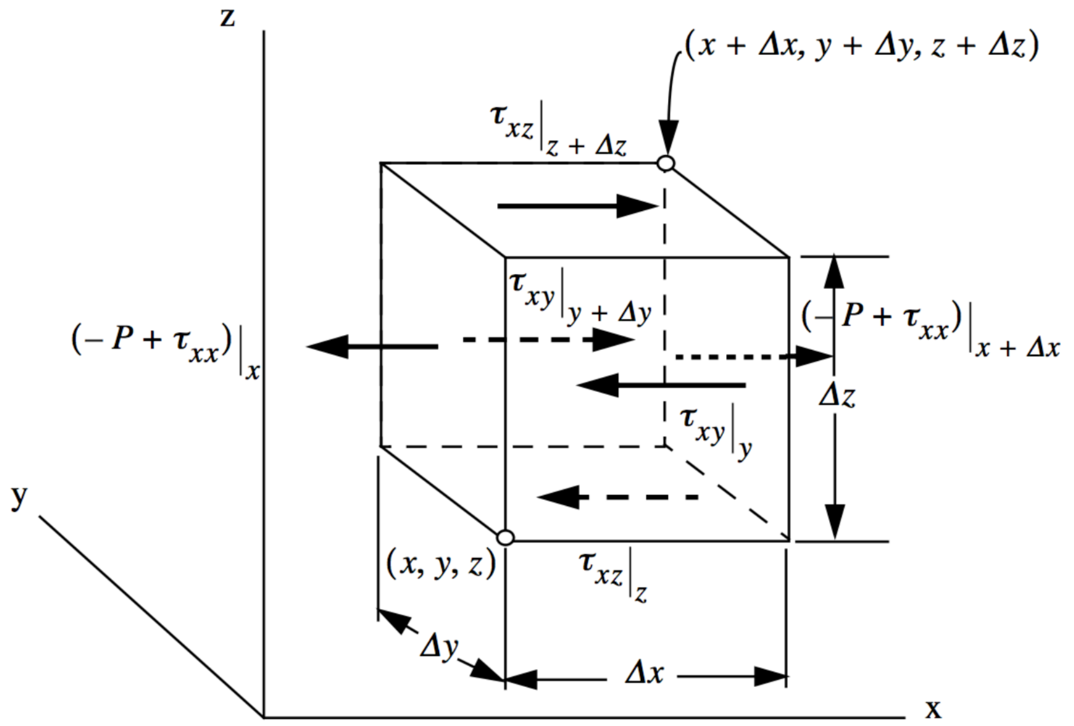
Pressure and viscous forces act on the control volume surface



$$F_x = \Delta y \Delta z \left((-P+\tau_{xx})|_{x+\Delta x} - (-P+\tau_{xx})|_x \right) + \Delta x \Delta z \left(\tau_{xy}|_{y+\Delta y} - \tau_{xy}|_y \right) + \Delta x \Delta y \left(\tau_{xz}|_{z+\Delta z} - \tau_{xz}|_z \right)$$

$$F_y = \Delta y \Delta z \left(\tau_{xy}|_{x+\Delta x} - \tau_{xy}|_x \right) + \Delta x \Delta z \left((-P+\tau_{yy})|_{y+\Delta y} - (-P+\tau_{yy})|_y \right) + \Delta x \Delta y \left(\tau_{yz}|_{z+\Delta z} - \tau_{yz}|_z \right)$$

$$F_z = \Delta y \Delta z \left(\tau_{xz}|_{x+\Delta x} - \tau_{xz}|_x \right) + \Delta x \Delta z \left(\tau_{yz}|_{y+\Delta y} - \tau_{yz}|_y \right) + \Delta x \Delta y \left((-P+\tau_{zz})|_{z+\Delta z} - (-P+\tau_{zz})|_z \right)$$

Conservation of momentum equation

$$\Delta x\Delta y\Delta z\left(\frac{\partial\rho U}{\partial t}\right)=\Delta y\Delta z\left(\rho\, UU|_x-\rho\, UU|_{x+\Delta x}\right)+$$

$$\Delta x\Delta z\left(\rho U\; V|_y-\rho U\; V|_{y+\Delta y}\right)+\Delta x\Delta y\left(\rho\; UW|_z-\rho U\; W|_{z+\Delta z}\right)+$$

Rearrange

$$\Delta y\Delta z\left((-P+\tau_{xx})|_{x+\Delta x}-(-P+\tau_{xx})|_x\right)+$$

$$\Delta x\Delta z\left(\tau_{xy}|_{y+\Delta y}-\tau_{xy}|_y\right)+\Delta x\Delta y\left(\tau_{xz}|_{z+\Delta z}-\tau_{xz}|_z\right)$$

$$\frac{\partial\rho U}{\partial t}+\frac{\rho\, UU|_{x+\Delta x}-\rho\, UU|_x+(P-\tau_{xx})|_{x+\Delta x}-(P-\tau_{xx})|_x}{\Delta x}+$$

$$\frac{\rho U\; V|_{y+\Delta y}-\rho U\; V|_y-\left(\tau_{xy}|_{y+\Delta y}-\tau_{xy}|_y\right)}{\Delta y}+\frac{\rho\; UW|_{z+\Delta z}-\rho U\; W|_z-\left(\tau_{xz}|_{z+\Delta z}-\tau_{xz}|_z\right)}{\Delta z}=0 \quad (1.78)$$

Let $\Delta x\to 0,\Delta y\to 0,\Delta z\to 0$. In this limit (1.78) becomes

$$\frac{\partial\rho U}{\partial t}+\frac{\partial\left(\rho UU+P-\tau_{xx}\right)}{\partial x}+\frac{\partial\left(\rho UV-\tau_{xy}\right)}{\partial y}+\frac{\partial\left(\rho UW-\tau_{xz}\right)}{\partial z}=0. \quad (1.79)$$

x, y and z equations

$$\frac{\partial\rho V}{\partial t}+\frac{\partial\left(\rho VU-\tau_{xy}\right)}{\partial x}+\frac{\partial\left(\rho VV+P-\tau_{yy}\right)}{\partial y}+\frac{\partial\left(\rho VW-\tau_{yz}\right)}{\partial z}=0$$

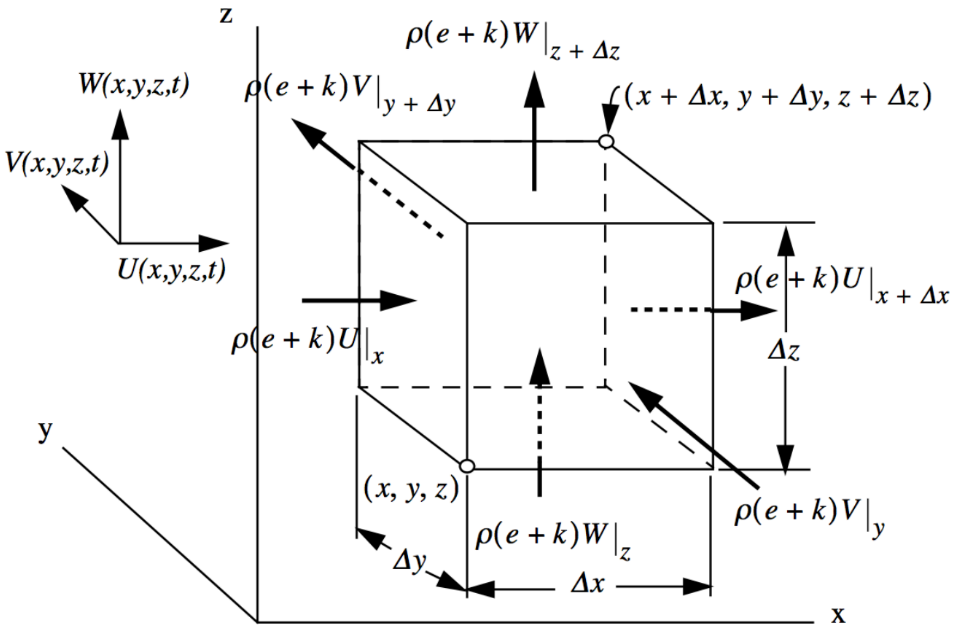
$$\frac{\partial\rho W}{\partial t}+\frac{\partial\left(\rho WU-\tau_{xz}\right)}{\partial x}+\frac{\partial\left(\rho WV-\tau_{yz}\right)}{\partial y}+\frac{\partial\left(\rho WW+P-\tau_{zz}\right)}{\partial z}=0$$

Conservation of energy

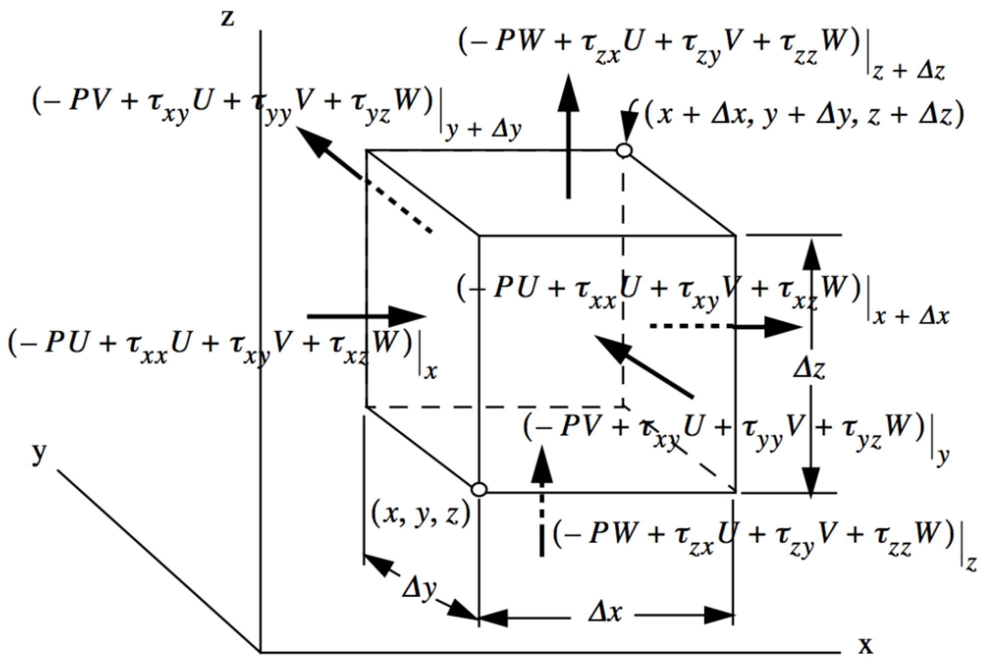
$$\left\{\begin{array}{l} \textit{rate of energy} \\ \textit{accumulation} \\ \textit{inside the control} \\ \textit{volume} \end{array}\right\} = \left\{\begin{array}{l} \textit{rate of energyflow} \\ \textit{into the} \\ \textit{control volume} \\ \textit{by convection} \end{array}\right\} -$$

$$\left\{\begin{array}{l} \textit{rate of energy flow} \\ \textit{out of the} \\ \textit{control volume} \\ \textit{by convection} \end{array}\right\} + \left\{\begin{array}{l} \textit{work done on the} \\ \textit{control volume} \\ \textit{by pressure and} \\ \textit{viscous forces} \end{array}\right\} +$$

$$\left\{\begin{array}{l} \textit{rate of energy addition} \\ \textit{due to heat conduction} \end{array}\right\} + \left\{\begin{array}{l} \textit{energy generation} \\ \textit{due to sources inside} \\ \textit{the control volume} \end{array}\right\}$$



