Distributed Optimization via
Alternating Direction Method of Multipliers

Stephen Boyd
Springer Lectures, UC Berkeley, 4/3/15

source:

Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers (Boyd, Parikh, Chu, Peleato, Eckstein)
Goals

robust methods for

- arbitrary-scale optimization
  - machine learning/statistics with huge data-sets
  - dynamic optimization on large-scale network
  - computer vision

- decentralized optimization
  - devices/processors/agents coordinate to solve large problem, by passing relatively small messages
Outline

Dual decomposition

Method of multipliers

Alternating direction method of multipliers

Common patterns

Examples

Consensus and exchange

Conclusions
Outline

Dual decomposition

Method of multipliers

Alternating direction method of multipliers

Common patterns

Examples

Consensus and exchange

Conclusions
Dual problem

- convex equality constrained optimization problem
  
  \[
  \begin{align*}
  \text{minimize} & \quad f(x) \\
  \text{subject to} & \quad Ax = b
  \end{align*}
  \]

- Lagrangian: \( L(x, y) = f(x) + y^T(Ax - b) \)

- dual function: \( g(y) = \inf_x L(x, y) \)

- dual problem: maximize \( g(y) \)

- recover \( x^* = \arg\min_x L(x, y^*) \)
Dual ascent

- gradient method for dual problem: $y^{k+1} = y^k + \alpha^k \nabla g(y^k)$

- $\nabla g(y^k) = A\tilde{x} - b$, where $\tilde{x} = \arg\min_x L(x, y^k)$

- dual ascent method is

\[
\begin{align*}
x^{k+1} &:= \arg\min_x L(x, y^k) \quad \text{// } x\text{-minimization} \\
y^{k+1} &:= y^k + \alpha^k (Ax^{k+1} - b) \quad \text{// dual update}
\end{align*}
\]

- works, with lots of strong assumptions
suppose $f$ is separable:

$$f(x) = f_1(x_1) + \cdots + f_N(x_N), \quad x = (x_1, \ldots, x_N)$$

then $L$ is separable in $x$: $L(x, y) = L_1(x_1, y) + \cdots + L_N(x_N, y) - y^T b,$

$$L_i(x_i, y) = f_i(x_i) + y^T A_i x_i$$

$x$-minimization in dual ascent splits into $N$ separate minimizations

$$x_i^{k+1} := \arg\min_{x_i} L_i(x_i, y^k)$$

which can be carried out in parallel
Dual decomposition

- dual decomposition (Everett, Dantzig, Wolfe, Benders 1960–65)

\[ x_i^{k+1} := \arg\min_{x_i} L_i(x_i, y^k), \quad i = 1, \ldots, N \]

\[ y^{k+1} := y^k + \alpha^k (\sum_{i=1}^{N} A_i x_i^{k+1} - b) \]

- scatter \( y^k \); update \( x_i \) in parallel; gather \( A_i x_i^{k+1} \)

- solve a large problem
  - by iteratively solving subproblems (in parallel)
  - dual variable update provides coordination

- works, with lots of assumptions; often slow
Outline

Dual decomposition

Method of multipliers

Alternating direction method of multipliers

Common patterns

Examples

Consensus and exchange

Conclusions
Method of multipliers

- a method to robustify dual ascent

- use augmented Lagrangian (Hestenes, Powell 1969), $\rho > 0$

$$L_\rho(x, y) = f(x) + y^T(Ax - b) + (\rho/2)\|Ax - b\|^2_2$$

- method of multipliers (Hestenes, Powell; analysis in Bertsekas 1982)

$$x^{k+1} := \arg\min_x L_\rho(x, y^k)$$

$$y^{k+1} := y^k + \rho(Ax^{k+1} - b)$$

(note specific dual update step length $\rho$)
Method of multipliers dual update step

- optimality conditions (for differentiable $f$):

$$Ax^* - b = 0, \quad \nabla f(x^*) + A^T y^* = 0$$

(primal and dual feasibility)

- since $x^{k+1}$ minimizes $L_\rho(x, y^k)$

$$0 = \nabla_x L_\rho(x^{k+1}, y^k)$$
$$= \nabla_x f(x^{k+1}) + A^T (y^k + \rho(Ax^{k+1} - b))$$
$$= \nabla_x f(x^{k+1}) + A^T y^{k+1}$$

- dual update $y^{k+1} = y^k + \rho(x^{k+1} - b)$ makes $(x^{k+1}, y^{k+1})$ dual feasible

- primal feasibility achieved in limit: $Ax^{k+1} - b \to 0$
Method of multipliers

(compared to dual decomposition)

