AA103 Homework 3, 2020 - 2021

Cantwell Spring 2020-21

Due April 19, 2021

Suggested Reading

Read AA210 Course reader Chapters 1 and chapter 8 sections 8.1, 8.2 and 8.3.

The problems in this homework are designed to help you familiarize yourself with the equations governing fluid flow.

Problem 1

The figure below shows a sketch of plane, incompressible Couette flow (AA210 Course reader Figure 1.9) that we discussed last week. Assume that the flow is steady and that there is no variation of the flow

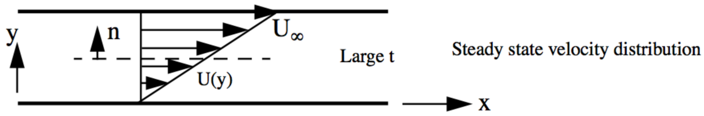

Figure 1: Steady, incompressible Couette flow after a period of start-up

velocity or pressure in either the x or z (into the paper) directions.

$$V = W = 0 \tag{1}$$

$$\frac{\partial()}{\partial t} = \frac{\partial()}{\partial x} = \frac{\partial()}{\partial z} = 0 \tag{2}$$

Use these assumptions to eliminate various terms in the incompressible equations of motion.
a) Show that the y momentum equation reduces to $\frac{\partial P}{\partial y} = 0$.
b) Show that the x momentum equation reduces to $\frac{\partial^2 U}{\partial y^2} = 0$.
c) Integrate the result to show that $U = U_\infty (y/d)$ where d is the distance between the plates.

Problem 2

The figure below shows a sketch of plane compressible Couette flow (AA210 Course reader Figure 8.2) Use the same assumptions to eliminate various terms in the compressible equations of motion.
a) Show that the y momentum equation again reduces to $\frac{\partial P}{\partial y} = 0$.
b) Show that the x momentum equation reduces to

$$\frac{\partial \left(\mu \frac{\partial u}{\partial y}\right)}{\partial y} = 0 \tag{3}$$

where μ is the viscosity of the gas.
3) Show that the energy equation reduces to

$$\frac{\partial \left(\mu U \frac{\partial U}{\partial y} + \kappa \frac{\partial T}{\partial y}\right)}{\partial y} = 0 \tag{4}$$

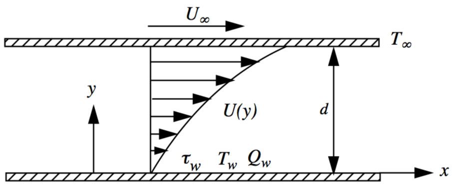

Figure 8.2: *Flow produced between two parallel plates in relative motion.*

Figure 2: Steady compressible Couette flow after a period of start-up

where κ is the thermal conductivity of the gas. Compare with the results in the AA210 Course reader, Equation (8.6).

Problem 3

The expansion into vacuum of a spherical cloud of a monatomic gas such as helium has a well-known exact solution of the equations for compressible isentropic flow. The velocity field is

$$U = \frac{xt}{t_0^2 + t^2} \qquad V = \frac{yt}{t_0^2 + t^2} \qquad W = \frac{zt}{t_0^2 + t^2}. \tag{5}$$

The density and pressure are

$$\begin{aligned} \frac{\rho}{\rho_0} &= \frac{t_0^3}{\left(t_0^2+t^2\right)^{3/2}} \left(1 - \frac{t_0^2}{R_{initial}^2} \left(\frac{x^2+y^2+z^2}{t_0^2+t^2}\right)\right)^{3/2} \\ \frac{P}{P_0} &= \left(\frac{\rho}{\rho_0}\right)^{5/3} \end{aligned} \tag{6}$$

where $R_{initial} = ({x_0}^2 + {y_0}^2 + {z_0}^2)^{1/2}$ is the initial radius of the cloud at $t = 0$ when the cloud is all at rest. The parameter t_0 is a timescale that determines how fast the cloud front accelerates into the vacuum of space. This problem has served as a model of the late stage of evolution of an expanding gas nebula from an exploding star.

1) Determine the particle paths $(x(t)/x_0, y(t)/y_0, z(t)/z_0)$ of fluid elements initially located at $R_{initial}$.
2) Plot the radius of the cloud, $R/R_{initial}$, as a function of time.
3) Plot the distribution of density at $t = 0, t = t_0, t = 3t_0$.
4) Work out the substantial derivative of the density, $D\rho/Dt = \partial\rho/\partial t + U \cdot \nabla\rho$.

You can answer item (4) the easy way or the hard way. If you decide to do it the hard way, I suggest you use Mathematica or Matlab to do the needed differentiation.