Power input by viscous and pressure forces



$$Power\ input\ to\ the\ control\ volume = \bar{F}\cdot\bar{U} =$$

$$\Delta y\Delta z\left\{\begin{array}{l}\left((-P+\tau_{xx})|_{x+\Delta x}-(-P+\tau_{xx})|_{x}\right)U+\\ \left(\tau_{xy}|_{x+\Delta x}-\tau_{xy}|_{x}\right)V+\left(\tau_{xz}|_{x+\Delta x}-\tau_{xz}|_{x}\right)W\end{array}\right\}+$$

$$\Delta x\Delta z\left\{\begin{array}{l}\left(\tau_{xy}|_{y+\Delta y}-\tau_{xy}|_{y}\right)U+\left((-P+\tau_{yy})|_{y+\Delta y}-(-P+\tau_{yy})|_{y}\right)V+\\ \left(\tau_{yz}|_{y+\Delta y}-\tau_{yz}|_{y}\right)W\end{array}\right\}+$$

$$\Delta x\Delta y\left\{\begin{array}{l}\left(\tau_{xz}|_{z+\Delta z}-\tau_{xz}|_{z}\right)U+\left(\tau_{yz}|_{z+\Delta z}-\tau_{yz}|_{z}\right)V+\\ \left((-P+\tau_{zz})|_{z+\Delta z}-(-P+\tau_{zz})|_{z}\right)W\end{array}\right\}.$$

Power input by viscous and pressure forces, cont'd

Rearrange

$$Power\ input\ to\ the\ control\ volume = \bar{F}\cdot\bar{U} =$$

$$\Delta y\Delta z\left\{\begin{array}{l}(-PU+\tau_{xx}U+\tau_{xy}V+\tau_{xz}W)|_{x+\Delta x}-\\(-PU+\tau_{xx}U+\tau_{xy}V+\tau_{xz}W)|_{x}\end{array}\right\}+$$

$$\Delta x\Delta z\left\{\begin{array}{l}(-PV+\tau_{xy}U+\tau_{yy}V+\tau_{yz}W)|_{y+\Delta y}-\\(-PV+\tau_{xy}U+\tau_{yy}V+\tau_{yz}W)|_{y}\end{array}\right\}+$$

$$\Delta x\Delta y\left\{\begin{array}{l}(-PW+\tau_{zx}U+\tau_{zy}V+\tau_{zz}W)|_{z+\Delta z}-\\(-PW+\tau_{zx}U+\tau_{zy}V+\tau_{zz}W)|_{z}\end{array}\right\}$$

Equation for conservation of energy

$$\Delta x \Delta y \Delta z \frac{\partial \rho (e+k)}{\partial t} =$$

$$\Delta y \Delta z \left(\rho (e+k) \, U|_x - \rho (e+k) \, U|_{x+\Delta x} \right) +$$

$$\Delta x \Delta z \left(\rho (e+k) \, V|_y - \rho (e+k) \, V|_{y+\Delta y} \right) +$$

$$\Delta x \Delta y \left(\rho (e+k) \, W|_z - \rho (e+k) \, W|_{z+\Delta z} \right) +$$

$$\Delta y \Delta z \left\{ \begin{array}{l} \left((-P+\tau_{xx})|_{x+\Delta x} - (-P+\tau_{xx})|_x \right) U + \\ \left(\tau_{xy}|_{x+\Delta x} - \tau_{xy}|_x \right) V + \left(\tau_{xz}|_{x+\Delta x} - \tau_{xz}|_x \right) W \end{array} \right\} +$$

$$\Delta x \Delta z \left\{ \begin{array}{l} \left(\tau_{xy}|_{y+\Delta y} - \tau_{xy}|_y \right) U + \left((-P+\tau_{yy})|_{y+\Delta y} - (-P+\tau_{yy})|_y \right) V + \\ \left(\tau_{yz}|_{y+\Delta y} - \tau_{yz}|_y \right) W \end{array} \right\} +$$

$$\Delta x \Delta y \left\{ \begin{array}{l} \left(\tau_{xz}|_{z+\Delta z} - \tau_{xz}|_z \right) U + \left(\tau_{yz}|_{z+\Delta z} - \tau_{yz}|_z \right) V + \\ \left((-P+\tau_{zz})|_{z+\Delta z} - (-P+\tau_{zz})|_z \right) W \end{array} \right\} +$$

$$\Delta y \Delta z \left(Q_x|_x - Q_x|_{x+\Delta x} \right) + \Delta x \Delta z \left(Q_y|_y - Q_y|_{y+\Delta y} \right) +$$

$$\Delta x \Delta y \left(Q_z|_z - Q_z|_{z+\Delta z} \right) +$$

$$\{\textit{power generation due to energy sources inside the control volume}\}.$$

$$\frac{\partial \rho (e+k)}{\partial t} +$$

$$\frac{\partial \left(\rho (e+k) U + PU - \tau_{xx} U - \tau_{xy} V - \tau_{xz} W + Q_x \right)}{\partial x} +$$

$$\frac{\partial \left(\rho (e+k) V + PV - \tau_{yx} U - \tau_{yy} V - \tau_{yz} W + Q_y \right)}{\partial y} +$$

$$\frac{\partial \left(\rho (e+k) W + PW - \tau_{zx} U - \tau_{zy} V - \tau_{zz} W + Q_z \right)}{\partial z} =$$

$$\{\textit{power generation due to energy sources inside the control volume}\}$$

Heat flux vector

$$Q_i = -\kappa (\partial T / \partial x_i)$$

Equations of motion in differential form

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Conservation of mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial y} = 0$$

Conservation of momentum

$$\frac{\partial \rho U}{\partial t} + \frac{\partial\left(\rho UU + P - \tau_{xx}\right)}{\partial x} + \frac{\partial\left(\rho UV - \tau_{xy}\right)}{\partial y} + \frac{\partial\left(\rho UW - \tau_{xz}\right)}{\partial z} = 0$$

$$\frac{\partial \rho V}{\partial t} + \frac{\partial\left(\rho VU - \tau_{xy}\right)}{\partial x} + \frac{\partial\left(\rho VV + P - \tau_{yy}\right)}{\partial y} + \frac{\partial\left(\rho VW - \tau_{yz}\right)}{\partial z} = 0$$

$$\frac{\partial \rho W}{\partial t} + \frac{\partial\left(\rho WU - \tau_{xz}\right)}{\partial x} + \frac{\partial\left(\rho WV - \tau_{yz}\right)}{\partial y} + \frac{\partial\left(\rho WW + P - \tau_{zz}\right)}{\partial z} = 0$$

Conservation of energy

$$\frac{\partial \rho\left(e+k\right)}{\partial t} +$$

$$\frac{\partial\left(\rho\left(e+k\right)U + PU - \tau_{xx}U - \tau_{xy}V - \tau_{xz}W + Q_x\right)}{\partial x}$$

$$\frac{\partial\left(\rho\left(e+k\right)V + PV - \tau_{yx}U - \tau_{yy}V - \tau_{yz}W + Q_y\right)}{\partial y}$$

$$\frac{\partial\left(\rho\left(e+k\right)W + PW - \tau_{zx}U - \tau_{zy}V - \tau_{zz}W + Q_z\right)}{\partial z} =$$

$\{$*Power generation due to energy sources inside the control volume*$\}$

The coordinate-independent form of the equations of motion is

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\bar{U}\right) = 0$$

$$\frac{\partial \rho\bar{U}}{\partial t} + \nabla\cdot\left(\rho\bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}\right) - \rho\bar{G} = 0$$

$$\frac{\partial \rho\left(e+k\right)}{\partial t} + \nabla\cdot\left(\rho\bar{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}}\cdot\bar{U} + \bar{Q}\right) - \rho\bar{G}\cdot\bar{U} = 0.$$

Using index notation the same equations in Cartesian coordinates are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho U_j\right) = 0$$

$$\frac{\partial \rho U_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho U_i U_j + P\delta_{ij} - \tau_{ij}\right) - \rho G_i = 0$$

$$\frac{\partial \rho\left(e+k\right)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho U_i\left(e + \frac{P}{\rho} + k\right) - \tau_{ij}U_j + Q_i\right) - \rho G_i U_i = 0.$$