- **good news**: converges under much more relaxed conditions
  \( f \) can be nondifferentiable, take on value \( +\infty, \ldots \)

- **bad news**: quadratic penalty destroys splitting of the \( x \)-update, so can’t do decomposition
Alternating direction method of multipliers

- a method
  - with good robustness of method of multipliers
  - which can support decomposition

- “robust dual decomposition” or “decomposable method of multipliers”

- proposed by Gabay, Mercier, Glowinski, Marrocco in 1976
Alternating direction method of multipliers

- ADMM problem form (with $f$, $g$ convex)

  minimize $f(x) + g(z)$
  subject to $Ax + Bz = c$

- two sets of variables, with separable objective

  $L_\rho(x, z, y) = f(x) + g(z) + y^T(Ax + Bz - c) + (\rho/2)\|Ax + Bz - c\|_2^2$

- ADMM:

  $x^{k+1} := \text{argmin}_x L_\rho(x, z^k, y^k)$ // $x$-minimization

  $z^{k+1} := \text{argmin}_z L_\rho(x^{k+1}, z, y^k)$ // $z$-minimization

  $y^{k+1} := y^k + \rho(Ax^{k+1} + Bz^{k+1} - c)$ // dual update
Alternating direction method of multipliers

- if we minimized over $x$ and $z$ jointly, reduces to method of multipliers
- instead, we do one pass of a Gauss-Seidel method
- we get splitting since we minimize over $x$ with $z$ fixed, and vice versa
ADMM and optimality conditions

- optimality conditions (for differentiable case):
  - primal feasibility: $Ax + Bz - c = 0$
  - dual feasibility: $\nabla f(x) + A^T y = 0, \quad \nabla g(z) + B^T y = 0$

- since $z^{k+1}$ minimizes $L_\rho(x^{k+1}, z, y^k)$ we have
  \[
  0 = \nabla g(z^{k+1}) + B^T y^k + \rho B^T (Ax^{k+1} + Bz^{k+1} - c) \\
  = \nabla g(z^{k+1}) + B^T y^{k+1}
  \]

- so with ADMM dual variable update, $(x^{k+1}, z^{k+1}, y^{k+1})$ satisfies second dual feasibility condition

- primal and first dual feasibility are achieved as $k \to \infty$
ADMM with scaled dual variables

- combine linear and quadratic terms in augmented Lagrangian

\[ L_\rho(x, z, y) = f(x) + g(z) + y^T (Ax + Bz - c) + \frac{\rho}{2} \| Ax + Bz - c \|^2 \]

\[ = f(x) + g(z) + \frac{\rho}{2} \| Ax + Bz - c + u \|^2 + \text{const.} \]

with \( u^k = (1/\rho)y^k \)

- ADMM (scaled dual form):

\[ x^{k+1} := \arg\min_x (f(x) + \frac{\rho}{2} \| Ax + Bz^k - c + u^k \|^2) \]

\[ z^{k+1} := \arg\min_z (g(z) + \frac{\rho}{2} \| Ax^{k+1} + Bz - c + u^k \|^2) \]

\[ u^{k+1} := u^k + (Ax^{k+1} + Bz^{k+1} - c) \]
Convergence

- assume (very little!)
  - $f, g$ convex, closed, proper
  - $L_0$ has a saddle point

- then ADMM converges:
  - iterates approach feasibility: $Ax^k + Bz^k - c \to 0$
  - objective approaches optimal value: $f(x^k) + g(z^k) \to p^*$
Related algorithms

- operator splitting methods
  (Douglas, Peaceman, Rachford, Lions, Mercier, ... 1950s, 1979)

- Dykstra's alternating projections algorithm (1983)

- Spingarn's method of partial inverses (1985)

- Rockafellar-Wets progressive hedging (1991)

- proximal methods (Rockafellar, many others, 1976–)

- saddle-point proximal methods (Chambolle, Pock 2005–)

- Bregman iterative methods (2008–)

- most of these are special cases of the proximal point algorithm
  (Rockafellar 1976)
Outline

Dual decomposition

Method of multipliers

Alternating direction method of multipliers

Common patterns

Examples

Consensus and exchange

Conclusions
Common patterns

- $x$-update step requires minimizing $f(x) + (\rho/2)\|Ax - v\|_2^2$
  (with $v = Bz^k - c + u^k$, which is constant during $x$-update)
- similar for $z$-update
- several special cases come up often
- can simplify update by exploiting structure in these cases
Decomposition

▷ suppose \( f \) is block-separable,

\[
f(x) = f_1(x_1) + \cdots + f_N(x_N), \quad x = (x_1, \ldots, x_N)
\]

