Hausdorff Measures on the Line

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In 1918 Hausdorff [1] defined a set of measures in metric spaces which included the Lebesgue n-dimensional measures, counting measures, as well as various non-integral-dimensional measures. These measures are the basis for various theories of generalized dimension, including Besicovitch's theory of fractionally dimensioned sets (now called fractals). I will restrict my study to these measures on the line, and a few foundational questions.

If \( h \) is defined for \( t \geq 0 \), \( h(t) \geq 0 \), increasing and continuous on the right, it is called a Hausdorff function. Let us reserve the symbols \( h \) and \( g \) for Hausdorff functions, i.e., \( h \) and \( g \) will always denote Hausdorff functions. Given \( h \) and \( E \subseteq \mathbb{R} \) (not necessarily measurable), we form the Hausdorff outer \( h \)-measure \( \mathcal{H}^h(E) \) as follows:

\[
\mathcal{H}^h(E) = \lim_{d \to 0} \inf \left\{ \sum_{i=1}^{\infty} h(b_i-a_i) \left| \bigcup_{i=1}^{\infty} (a_i, b_i) \right|, b_i - a_i < d \right\}
\]

Note that as \( d \to 0 \) the class of sums over which we take the infimum decreases, hence \( \mathcal{H}^h(E) \) increases and therefore does indeed converge (possibly to infinity). It is easy to see that \( \mathcal{H}^h \) is a metric outer measure, that is, additive on sets separated by positive distance. Hence the field of \( \mathcal{H}^h \) measurable sets includes the Borel sets, in particular the closed sets I construct are measurable. We shall also
denote the measure $m^h$, and shall assume that all sets mentioned are measurable.

Since $h$ is continuous on the right, it is clear that $m^h$ can be calculated using closed intervals, in fact using any sets $S$, if we replace $h_{1-\epsilon}$ by $\max(0, h)$. It is for this reason that $h$ is required to be continuous on the right. The definition I have given follows Rogers [2] and is the most general definition used. Often $h$ is required to be continuous, or satisfy $h(0)=0$. For example Hausdorff himself considered only concave, continuous $h$ with $h(0)=0$. I shall show that, for measures in $\mathcal{M}$, we may as well assume $h$ to be continuous and subadditive, that is, satisfy $h(xy) \leq h(x) + h(y)$.

A few questions arise immediately. It is clear that for $h(t)=t$, $m^h$ is ordinary Lebesgue measure and that for $h(t)=1$, $m^h$ is counting measure. Are there any other non-trivial Hausdorff measures? In his original paper, Hausdorff showed that if $h$ is continuous, concave, and satisfies $h(0)=0$, then there is a set $S$ such that $m^h(S)=1$ and proved as a specific example that $m^h(C)\leq 1$, where $C$ is the Cantor middle third set and $h(t)=\frac{1}{3}t$. The necessary and sufficient condition on $h$ that there exist a set with finite positive measure was a long-standing problem, solved by A. Dvoretzky [3] in 1948. The condition is only $\lim_{t \to 0} h(t) > 0$.

A more vague question: what is the relationship between $h$ and the measure it generates? For example, when $d_0
different functions generate the same measure? This is a
difficult question which I shall answer in the case the
functions are concave.

(2)

It is clear that \( a^1 \) is determined by its behavior near
0. I start with:

**Prop. 2.1:** If \( \limsup_{t \to 0} \frac{a(t)}{h(t)} = b \), then for all \( E \),
\( a^0(E) \leq ah(E) \).

**Proof:** If \( b = \infty \), the inequality is trivial. Suppose \( b \) is
finite, \( E \in \mathcal{B} \). Given \( e > 0 \) choose \( d_0 \) such that

\[
\frac{a(t)}{h(t)} \leq b + e \quad \text{for } t < \frac{d_0}{2}.
\]

If \( d < d_0 \) and \( \bigcup_{i=1}^{\infty} (a_i, b_i) \supset E \), \( b_i - a_i < d \), then

\[
a_d^0(E) \leq \sum_{i=1}^{\infty} g(b_i - a_i) \leq (b + e) \sum_{i=1}^{\infty} h(b_i - a_i)
\]

Since this holds for all \( d \)-coverings of \( E \) where \( d < d_0 \),

\[
a_d^0(E) \leq (b + e)h_d^0(E) \quad (d < d_0)
\]

\[
\therefore a^0(E) \leq (b + e)a^0(E)
\]
As ε was arbitrary, $m^h(E) \leq m^\varepsilon(E) \leq m^h(E)$

Corollary 2.1: If $\liminf_{t \to 0} \frac{h(t)}{h(0)} = a$, and $\limsup_{t \to 0} \frac{h(t)}{h(0)} = b$, then for all $E$

$$a m^h(E) \leq m^\varepsilon(E) \leq b m^h(E)$$

I will show later that these bounds are the best possible, when $h$ and $\varepsilon$ are concave.