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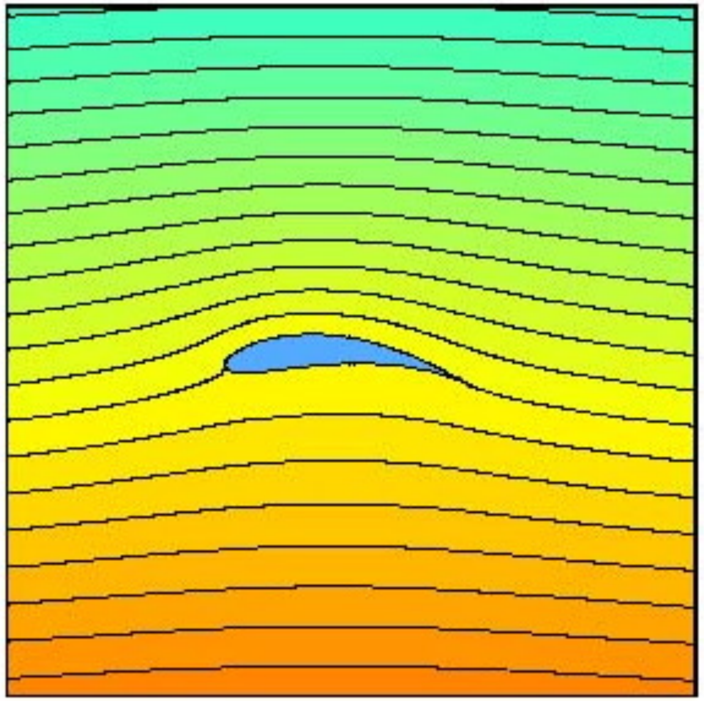
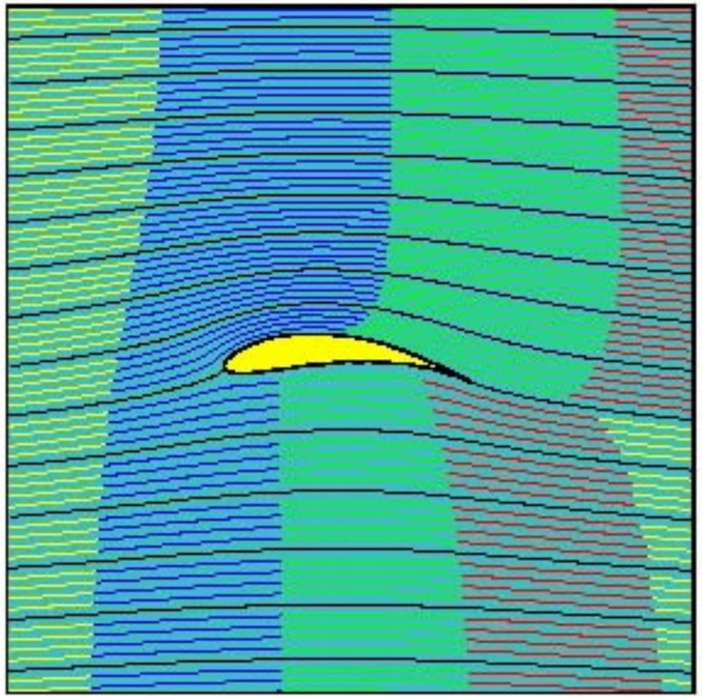
Particle paths, streamlines and streaklines

Streamlines



Streaklines



Steady flow over a wing flap.

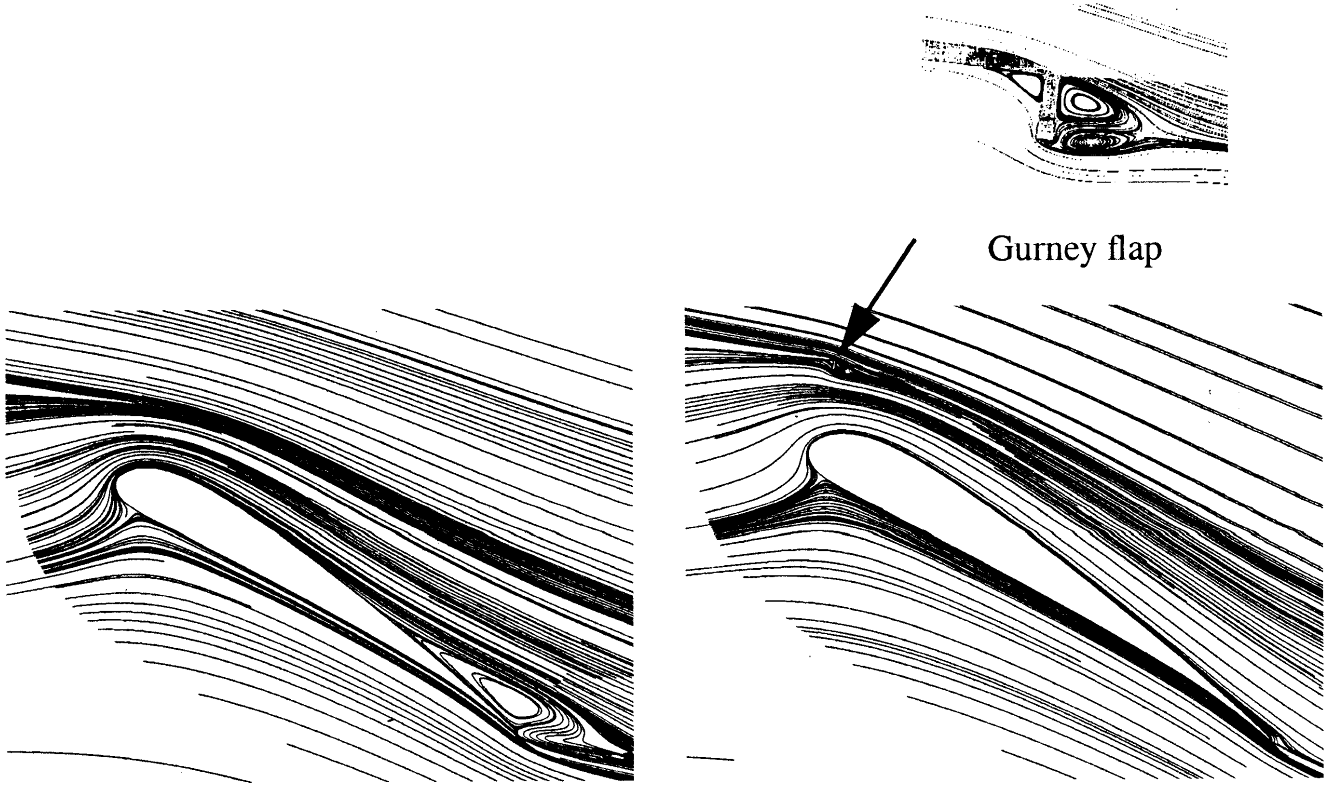
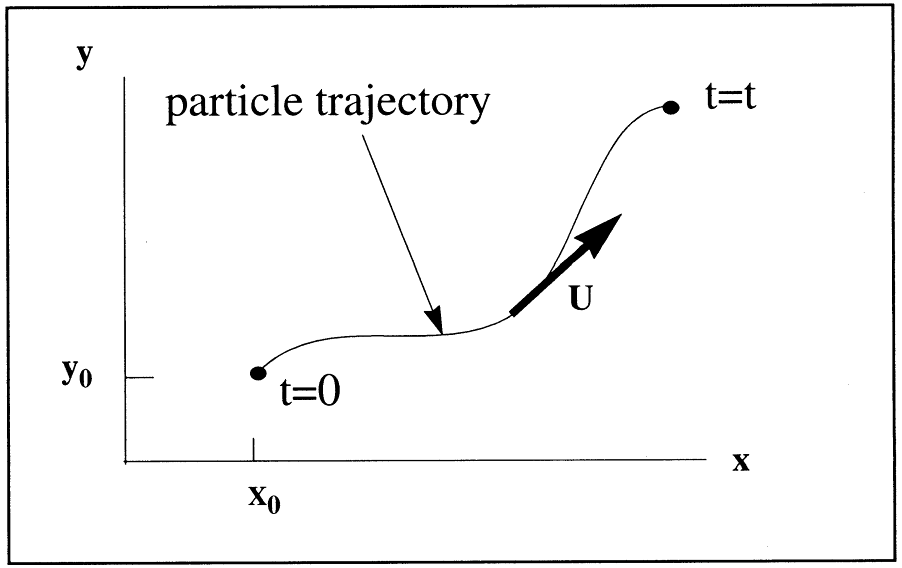


Figure 1.3 Computed streamlines over a wing flap.

Particle paths in 2D

The figure below shows the trajectory in space of a fluid element moving under the action of a two-dimensional steady velocity field



The equations that determine the trajectory are:

$$\left.\begin{aligned}\frac{dx(t)}{dt} &= U(x(t), y(t)) \\ \frac{dy(t)}{dt} &= V(x(t), y(t))\end{aligned}\right\}.$$

Particle paths in 3D

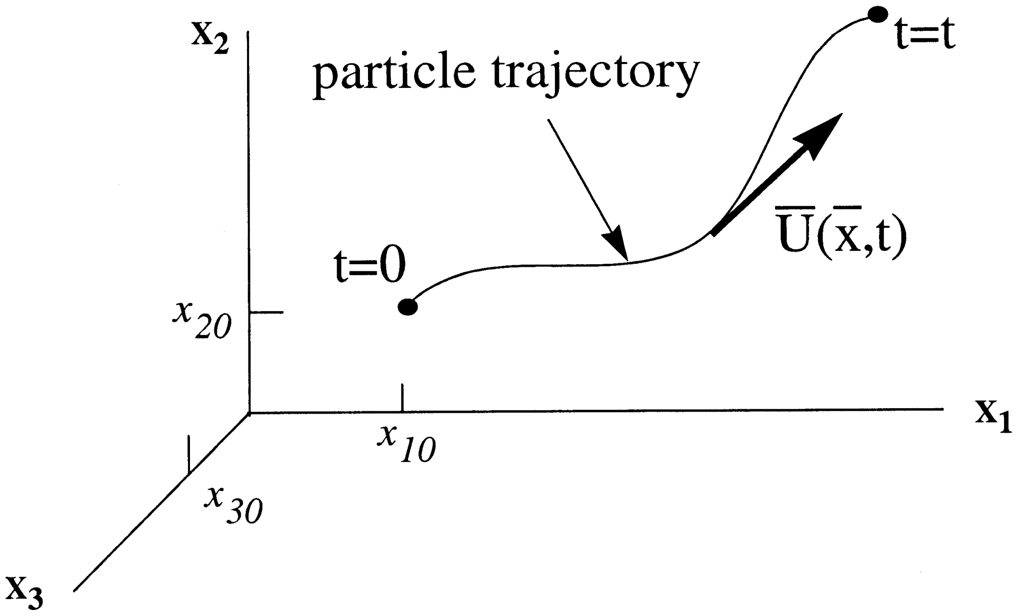


Figure 1.6 Particle trajectory in three dimensions

The figure above shows the trajectory in space traced out by a particle under the action of a general three dimensional unsteady flow, The equations governing the motion of the particle are:

$$\frac{dx_i(t)}{dt} = U_i(x_1(t), x_2(t), x_3(t), t) \quad ; \qquad i = 1, 2, 3$$