▷ \( A \) is conformably block separable: \( A^T A \) is block diagonal

▷ then \( x \)-update splits into \( N \) parallel updates of \( x_i \)
Proximal operator

- consider \( x \)-update when \( A = I \)

\[
x^+ = \arg\min_x (f(x) + (\rho/2)\|x - v\|^2_2) = \text{prox}_{f,\rho}(v)
\]

- some special cases:

\[
f = I_C \text{ (indicator funct. of set } C) \quad x^+ := \Pi_C(v) \text{ (projection onto } C) \\
f = \lambda \| \cdot \|_1 \text{ (} \ell_1 \text{ norm)} \quad x^+_i := S_{\lambda/\rho}(v_i) \text{ (soft thresholding)} \\
(S_a(v) = (v - a)_+ - (-v - a)_+)
\]
Quadratic objective

\[ f(x) = \frac{1}{2}x^T P x + q^T x + r \]

\[ x^+ := (P + \rho A^T A)^{-1}(\rho A^T v - q) \]

- use matrix inversion lemma when computationally advantageous

\[
(P + \rho A^T A)^{-1} = P^{-1} - \rho P^{-1} A^T (I + \rho A P^{-1} A^T)^{-1} A P^{-1}
\]

- (direct method) cache factorization of \( P + \rho A^T A \) (or \( I + \rho A P^{-1} A^T \))

- (iterative method) warm start, early stopping, reducing tolerances
Smooth objective

- $f$ smooth

- can use standard methods for smooth minimization
  - gradient, Newton, or quasi-Newton
  - preconditioned CG, limited-memory BFGS (scale to very large problems)

- can exploit
  - warm start
  - early stopping, with tolerances decreasing as ADMM proceeds
Outline

Dual decomposition

Method of multipliers

Alternating direction method of multipliers

Common patterns

Examples

Consensus and exchange

Conclusions
Constrained convex optimization

- consider ADMM for generic problem

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in \mathcal{C}
\end{align*}
\]

- ADMM form: take \( g \) to be indicator of \( \mathcal{C} \)

\[
\begin{align*}
\text{minimize} & \quad f(x) + g(z) \\
\text{subject to} & \quad x - z = 0
\end{align*}
\]

- algorithm:

\[
\begin{align*}
x^{k+1} & := \arg\min_x \left( f(x) + \left( \frac{\rho}{2} \right) \| x - z^k + u^k \|_2^2 \right) \\
z^{k+1} & := \Pi_{\mathcal{C}}(x^{k+1} + u^k) \\
u^{k+1} & := u^k + x^{k+1} - z^{k+1}
\end{align*}
\]
Lasso

- lasso problem:

\[
\text{minimize} \quad \frac{1}{2} \|Ax - b\|_2^2 + \lambda \|x\|_1
\]

- ADMM form:

\[
\text{minimize} \quad \frac{1}{2} \|Ax - b\|_2^2 + \lambda \|z\|_1 \\
\text{subject to} \quad x - z = 0
\]

- ADMM:

\[
x^{k+1} := (A^T A + \rho I)^{-1}(A^T b + \rho z^k - y^k) \\
z^{k+1} := \frac{S_{\lambda/\rho}}{}(x^{k+1} + y^k / \rho) \\
y^{k+1} := y^k + \rho(x^{k+1} - z^{k+1})
\]
Lasso example

- example with dense $A \in \mathbb{R}^{1500 \times 5000}$
  (1500 measurements; 5000 regressors)

- computation times

  - factorization (same as ridge regression) 1.3s
  - subsequent ADMM iterations 0.03s
  - lasso solve (about 50 ADMM iterations) 2.9s
  - full regularization path (30 $\lambda$’s) 4.4s

- not bad for a very short Matlab script
Sparse inverse covariance selection

- $S$: empirical covariance of samples from $\mathcal{N}(0, \Sigma)$, with $\Sigma^{-1}$ sparse (i.e., Gaussian Markov random field)

- estimate $\Sigma^{-1}$ via $\ell_1$ regularized maximum likelihood

  $$\text{minimize} \quad \text{Tr}(SX) - \log \det X + \lambda \|X\|_1$$

- methods: COVSEL (Banerjee et al 2008), graphical lasso (FHT 2008)
Sparse inverse covariance selection via ADMM

► ADMM form:

\[
\begin{align*}
\text{minimize} & \quad \text{Tr}(SX) - \log \det X + \lambda \|Z\|_1 \\
\text{subject to} & \quad X - Z = 0
\end{align*}
\]