Corollary 2.2: If $\lim_{t \to 0} \frac{h(t)}{h(t)} = a$, then for all $E$

$$m^\varepsilon(E) = a m^h(E)$$

In particular, if $\lim_{t \to 0} \frac{h(t)}{h(t)} = 1$, $\varepsilon$ and $h$ generate the same measure. The converses of the above are quite difficult and their consideration will be postponed. I turn now to show that $h$ may be assumed to be continuous and subadditive.

[1]

Lemma 3.1: If $h(0) = 0$, $h$ generates the same measure as

$$\pi(t) = \inf \left\{ \sum_{i=1}^{\infty} \frac{h(c_i)}{c_i} \mid \sum_{i=1}^{\infty} c_i = 1, 0 \leq c_i \leq 1 \right\}$$

Proof: Let $t \leq R$, $h(0) = 0$. $\pi(t) \leq h(t)$ (just let $c_i = 1$, $\varepsilon = 0$, for $j \leq 1$). Hence $\limsup_{t \to 0} \frac{\pi(t)}{h(t)} \leq 1$, so by Prop.
2.1, \( n^i(E) \leq n^i(\mathbb{E}) \). I'll now establish the opposite inequality. If \( n^i(\mathbb{E}) = \infty \), the inequality is trivial, so assume now \( n^i(\mathbb{E}) \) is finite. Given \( \epsilon, \delta > 0 \), choose a cover \( \bigcup_{i=1}^{\infty} (a_i, b_i) \supseteq \mathbb{E}, b_i - a_i < \delta \), such that

\[
\sum_{i=1}^{\infty} \pi_i(a_i - s_i) = \pi_d^i(\mathbb{E}) < \frac{\epsilon}{2}
\]

We can do this by the definition of \( \pi_d^i(\mathbb{E}) \). Choose \( c_{ik}, i,k=1,2,... \) such that \( 0 \leq c_{ik} \leq 1, \sum_{k=1}^{\infty} c_{ik} = 1, \) and

\[
\sum_{k=1}^{\infty} h(c_{ik}(b_i - a_i)) - \pi_i(b_i - a_i) < \epsilon 2^{-i-1}
\]

We can do this by the definition of \( h \). Consider the closed intervals \( [a_{ik}, b_{ik}], i,k=1,2,... \) given by

\[
a_{ik} = s_i + (\sum_{j=1}^{i} c_{ij})(b_i - a_i)
\]

\[
b_{ik} = s_i + (\sum_{j=1}^{i} c_{ij})(b_i - a_i)
\]

These are just a subdivision of \([a_i, b_i] \). Thus

\[
\bigcup_{i,k=1}^{\infty} [a_{ik}, b_{ik}] \supseteq \bigcup_{i=1}^{\infty} [a_i, b_i] \supseteq \mathbb{E} \quad \text{and}
\]

\[
b_{ik} - a_{ik} = c_{ik}(b_i - a_i) \leq b_i - a_i < \delta
\]
Hence

\[ w^h_d(\varepsilon) \leq \sum_{i,k=1}^{\infty} h(b_{ik} - a_{ik}) \leq \sum_{i=1}^{\infty} \{ n(b_{ik} - a_{ik}) + e 2^{-i-1} \} = \]

\[ = \sum_{i=1}^{\infty} n(b_{ik} - a_{ik}) + \frac{e}{2} \leq n^h_d(\varepsilon) + \varepsilon \]

As \( \varepsilon \) was arbitrary, we conclude \( n^h_d(\varepsilon) \leq n^h(\varepsilon) \). Consequently \( n^h(\varepsilon) \leq n^h_0(\varepsilon) \), therefore \( n^h(\varepsilon) = n^h_0(\varepsilon) \).

**Lemma 3.2:** If \( \sum_{i=1}^{\infty} e_i = 1 \), \( 0 \leq e_i \leq 1 \), then

\[ \sum_{i=1}^{\infty} n(e_i) t \geq n(t) \] (\( n^a \) as in lemma 3.1).