Formally, these equations are solved by integrating the velocity field.

$$x_i(t) = x_{i0} + \int_0^t U_i(x_1(t), x_2(t), x_3(t), t)dt \quad ; \qquad i = 1, 2, 3$$

The substantial derivative operator D

The acceleration of a particle is

$$\frac{d^2 x_i(t)}{dt^2} = \frac{d}{dt} U_i(x_1(t), x_2(t), x_3(t), t) = \frac{\partial U_i}{\partial t} + \frac{\partial U_i}{\partial x_k}\frac{dx_k}{dt}$$

Insert the velocities. The result is called the substantial or material derivative and is usually denoted by

$$\frac{DU_i}{Dt} = \frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} = \frac{\partial \bar{U}}{\partial t} + \bar{U} \bullet \nabla \bar{U}$$

The time derivative of any flow variable evaluated on a fluid element is given by a similar formula. For example the rate of change of density following a fluid particle is

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + U_k \frac{\partial \rho}{\partial x_k} = \frac{\partial \rho}{\partial t} + \bar{U} \bullet \nabla \rho$$

Leibniz rule for differentiation of integrals

Differentiation under the integral sign. According to the fundamental theorem of calculus if

$$I(x) = \int_{constant}^{x} f(x')dx'$$

then

$$\frac{dI}{dx} = f(x).$$

Similarly if

$$I(x) = \int_{x}^{constant} f(x')dx'$$

then

$$\frac{dI}{dx} = -f(x)$$

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Suppose the function depends on two variables

$$I(t) = \int_a^b f(x', t)dx'$$

where the limits of integration are constant.

The derivative of the integral with respect to time is

$$\frac{dI(t)}{dt} = \int_a^b \frac{\partial}{\partial t} f(x', t)dx'$$

But suppose the limits of the integral depend on time.

$$I(t, a(t), b(t)) = \int_{a(t)}^{b(t)} f(x', t)dx'$$

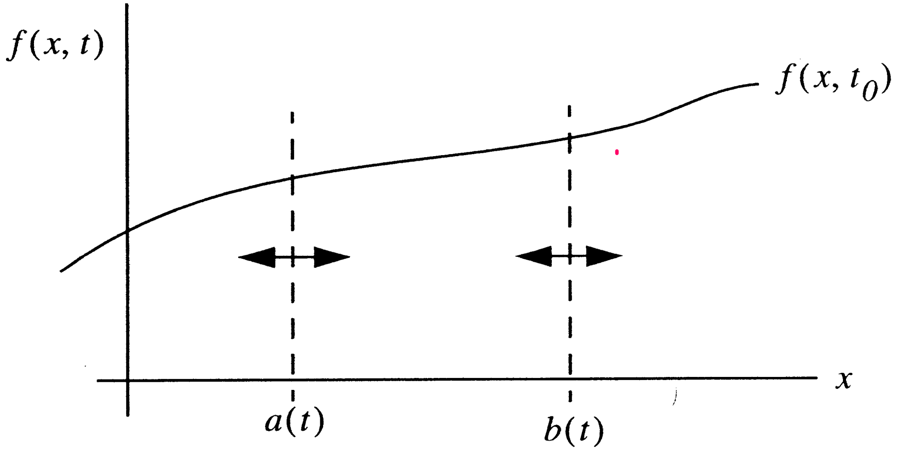

Figure 5.1 Integration with a moving boundary. The function $f(x, t)$ is shown at one instant in time.

From the chain rule.

$$\frac{DI}{Dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a}\frac{da}{dt} + \frac{\partial I}{\partial b}\frac{db}{dt}$$

In this case the derivative of the integral with respect to time is

$$\frac{DI}{Dt} = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x', t)dx' + f(b(t), t)\frac{db}{dt} - f(a(t), t)\frac{da}{dt}.$$

Time rate of change due to movement of the boundaries.

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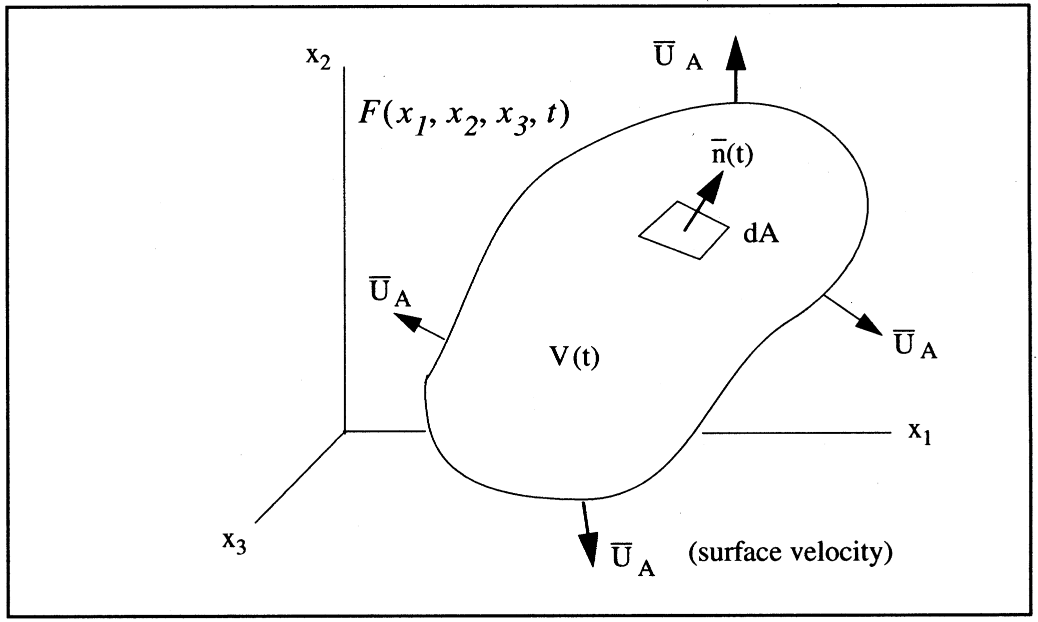
In three dimensions Leibniz' rule describes the time rate of change of the integral of some function of space and time, F, contained inside a control volume V.



$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \frac{\partial F}{\partial t}\,dV + \int_{A(t)} F\,\bar{U}_A \bullet \bar{n}\,dA.$$

Rate of change of the total amount of F in V

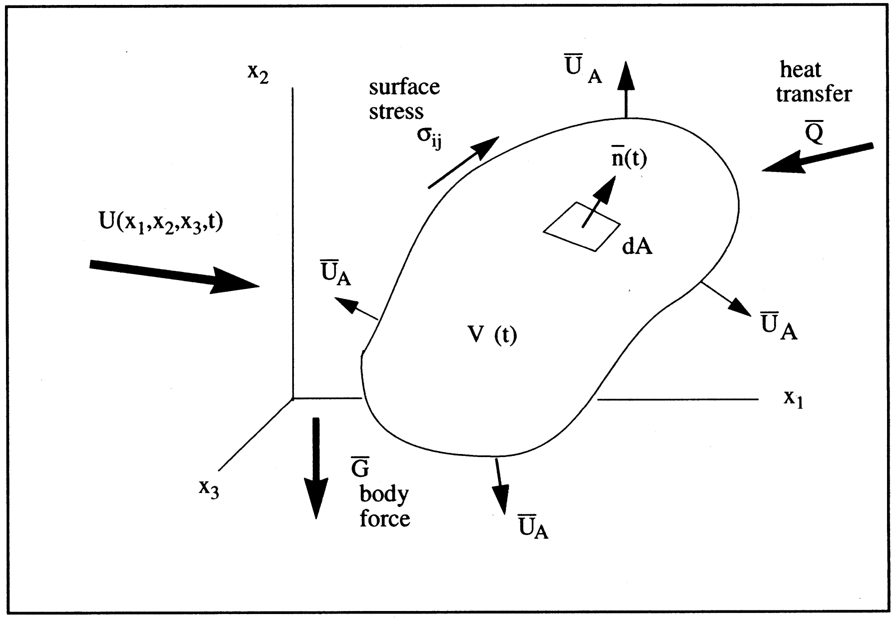
=

Rate due to changes of F within V

+

Rate due to movement of the surface of V

Consider a fluid with the velocity field defined at every point.



Let the velocity of each surface element coincide with the fluid velocity. This is called a Lagrangian control volume.

$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \frac{\partial F}{\partial t}dV + \int_{A(t)} F\,\bar{U}\bullet\bar{n}\,dA\,.$$

Use Gauss's theorem to convert the surface integral to a volume integral.

Reynolds transport theorem

$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \left(\frac{\partial F}{\partial t} + \nabla\bullet(F\bar{U})\right)dV\,.$$

Conservation of mass

The Reynolds transport theorem applied to the density is

$$\frac{D}{Dt}\int_{V(t)} \rho dV = \int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \bar{U})\right) dV .$$

Since there are no sources of mass contained in the control volume and the choice of control volume is arbitrary the kernel of the integral must be zero.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \bar{U}) = 0} .$$

This is the general procedure that we will use to derive the differential form of the equations of motion.

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Incompressible flow

Expand the continuity equation.

$$\frac{\partial\rho}{\partial t} + \overline{U} \bullet \nabla\rho + \rho\nabla \bullet \overline{U} = 0.$$

If the density is constant then the continuity equation reduces to

$$\rho\nabla \bullet \overline{U} = 0.$$

Conservation of momentum

Newtonian stress

The stress tensor in a fluid is composed of two parts; an isotropic part due to the pressure and a symmetric part due to viscous friction.