► ADMM:

\[
\begin{align*}
X^{k+1} & := \arg\min_X \left( \text{Tr}(SX) - \log \det X + \frac{\rho}{2} \|X - Z^k + U^k\|_F^2 \right) \\
Z^{k+1} & := S_{\lambda/\rho}(X^{k+1} + U^k) \\
U^{k+1} & := U^k + (X^{k+1} - Z^{k+1})
\end{align*}
\]
Analytical solution for $X$-update

- compute eigendecomposition
  \[ \rho(Z^k - U^k) - S = Q\Lambda Q^T \]

- form diagonal matrix $\tilde{X}$ with
  \[ \tilde{X}_{ii} = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4\rho}}{2\rho} \]

- let $X^{k+1} := Q\tilde{X}Q^T$

- cost of $X$-update is an eigendecomposition
Sparse inverse covariance selection example

- $\Sigma^{-1}$ is $1000 \times 1000$ with $10^4$ nonzeros
  - graphical lasso (Fortran): 20 seconds – 3 minutes
  - ADMM (Matlab): 3 – 10 minutes
  - (depends on choice of $\lambda$)

- very rough experiment, but with no special tuning, ADMM is in ballpark of recent specialized methods

- (for comparison, COVSEL takes 25+ min when $\Sigma^{-1}$ is a $400 \times 400$ tridiagonal matrix)
Outline

Dual decomposition

Method of multipliers

Alternating direction method of multipliers

Common patterns

Examples

Consensus and exchange

Conclusions
Consensus optimization

- want to solve problem with $N$ objective terms

$$\text{minimize } \sum_{i=1}^{N} f_i(x)$$

- e.g., $f_i$ is the loss function for $i$th block of training data

- ADMM form:

$$\text{minimize } \sum_{i=1}^{N} f_i(x_i)$$

subject to

$$x_i - z = 0$$

- $x_i$ are local variables
- $z$ is the global variable
- $x_i - z = 0$ are consistency or consensus constraints
- can add regularization using a $g(z)$ term
Consensus optimization via ADMM

$\triangleright \quad L_\rho(x, z, y) = \sum_{i=1}^{N} \left( f_i(x_i) + y_i^T (x_i - z) + (\rho/2)\|x_i - z\|_2^2 \right)$

$\triangleright \quad \text{ADMM:}$

\begin{align*}
x_{i}^{k+1} & := \arg\min_{x_i} (f_i(x_i) + y_i^k (x_i - z^k) + (\rho/2)\|x_i - z^k\|_2^2) \\
z^{k+1} & := \frac{1}{N} \sum_{i=1}^{N} (x_{i}^{k+1} + (1/\rho)y_i^k) \\
y_{i}^{k+1} & := y_i^k + \rho(x_i^{k+1} - z^{k+1})
\end{align*}

$\triangleright \quad \text{with regularization, averaging in } z \text{ update is followed by } \text{prox}_{g,\rho}$
Consensus optimization via ADMM

- using \( \sum_{i=1}^{N} y_i^k = 0 \), algorithm simplifies to

\[
x_i^{k+1} := \arg\min_{x_i} \left( f_i(x_i) + y_i^k x_i + \left( \frac{\rho}{2} \right) \| x_i - \bar{x}^k \|_2^2 \right)
\]

\[
y_i^{k+1} := y_i^k + \rho (x_i^{k+1} - \bar{x}^{k+1})
\]

where \( \bar{x}^k = \left( \frac{1}{N} \right) \sum_{i=1}^{N} x_i^k \)

- in each iteration
  - gather \( x_i^k \) and average to get \( \bar{x}^k \)
  - scatter the average \( \bar{x}^k \) to processors
  - update \( y_i^k \) locally (in each processor, in parallel)
  - update \( x_i \) locally
Statistical interpretation

- $f_i$ is negative log-likelihood for parameter $x$ given $i$th data block

- $x_{i}^{k+1}$ is MAP estimate under prior $\mathcal{N}(\overline{x}^{k} + (1/\rho)y_{i}^{k}, \rho I)$

- prior mean is previous iteration’s consensus shifted by ‘price’ of processor $i$ disagreeing with previous consensus

- processors only need to support a Gaussian MAP method
  - type or number of data in each block not relevant
  - consensus protocol yields global maximum-likelihood estimate
Consensus classification

- data (examples) \((a_i, b_i), i = 1, \ldots, N, a_i \in \mathbb{R}^n, b_i \in \{-1, +1\}\)

- linear classifier \(\text{sign}(a^T w + v)\), with weight \(w\), offset \(v\)