**Proof:** Since \( n(e_i t) \) is finite, given \( \varepsilon > 0 \), choose \( e_{ik}, \ 0 \leq e_{ik} \leq 1 \), such that

\[ \sum_{k=1}^{\infty} e_{ik} = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} n(e_{ik} c_i t) - n(c_i t) < e 2^{-i} \]

Then we note \( \sum_{i=1}^{\infty} e_{ik} c_i = \sum_{i=1}^{\infty} c_i \sum_{k=1}^{\infty} e_{ik} = 1 \), and \( 0 \leq e_{ik} c_i \leq 1 \), so by the definition of \( n(t) \),

\[ n(t) \leq \sum_{i=1}^{\infty} n(e_{ik} c_i t) \leq \sum_{i=1}^{\infty} (n(e_{ik} t) + e 2^{-i}) = \sum_{i=1}^{\infty} n(c_i t) + \varepsilon \]

As \( \varepsilon \) was arbitrary, lemma 3.2 is established.

**Lemma 3.3:** \( n \) is increasing (\( n^a \) as in lemma 3.1 ).
Proof: Suppose \( x < y \), but \( \mathcal{H}(y) < \mathcal{H}(x) \). Choose \( c_i \) such that
\[
0 \leq c_i \leq 1, \quad \sum_{i=1}^{\infty} c_i = 1, \quad \text{and} \quad \sum_{i=1}^{\infty} h(c_i, y) < \mathcal{H}(x).
\]
By monotonicity of \( h \) and \( x < y \),
\[
\sum_{i=1}^{\infty} h(c_i, x) \leq \sum_{i=1}^{\infty} h(c_i, y) < \mathcal{H}(x)
\]
contradicting the definition of \( \mathcal{H} \). Therefore \( \mathcal{H} \) is increasing.

Lemma 3.4: \( \mathcal{H} \) is continuous (as in Lemma 3.1). Proof:

If \( x < y \), \( h(x) \leq \mathcal{H}(y) \leq \mathcal{H}(x) + \mathcal{H}(y-x) \), hence

\[
|\mathcal{H}(y) - \mathcal{H}(x)| \leq \mathcal{H}(y-x)
\]

which goes to 0 as \( x \) goes to \( y \), establishing the continuity of \( \mathcal{H} \).

Theorem 3.1: Every Hausdorff measure in \( \mathbb{R} \) is generated by a continuous, subadditive \( h \).

Proof: Given \( \mathcal{H} \), if \( h(0) > 0 \) then \( \mathcal{H} = h(0) \), and \( h(0) \) is certainly continuous and subadditive. If \( h(0) = 0 \), by Lemmas 3.1 through 3.4, \( \mathcal{H} = \mathcal{H} \) and \( \mathcal{H} \) is continuous and subadditive.

Subadditive is weaker than concave, for if \( h \) is concave, \( \frac{h}{h(t)} \) is increasing, hence

\[
h(x+y) \leq \frac{h(x)}{y} (x+y) = h(y) + \frac{x}{y} h(y)
\]
By symmetry we may assume $xy$, hence

$$h(y) \leq h(x)^{\frac{y}{x}} \quad h(x+y) \leq h(x) + h(y)$$

Thus $h$ is subadditive. I am not sure whether every Hausdorff measure in $\mathbb{R}$ is generated by a concave continuous function, but I suspect that this is not the case.

[4]

The converse of Corollary 2.1:

Theorem 4.1: If $h$ is concave, continuous, $\lim_{t \to 0} \frac{h(t)}{t} = a$, and $a > 0$, then there is a set $S \subseteq \mathbb{R}$ such that

$$0 < a^h(S) < \infty$$

$$a^{h^n}(S) \leq a(S) \leq (1+c)a^{h^n}(S)$$

Theorem 4.1 does not appear in the literature, though it may be known. The proof is a combination of A. Dvoretsky [3] and a generalization of Hausdorff [1], though more involved than either. I have chosen the notation to agree with these sources, so that their contributions are clear.

Proof of Theorem 4.1: We assume first that $a$ is finite, and $h$ and $g$ satisfy the hypotheses. If $h(0) > 0$, then $a^h$ is counting measure, so we let $S$ be any finite set. It is easy
to check that the conclusion of theorem 4.1 is then satisfied. If \( \lim_{t \to 0} h(t) = \infty \), then \( a \) is Lebesgue measure and a simple argument shows, so is \( a' \); in this case we can take \( S = (0, 1] \). So assume now that \( \lim_{t \to 0} h(t) = \infty \), and \( h(0) = 0 \).