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$$

where

$$\bar{\bar{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{cases} \delta_{ij} = 1 \ ; & i = j \\ \delta_{ij} = 0 \ ; & i \neq j \end{cases}$$

We deal only with Newtonian fluids for which the stress is linearly related to the rate-of-strain.

$$\tau_{ij} = 2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_v\right)\delta_{ij} S_{kk}$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

Notice that viscous forces contribute to the normal stresses through the non-zero diagonal terms in the stress tensor.

$$\tau_{ij} = 2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_v\right)\delta_{ij}S_{kk}$$

Sum the diagonal terms to generate the mean normal stress

$$\sigma_{mean} = (1/3)\sigma_{ii} = -P + \mu_v S_{kk}.$$

The "bulk viscosity" that appears here is often assumed to be zero. This is the so-called Stokes hypothesis. In general the bulk viscosity is not zero except for monatomic gases but the Stokes hypothesis is often invoked anyway.

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The rate of change of the total amount of momentum inside the control volume is determined by the external forces that act on the control volume surface.

$$\frac{D}{Dt}\int_{V(t)} \rho \bar{U} dV = \int_{A(t)} (-P\bar{\bar{I}} + \bar{\bar{\tau}}) \bullet \bar{n} dA + \int_{V(t)} \rho \bar{G} dV .$$

Use the Reynolds transport theorem to replace the left-hand-side and Gauss's theorem to replace the surface integrals.

$$\int_{V(t)} \left(\frac{\partial \rho \bar{U}}{\partial t} + \nabla \bullet (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) - \rho \bar{G} \right) dV = 0 .$$

Since there are no sources of momentum inside the control volume and the choice of control volume is arbitrary, the kernel must be zero.

$$\boxed{\frac{\partial \rho \bar{U}}{\partial t} + \nabla \bullet (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) - \rho \bar{G} = 0} .$$

Conservation of energy

The rate of change of the total energy inside the control volume is determined by the rate at which the external forces do work on the control volume plus the rate of heat transfer across the control volume surface.

$$\frac{D}{Dt}\int_{V(t)} \rho(e + k)dV = \int_{A(t)} ((-P\bar{\bar{I}} + \bar{\bar{\tau}}) \bullet \bar{U} - \bar{Q}) \bullet \bar{n}dA + \int_{V(t)} (\rho\bar{G} \bullet \bar{U})dV .$$

In a linear heat conducting medium

$$Q_i = -\kappa(\partial T / \partial x_i)$$

Again, use the Reynolds transport theorem to replace the left-hand-side and Gauss's theorem to replace the surface integrals.

$$\int_{V(t)} \left(\frac{\partial\rho(e + k)}{\partial t} + \nabla \bullet \left(\rho\bar{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}\right) - \rho\bar{G} \bullet \bar{U}\right)dV = 0 .$$

Since there are no sources of energy inside the control volume and the choice of control volume is arbitrary, the kernel must be zero.

$$\frac{\partial\rho(e+k)}{\partial t} + \nabla\bullet\left(\rho\bar{U}\left(e+\frac{P}{\rho}+k\right)-\bar{\bar{\tau}}\bullet\bar{U}+\bar{Q}\right)-\rho\bar{G}\bullet\bar{U} = 0.$$

Stagnation enthalpy

$$h_t = e+\frac{P}{\rho}+k = h+\frac{1}{2}U_iU_i$$

Typical gas transport properties at 300K and one atmosphere.

Fluid	$\mu \times 10^5$, kg/(m)(s)	μ_v/μ	$\kappa \times 10^2$, J/(m)(s)(K)	$\frac{\mu}{\rho} \times 10^5$, m^2/s	Pr
He	1.98	0	15.0	12.2	0.67
Ar	2.27	0	1.77	1.40	0.67
H_2	0.887	32	17.3	10.8	0.71
N_2	1.66	0.8	2.52	1.46	0.71
O_2	2.07	0.4	2.58	1.59	0.72
CO_2	1.50	1,000	1.66	0.837	0.75
Air	1.85	0.6	2.58	1.57	0.71
H_2O (*liquid*)	85.7	3.1	61	0.0857	6.0
Ethyl alcohol	110	4.5	18.3	0.14	15
Glycerine	134,000	0.4	29	109	11,000

$$Pr = \frac{\mu C_p}{\kappa}$$

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Summary - differential equations of motion

$$\left.\begin{aligned}
&\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\bar{U}) = 0\\
&\frac{\partial\rho\bar{U}}{\partial t} + \nabla\bullet(\rho\bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) - \rho\bar{G} = 0\\
&\frac{\partial\rho(e+k)}{\partial t} + \nabla\bullet\left(\rho\bar{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}}\bullet\bar{U} + \bar{Q}\right) - \rho\bar{G}\bullet\bar{U} = 0
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i) = 0\\
&\frac{\partial\rho U_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_i U_j + P\delta_{ij} - \tau_{ij}) - \rho G_i = 0\\
&\frac{\partial\rho(e+k)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho U_i\left(e + \frac{P}{\rho} + k\right) - \tau_{ij}U_j + Q_i\right) - \rho G_i U_i = 0
\end{aligned}\right\}.$$

Integral form of the equations of motion

Recall the Leibniz rule

$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \frac{\partial F}{\partial t}dV + \int_{A(t)} F\,\bar{U}_A \bullet \bar{n}\,dA\,.$$

Integral equations on an Eulerian control volume

If the surface of the control volume is fixed in space, ie, the velocity of the surface is zero then

$$\frac{d}{dt}\int_V F\,dV = \int_V \frac{\partial F}{\partial t}dV$$

This is called an Eulerian control volume.

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The integral form of the continuity equation on an Eulerian control volume is derived as follows.

$$\frac{d}{dt}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV$$

Use the differential equation for continuity to replace the partial derivative inside the integral on the right-hand-side

$$\frac{d}{dt}\int_V \rho dV = -\int_V \nabla \bullet (\rho \bar{U}) dV$$

Use the Gauss theorem to convert the volume integral to a surface integral. The integral form of the continuity equation is:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \bar{U} \bullet \bar{n} dA = 0$$

The integral form of the conservation equations on an Eulerian control volume is

$$\frac{d}{dt}\int_V \rho dV + \int_A (\rho\bar{U}) \bullet \bar{n} dA = 0$$

$$\frac{d}{dt}\int_V \rho\bar{U} dV + \int_A (\rho\bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA - \int_V \rho\bar{G} dV = 0$$

$$\frac{d}{dt}\int_V \rho(e+k) dV + \int_A \left(\rho\bar{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}\right) \bullet \bar{n} dA - \int_V (\rho\bar{G} \bullet \bar{U}) dV = 0$$

Mixed Eulerian-Lagrangian control volumes

The integral form of the continuity equation on a Mixed Eulerian-Lagrangian control volume is derived as follows. Let F in Liebniz rule be the fluid density.

$$\frac{D}{Dt}\int_{V(t)} \rho dV = \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{A(t)} \rho \, \bar{U}_A \bullet \bar{n} dA$$

Use the differential equation for continuity to replace the partial derivative inside the first integral on the right-hand-side and use the Gauss theorem to convert the volume integral to a surface integral. The integral form of the continuity equation on a Mixed Eulerian-Lagrangian control volume is

$$\frac{D}{Dt}\int_{V(t)} \rho dV = -\int_{A(t)} \rho \bar{U} \bullet \bar{n} dA + \int_{A(t)} \rho \, \bar{U}_A \bullet \bar{n} dA$$

The integral equations of motion on a general control volume where the surface velocity is not the same as the fluid velocity are derived in a similar way.

The most general integral form of the conservation equations is

$$\frac{D}{Dt}\int_{V(t)} \rho dV + \int_{A(t)} \rho(\bar{U} - \bar{U}_A) \bullet \bar{n} dA = 0$$

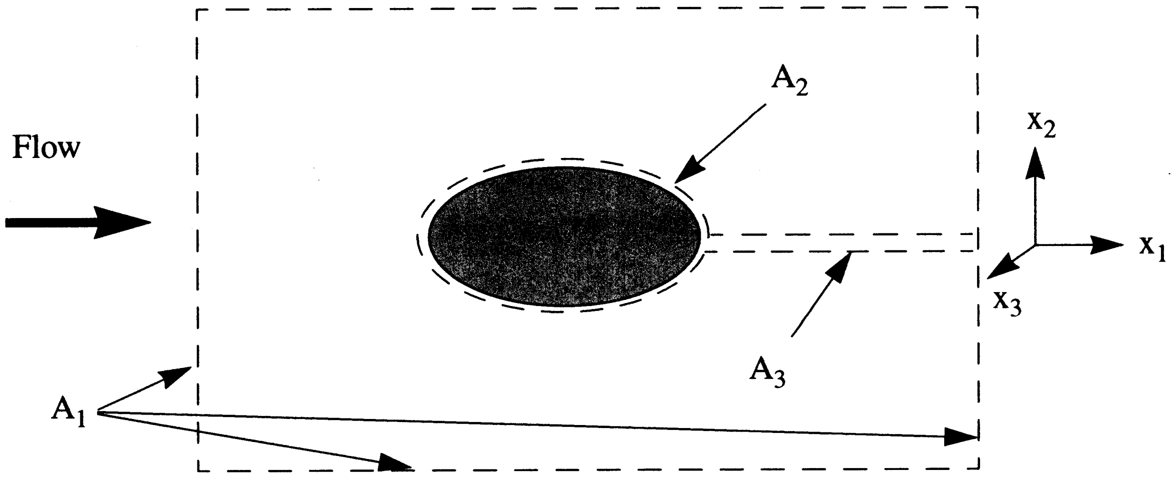
$$\frac{D}{Dt}\int_{V(t)} \rho\bar{U} dV + \int_{A(t)} (\rho\bar{U}(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA - \int_{V(t)} \rho\bar{G} dV = 0$$

$$\frac{D}{Dt}\int_{V(t)} \rho(e+k) dV + \int_{A(t)} (\rho(e+k) \bullet (\bar{U} - \bar{U}_A) + P\bar{\bar{I}} \bullet \bar{U} - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}) \bullet \bar{n} dA - \int_{V(t)} (\rho\bar{G} \bullet \bar{U}) dV = 0$$

Remember U_A is the velocity of the control volume surface.