- margin for \(i\)th example is \(b_i(a_i^T w + v)\); want margin to be positive

- loss for \(i\)th example is \(l(b_i(a_i^T w + v))\)
  - \(l\) is loss function (hinge, logistic, probit, exponential, \ldots)

- choose \(w, v\) to minimize \(\frac{1}{N} \sum_{i=1}^{N} l(b_i(a_i^T w + v)) + r(w)\)
  - \(r(w)\) is regularization term (\(\ell_2, \ell_1, \ldots\))

- split data and use ADMM consensus to solve
Consensus SVM example

- hinge loss $l(u) = (1 - u)^+$ with $\ell_2$ regularization

- baby problem with $n = 2$, $N = 400$ to illustrate

- examples split into 20 groups, in worst possible way: each group contains only positive or negative examples
Iteration 5

Consensus and exchange
Distributed lasso example

- example with dense $A \in \mathbb{R}^{4000 \times 8000}$ (roughly 30 GB of data)
  - distributed solver written in C using MPI and GSL
  - no optimization or tuned libraries (like ATLAS, MKL)
  - split into 80 subsystems across 10 (8-core) machines on Amazon EC2

- computation times
  
  loading data \hspace{1cm} 30s
  factorization \hspace{1cm} 5m
  subsequent ADMM iterations \hspace{1cm} 0.5–2s
  lasso solve (about 15 ADMM iterations) \hspace{1cm} 5–6m
Exchange problem

minimize \( \sum_{i=1}^{N} f_i(x_i) \)
subject to \( \sum_{i=1}^{N} x_i = 0 \)

- another canonical problem, like consensus
- in fact, it’s the dual of consensus
- can interpret as \( N \) agents exchanging \( n \) goods to minimize a total cost
- \( (x_i)_j \geq 0 \) means agent \( i \) receives \( (x_i)_j \) of good \( j \) from exchange
- \( (x_i)_j < 0 \) means agent \( i \) contributes \( |(x_i)_j| \) of good \( j \) to exchange
- constraint \( \sum_{i=1}^{N} x_i = 0 \) is equilibrium or market clearing constraint
- optimal dual variable \( y^* \) is a set of valid prices for the goods
Exchange ADMM

- solve as a generic constrained convex problem with constraint set

\[ C = \{ x \in \mathbb{R}^{nN} \mid x_1 + x_2 + \cdots + x_N = 0 \} \]

- scaled form:

\[ x_i^{k+1} := \arg\min_{x_i} (f_i(x_i) + (\rho/2)\|x_i - x_i^k + \bar{x}^k + u^k\|_2^2) \]

\[ u^{k+1} := u^k + \bar{x}^{k+1} \]

- unscaled form:

\[ x_i^{k+1} := \arg\min_{x_i} (f_i(x_i) + y^{kT}x_i + (\rho/2)\|x_i - (x_i^k - \bar{x}^k)\|_2^2) \]

\[ y^{k+1} := y^k + \rho\bar{x}^{k+1} \]
Interpretation as tâtonnement process

- *tâtonnement process*: iteratively update prices to clear market
- work towards equilibrium by increasing/decreasing prices of goods based on excess demand/supply
- dual decomposition is the simplest tâtonnement algorithm
- ADMM adds proximal regularization
  - incorporate agents’ prior commitment to help clear market
  - convergence far more robust convergence than dual decomposition
Distributed dynamic energy management

- \( N \) devices exchange power in time periods \( t = 1, \ldots, T \)
- \( x_i \in \mathbb{R}^T \) is power flow profile for device \( i \)
- \( f_i(x_i) \) is cost of profile \( x_i \) (and encodes constraints)
- \( x_1 + \cdots + x_N = 0 \) is energy balance (in each time period)
- dynamic energy management problem is exchange problem
- exchange ADMM gives distributed method for dynamic energy management
- each device optimizes its own profile, with quadratic regularization for coordination
- residual (energy imbalance) is driven to zero
Example

- network with 8000 devices exchanging power at 3000 nodes (mixture of generators, batteries, smart loads, transmission lines, ...)
- coordinate devices over 96 time periods
- ~ 1 million variables in optimization problem
Solve time scaling

- serial multi-threaded implementation on 32-core machine with 64 independent threads
- best fit exponent is 0.996
- fully decentralized computation would result in sub second solve time for any size network
Outline

Dual decomposition

Method of multipliers

Alternating direction method of multipliers

Common patterns

Examples

Consensus and exchange

Conclusions
Summary and conclusions

ADMM

- is the same as, or closely related to, many methods with other names
- has been around since the 1970s
- gives simple single-processor algorithms that can be competitive with state-of-the-art
- can be used to coordinate many processors, each solving a substantial problem, to solve a very large problem