Examples of control volume analysis

Example 1 - Solid body at rest, steady flow



Integral form of mass conservation

$$\int_{A_1} (\rho \bar{U}) \bullet \bar{n} dA = 0.$$

Integral form of momentum conservation

$$\int_{A_1} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA + \int_{A_2} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA = 0$$

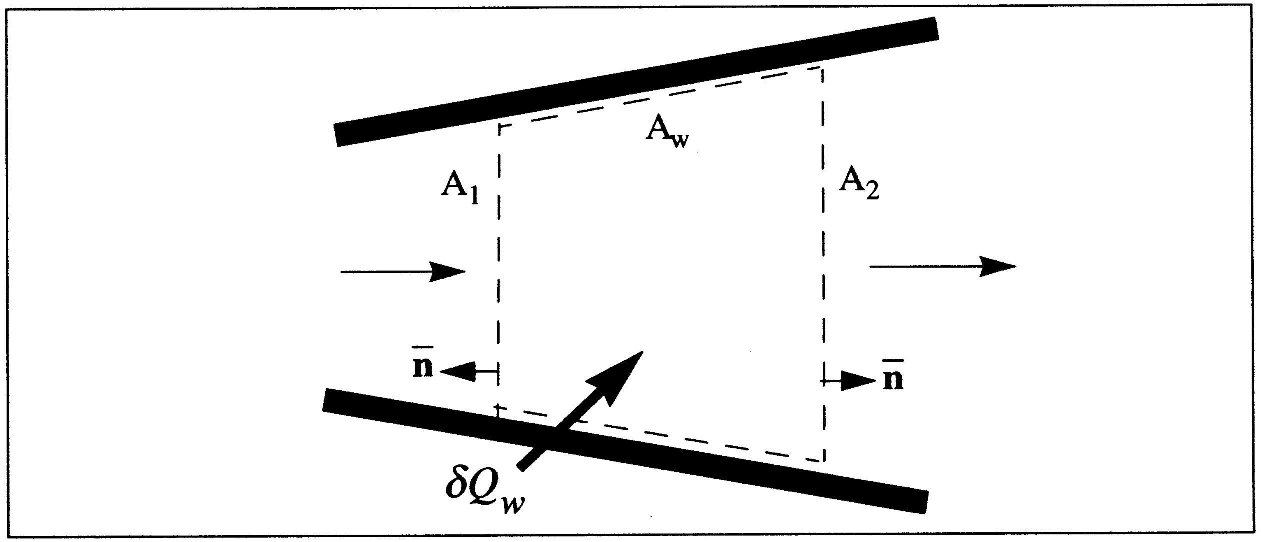
Momentum fluxes in the streamwise and normal directions are equal to the lift and drag forces exerted by the flow on the body.

$$Drag = \int_{A_2} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n}dA \Big|_{x_1} \quad ; \qquad Lift = \int_{A_2} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n}dA \Big|_{x_2} .$$

$$\int_{A_1} (\rho\bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n}dA \Big|_{x_1} + Drag = 0.$$

$$\int_{A_1} (\rho\bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n}dA \Big|_{x_2} + Lift = 0.$$

Example 2 - Channel flow with heat addition



Mass conservation

$$\int_{A_1} (\rho \bar{U}) \bullet \bar{n} dA + \int_{A_2} (\rho \bar{U}) \bullet \bar{n} dA = 0.$$

Energy conservation

$$\int_{A} (\rho \bar{U}(e + k) + P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}) \bullet \bar{n} dA = 0.$$

To a good approximation the energy balance becomes

$$\int_A \rho\bar{U}\left(e + \frac{P}{\rho} + k\right) \bullet \bar{n}dA = -\int_A \bar{Q} \bullet ndA .$$

Most of the conductive heat transfer is through the wall.

$$-\int_A \bar{Q} \bullet ndA \cong -\int_{A_w} \bar{Q} \bullet ndA = \delta Q .$$

The energy balance reduces to

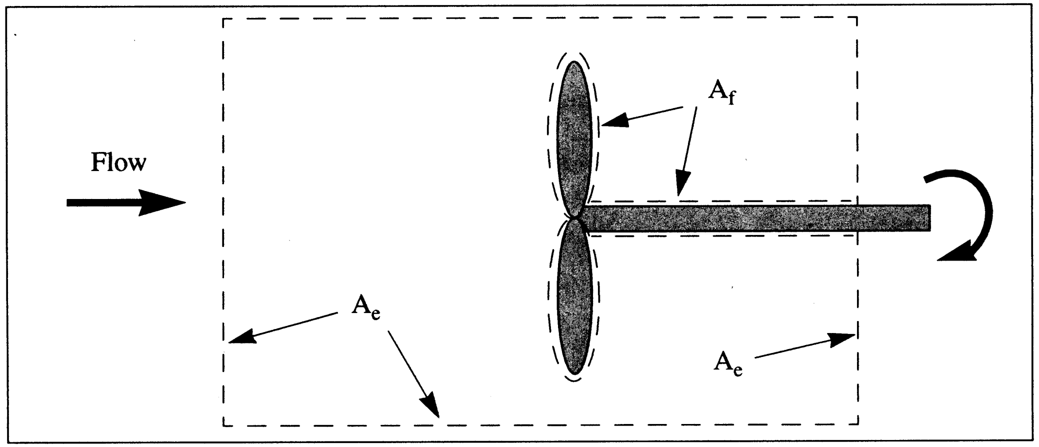
$$\int_{A_2} \rho h_t \bar{U} \bullet \bar{n}dA + \int_{A_1} \rho h_t \bar{U} \bullet \bar{n}dA = \delta Q .$$

When the vector multiplication is carried out the energy balance becomes

$$\int_{A_2} \rho_2 U_2 h_{t2} dA - \int_{A_1} \rho_1 U_1 h_{t1} dA = \delta Q .$$

The heat addition (or removal) per unit mass flow is equal to the change in stagnation enthalpy of the flow.

Example 3 - A rotating fan in a stationary flow



The control volume surface is attached to and moves with the fan surface.

The integrated mass fluxes are zero.

$$\int_{A_e} (\rho \bar{U}) \bullet \bar{n} dA = 0$$

Momentum fluxes are equal to the surface forces on the fan

$$\int_{A_e} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA + \int_{A_f} (\rho \bar{U}(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA = 0$$

$$\int_{A_e} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA + \bar{F} = 0$$

The vector force by the flow on the fan is

$$\bar{F} = \int_{A_f} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA$$

The flow and fan velocity on the fan surface are the same due to the no-slip condition.

The integrated energy fluxes are equal to the work done by the flow on the fan.

$$\int_{A_e} (\rho\bar{U}(e+k) + P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U}) \bullet \bar{n} dA + \int_{A_f} (P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U}) \bullet \bar{n} dA = 0.$$

$$Work = \int_{A_f} (P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U}) \bullet \bar{n} dA = \delta W$$

If the flow is adiabatic and work by viscous normal stresses is neglected the energy equation becomes.

$$\int_{A_e} \rho\left(e + \frac{P}{\rho} + k\right)\bar{U} \bullet \bar{n} dA + \delta W = 0$$

The work per unit mass flow is equal to the change in stagnation enthalpy of the flow.

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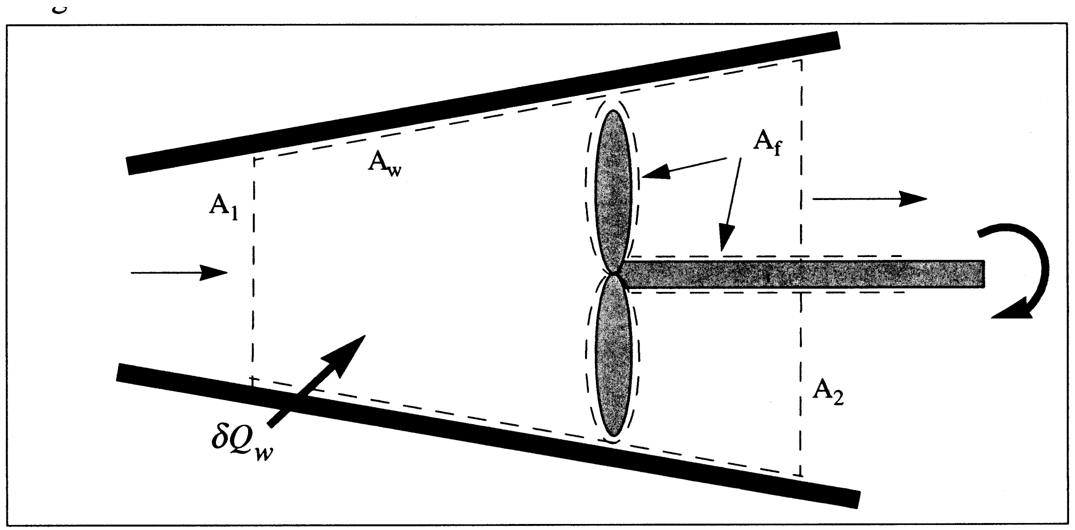
Example 4 - Combined heat transfer and work

In a general situation with heat transfer and work



$$\int_{A_2} \rho_2 U_2 h_{t2} dA - \int_{A_1} \rho_1 U_1 h_{t1} dA = \delta Q - \delta W.$$

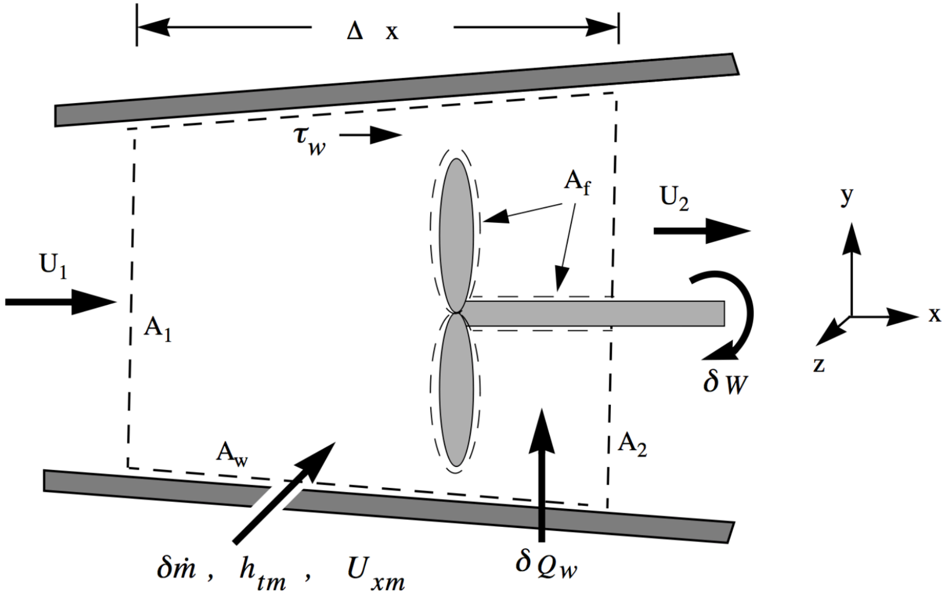
Quasi-one dimensional conservation equations

Recall the enthalpy

$$h = e + \frac{P}{\rho}$$

Stagnation enthalpy

$$h_t = h + \frac{1}{2}U^2$$



General steady channel flow with friction, heat exchange, mass exchange and work

mass conservation

$$\int_{A_2} \rho \bar{U} \cdot \bar{n} dA = -\int_{A_1} \rho \bar{U} \cdot \bar{n} dA + \delta \dot{m}$$

momentum conservation

$$\int_{A_2} \left(\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}} \right) \cdot \bar{n} dA = -\int_{A_1} \left(\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}} \right) \cdot \bar{n} dA - \int_{A_w} \left(P\bar{\bar{I}} - \bar{\bar{\tau}} \right) \cdot \bar{n} dA - \int_{A_f} \left(P\bar{\bar{I}} - \bar{\bar{\tau}} \right) \cdot \bar{n} dA + U_{xm} \delta \dot{m}$$

energy conservation

Work by the fan

$$\int_{A_2} \left(\rho h_t \bar{U} - \bar{\bar{\tau}} \cdot \bar{U} + \bar{Q} \right) \cdot \bar{n} dA = -\int_{A_1} \left(\rho h_t \bar{U} - \bar{\bar{\tau}} \cdot \bar{U} + \bar{Q} \right) \cdot \bar{n} dA + \int_{A_w} \bar{Q} \cdot \bar{n}\, dA + h_{tm} \delta \dot{m} + \int_{A_f} \left(P\bar{U} - \bar{\bar{\tau}} \cdot \bar{U} \right) \cdot \bar{n} dA$$

Area averaged flow

Average the flow across the channel.

$$\hat{\rho}(x) = \frac{1}{A(x)}\int \rho(x, y, z)dydz.$$

Define

$$\hat{T}(x) = \frac{1}{A(x)}\int T(x, y, z)dydz$$

$$\hat{P}(x) = \frac{1}{A(x)}\int P(x, y, z)dydz$$

$$\hat{s}(x) = \frac{1}{A(x)}\int s(x, y, z)dydz$$

$$\hat{U}(x) = \frac{1}{A(x)}\int U(x, y, z)dydz$$

$$\hat{\tau}_{xx}(x) = \frac{1}{A(x)}\int \tau_{xx}(x, y, z)dydz$$

$$\hat{Q}_x(x) = \frac{1}{A(x)}\int Q_x(x, y, z)dydz$$

Every variable in the flow can be written as a mean plus a fluctuation.

$$\rho(x, y, z) = \hat{\rho}(x) + \rho'(x, y, z)$$

$$T(x, y, z) = \hat{T}(x) + T'(x, y, z)$$

$$P(x, y, z) = \hat{P}(x) + P'(x, y, z)$$

$$s(x, y, z) = \hat{s}(x) + s'(x, y, z)$$

$$U(x, y, z) = \hat{U}(x) + U'(x, y, z)$$

$$\tau_{xx}(x, y, z) = \hat{\tau}_{xx}(x) + \tau_{xx}'(x, y, z)$$

$$Q_x(x, y, z) = \hat{Q}_x(x) + Q_x'(x, y, z)$$

Express the mass flux integral in terms of means and fluctuations.

$$\int_A \rho U dA = \int_A (\hat{\rho} + \rho')(\hat{U} + U')dA = \int_A \hat{\rho}\hat{U} dA + \int_A \rho' U' dA =$$

$$\hat{\rho}(x)\hat{U}(x)A(x) + \overline{\rho' U'}A(x)$$

As long as nonlinear correlations are small, the mean is an accurate approximation.

In terms of area averaged variables, the integral equations of motion are as follows.

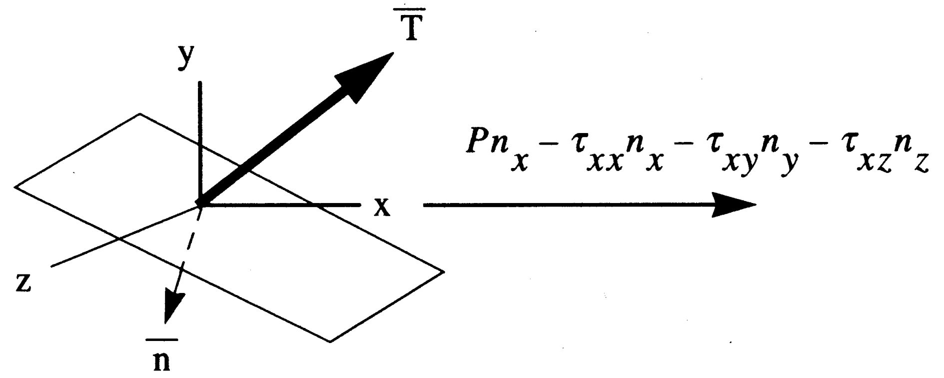
$$\hat{\rho}_2 \hat{U}_2 A_2 - \hat{\rho}_1 \hat{U}_1 A_1 = \delta \dot{m}$$

$$(\hat{\rho}_2 \hat{U}_2 \hat{U}_2 + \hat{P}_2 - \hat{\tau}_{xx2}) A_2 - (\hat{\rho}_1 \hat{U}_1 \hat{U}_1 + \hat{P}_1 - \hat{\tau}_{xx1}) A_1 + \left. \int_{A_w} (P\bar{\bar{\delta}} - \bar{\bar{\tau}}) \bullet \bar{n}\, dA \right|_x - U_{xm} \delta \dot{m} + \delta F_x = 0$$

$$(\hat{\rho}_2 \hat{H}_2 \hat{U}_2 - \hat{\tau}_{xx2} \hat{U}_2 + \hat{Q}_2) A_2 - (\hat{\rho}_1 \hat{H}_1 \hat{U}_1 - \hat{\tau}_{xx1} \hat{U}_1 + \hat{Q}_1) A_1 = \delta Q + h_{tm} \delta \dot{m} - \delta W$$

The traction vector

The pressure-stress integral on the wall.



The traction vector

$$\bar{T} = (P\bar{\bar{\delta}} - \bar{\bar{\tau}}) \bullet \bar{n} = \begin{bmatrix} Pn_x - \tau_{xx}n_x - \tau_{xy}n_y - \tau_{xz}n_z \\ -\tau_{xy}n_x + Pn_y - \tau_{yy}n_y - \tau_{yz}n_z \\ -\tau_{zx}n_z - \tau_{zy}n_y + Pn_z - \tau_{zz}n_z \end{bmatrix}$$

Imagine the length of the control volume made very small.

$$\int_{A_w} (P\bar{\bar{\delta}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \bigg|_x = \int_{A_w} (P n_x - \tau_{xx} n_x - \tau_{xy} n_y - \tau_{xz} n_z) dA \cong$$

$$\left(\frac{P_1 + P_2}{2}\right)(A_1 - A_2) - \left(\frac{\tau_{xx1} + \tau_{xx2}}{2}\right)(A_1 - A_2) + \tau_w \pi \left(\frac{D_1 + D_2}{2}\right)\Delta x$$

where

$$\int_{A_w} n_x dA = (A_1 - A_2)$$

and

$$\int_{A_w} (\tau_{xy} n_y + \tau_{xz} n_z) dA \equiv -\tau_w \pi \left(\frac{D_1 + D_2}{2}\right)\Delta x .$$

Introduce the hydraulic diameter

$$D = \left(\frac{4A}{\pi}\right)^{1/2}.$$

The integrated equations of motion now take the form

$$\rho_2 U_2 A_2 - \rho_1 U_1 A_1 = \delta\dot{m}$$

$$(\rho_2 U_2 U_2 + P_2 - \tau_{xx2})A_2 - (\rho_1 U_1 U_1 + P_1 - \tau_{xx1})A_1 -$$
$$\left(\frac{P_1 + P_2}{2}\right)(A_2 - A_1) + \left(\frac{\tau_{xx1} + \tau_{xx2}}{2}\right)(A_2 - A_1) + \tau_w \pi\left(\frac{D_1 + D_2}{2}\right)\Delta x =$$
$$U_{xm}\delta\dot{m} - \delta F_x$$

$$(\rho_2 h_{t2} U_2 - \tau_{xx2} U_2 + Q_{x2})A_2 - (\rho_1 h_{t1} U_1 - \tau_{xx1} U_1 + Q_{x1})A_1 =$$
$$\delta Q + h_{tm}\delta\dot{m} - \delta W$$

where the “hat” has been dropped.

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Let the length of the control volume go to zero.

$$\rho_2 U_2 A_2 - \rho_1 U_1 A_1 \quad \Rightarrow \quad d(\rho U A)$$

$$\rho_2 U_2^2 A_2 - \rho_1 U_1^2 A_1 \quad \Rightarrow \quad d(\rho U^2 A)$$

$$P_2 A_2 - P_1 A_1 \quad \Rightarrow \quad d(PA)$$

$$\tau_{xx2} A_2 - \tau_{xx1} A_1 \quad \Rightarrow \quad d(\tau_{xx} A)$$

$$\left(\frac{P_1 + P_2}{2}\right)(A_2 - A_1) \quad \Rightarrow \quad PdA$$

$$\left(\frac{\tau_{xx1} + \tau_{xx2}}{2}\right)(A_2 - A_1) \quad \Rightarrow \quad \tau_{xx} dA$$

$$\tau_w \pi \left(\frac{D_1 + D_2}{2}\right)\Delta x \quad \Rightarrow \quad \tau_w \pi D dx$$

$$\rho_2 U_2 h_{t2} A_2 - \rho_1 U_1 h_{t1} A_1 \quad \Rightarrow \quad d(\rho U h_t A)$$

$$\tau_{xx2} U_2 A_2 - \tau_{xx1} U_1 A_1 \quad \Rightarrow \quad d(\tau_{xx} U A)$$

$$Q_{x2} A_2 - Q_{x1} A_1 \quad \Rightarrow \quad d(Q_x A)$$

The integrated equations are now expressed in terms of differentials.

$$d(\rho U A) = \delta \dot{m}$$

$$d(\rho U^2 A) + d(PA) - d(\tau_{xx} A) - P dA + \tau_{xx} dA = -\tau_w \pi D dx + U_{xm} \delta \dot{m} - \delta F_x$$

$$d(\rho U h_t A) + (-d(\tau_{xx} U A)) + d(Q_x A) = \delta Q + h_{tm} \delta \dot{m} - \delta W$$

Use continuity to simplify the momentum and energy equations.

$$U \delta \dot{m} + \rho U A dU + A dP - A d\tau_{xx} = -\tau_w \pi D dx + U_{xm} \delta \dot{m} - \delta F_x$$

$$h_t \delta \dot{m} + \rho U A dh_t - \frac{\tau_{xx}}{\rho} \delta \dot{m} - \rho U A d\left(\frac{\tau_{xx}}{\rho}\right) + \frac{Q_x}{\rho U} \delta \dot{m} + \rho U A d\left(\frac{Q_x}{\rho U}\right) = \delta Q + h_{tm} \delta \dot{m} - \delta W$$

The 1-D (area averaged) equations of motion.

$$d(\rho U A) = \delta\dot{m}$$

$$d(P - \tau_{xx}) + \rho U dU = -\tau_w\left(\frac{\pi D dx}{A}\right) + \frac{(U_{xm} - U)\delta\dot{m}}{A} - \frac{\delta F_x}{A}$$

$$d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) = \frac{\delta Q}{\rho U A} - \frac{\delta W}{\rho U A} + \left(h_{tm} - \left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right)\right)\frac{\delta\dot{m}}{\rho U A}$$

Introduce the friction coefficient.

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

and the heat and work per unit mass flow

$$\delta q = \frac{\delta Q}{\rho U A} \quad ; \qquad \delta w = \frac{\delta W}{\rho U A}.$$

Finally the area averaged equations of motion take the concise form

$$d(\rho U A) = \delta \dot{m}$$

$$d(P - \tau_{xx}) + \rho U dU = -\frac{1}{2}\rho U^2\left(4C_f\frac{dx}{D}\right) + \frac{(U_{xm} - U)\delta\dot{m}}{A} - \frac{\delta F_x}{A}$$

$$d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) = \delta q - \delta w + \left(h_{tm} - \left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right)\right)\frac{\delta\dot{m}}{\rho U A}$$

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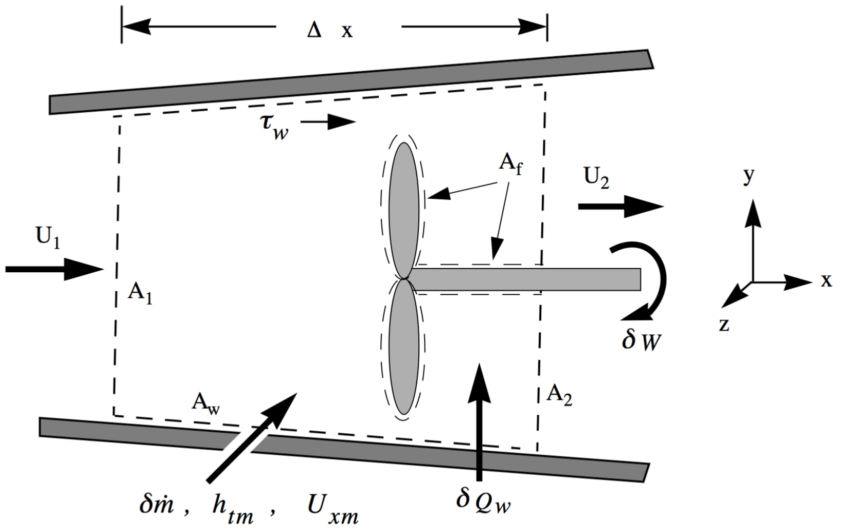
Working equations for quasi-one-dimensional flow



General <u>steady</u> channel flow with friction, heat exchange, mass exchange and work

$$\Delta x \rightarrow 0$$

mass conservation

$$d(\rho U A) = \delta \dot{m}$$

momentum conservation

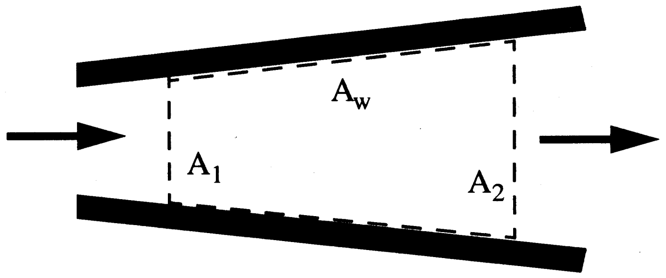
$$d(P - \tau_{xx}) + \rho U dU = -\frac{1}{2}\rho U^2 \left(\frac{4C_f dx}{D}\right) - \frac{\delta F_x}{A} + \left(\frac{U_{xm} - U}{A}\right)\delta \dot{m}$$

energy conservation

$$d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) = \frac{\delta Q}{\rho U A} - \frac{\delta W}{\rho U A} + \left(h_{tm} - \left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right)\right)\frac{\delta \dot{m}}{\rho U A}$$

By averaging flow properties over the channel cross-section, to a good approximation the full steady equations of motion can be reduced to a set of equations that describe differential changes of the flow.

Steady, gravity-free, adiabatic flow of a compressible fluid in a channel



Note, friction is not zero

For this case the energy equation takes the form of a perfect differential.

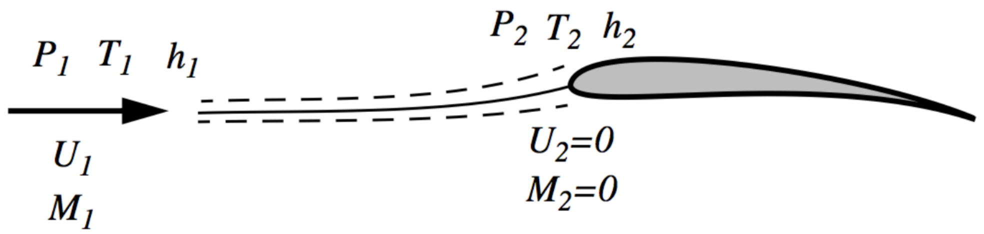
$$d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) = 0.$$

For most flow situations (outside of shock waves) the stress and heat conduction terms can be neglected. Thus

$$h_{t2} = h_{t1}.$$

Stagnation temperature and pressure,cont'd

Compressible flow stagnating against a wing leading edge



If the path from 1 to 2 is adiabatic $h_2 = h_t = h_1 + \frac{1}{2}U_1^2$

If the temperature change between 1 and 2 is not large then the heat capacity will be nearly constant. The stagnation temperature is defined by

$$C_p T_2 = C_p T_t = C_p T_1 + \frac{1}{2}U_1^2$$

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Stagnation temperature and pressure,cont'd

The stagnation temperature is defined by

$$C_p T_2 = C_p T_t = C_p T_1 + \frac{1}{2} U_1^2$$

We can express this in terms of the Mach number

$$\frac{T_t}{T_1} = 1 + \left(\frac{\gamma - 1}{2}\right) M_1^2$$

$$M = \frac{U}{a} = \frac{U}{(\gamma R T)^{1/2}}$$

If the path from 1 to 2 is isentropic $\quad P_2 = P_{t2}$

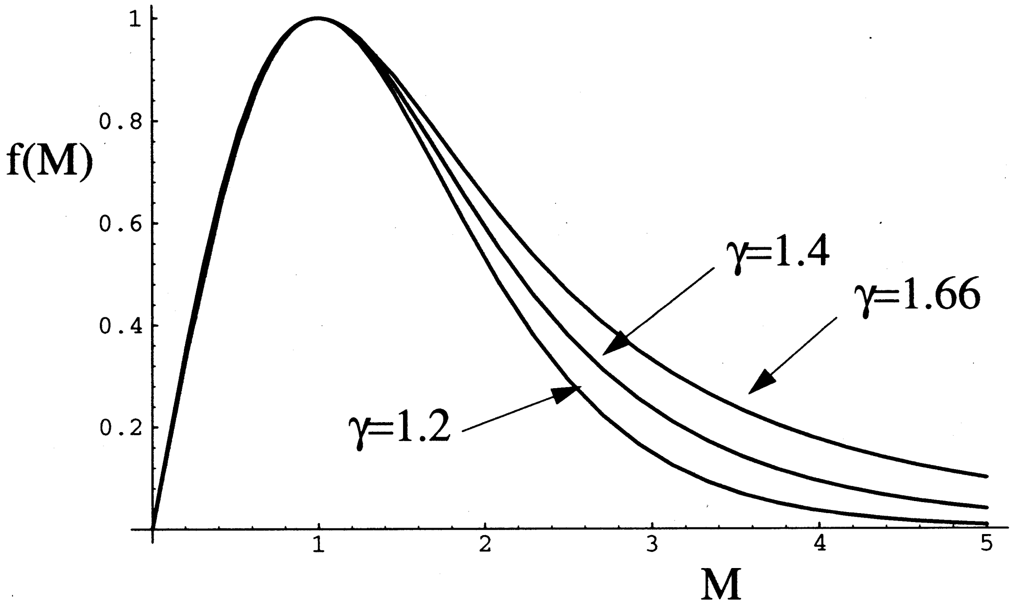
$$\frac{P_t}{P_1} = \left(\frac{T_t}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \left(\frac{\gamma - 1}{2}\right) M_1^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\gamma = \frac{C_p}{C_v}$$

Isentropic flow is realized If viscous friction and heat transfer are zero (adiabatic frictionless flow).

The mass flow at any point in a channel can be expressed in terms of the local stagnation pressure and temperature

$$\dot{m} = \rho U A = \frac{\gamma}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \left(\frac{P_t A}{\sqrt{\gamma R T_t}}\right) f(M)$$



$$f(M) = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{M}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}$$