I first choose two sequences which are close to each other and have nice properties with respect to \( h \) and \( h' \); from one of these sequences I construct the desired set \( S \).

Claim 1: We can choose two sequences \((x_j), (x_j')\) such that:

(i) \( x_0 = x_0' = 1 \)

(ii) \( x_{j+1} \leq x_{j+1}' \leq x_j \)

(iii) \( \frac{x_{j+1}'}{h(x_{j+1}')} = \frac{x_j}{h(x_j)} \)

(iv) \( \frac{r(x_j)}{h(x_j')} > \frac{a^j}{\ln(1+\varepsilon)} + 1 \)

(v) \( \frac{a}{h(x_{j+1}')} < \frac{1}{j+1} \)

(vi) \( x(x_{j+1}') = \frac{h(x_j)}{x_{j+1}'} \)

Where \( K_{j+1} = \left[ \frac{h(x_j)}{h(x_{j+1}')} \right] + 1 \) (denotes integer part)

Proof by induction. Suppose we've picked \( x_j, x_j' \) for \( j = 0, 1, \ldots, n > 0 \) satisfying (i)-(vi). I will show we may choose \( x_{n+1}, x_{n+1}' \) satisfying (i)-(vi). Since \( h(t) \to 0 \) as \( t \to 0 \), for sufficiently small \( x_{n+1} \), (iv) will be satisfied. Since \( h(t) \to 0 \) as \( t \to 0 \), for sufficiently small \( x_{n+1}' \), (iii) will be satisfied. The second half of (ii) is clearly satisfied for sufficiently small \( x_{n+1}' \). Since (v) is satisfied for
arbitrarily small $x^*_n$, we may choose $x^*_{n+1}$ satisfying (i)-(vi) simultaneously. Having chosen $x^*_{n+1}$ we choose $x^*_{n+1}$ so that (vi) is satisfied. This we may do because $h$ is continuous. Now

$$h(x^*_{n+1}) > \frac{h(x^*_n)}{n+1} > \frac{h(x^*_j)}{n+1} = h(x^*_{n+1})$$

by the definition of $x^*_{n+1}$, choice of $x^*_{n+1}$, and the inductive hypothesis. Since $h$ is monotone, we conclude $x^*_{n+1} < x^*_j$, so (i) through (vi) are satisfied, proving claim 1.

Claim 2:

(i) $k^j_{i+1} \geq 2$

(ii) $\frac{x^*_{j+1}}{x^*_j} < x^*_j$

(iii) $\frac{h(x^*_j)}{h(x^*_{j+1})} \leq (1+a)(\frac{1}{j+1})$

Proof of Claim 2:

(i) is immediate from (iv) of claim 1. Since $\frac{k}{h(x^*_j)}$ increases (as $h$ is concave), by claim 1 (ii),(iii),

$$\frac{x^*_{j+1}}{h(x^*_j)} \leq \frac{x^*_{j+1}}{h(x^*_j)} < \frac{x^*_j}{h(x^*_j)}$$

$$\frac{h(x^*_j)}{h(x^*_{j+1})} > (j+1)x^*_{j+1} < x^*_j$$


establishing (ii). Since

\[
\frac{h(x^*_j)}{h(x_{j+1}^*)} \leq \frac{h(x^*_j)}{h(x_{j+1}^*)} + 1
\]

and

\[
x_{j+1}^* = \frac{h(x_j^*)}{h(x_{j+1}^*)}.
\]

\[
\frac{h(x^*_j)}{h(x_{j+1}^*)} (1 - \frac{1}{x_{j+1}^*}) \geq \frac{h(x^*_j)}{h(x_{j+1}^*)} \geq \frac{h(x_1^*)}{h(x_{n+1}^*)}
\]

\[
\frac{h(x_{j+1}^*)}{h(x_j^*)} \leq \left( 1 + \frac{1}{x_{j+1}^*} \right)^{\frac{1}{1}} \leq \left( 1 + \frac{1}{x_j^*} \right)^{\frac{1}{1}} \leq \ldots
\]

\[
\leq \prod_{i=1}^{j+1} \left( 1 + \frac{1}{x_i^*} \right)
\]

But we've arranged \( x_1^* > \frac{2^{j+1}}{ln(1+e) + 1} \). Since \( \frac{1}{x_i^* - 1} < 1 \), it is easy to verify that:

\[
ln(1 + \frac{1}{x_1^* - 1}) \leq \frac{1}{x_1^* - 1} \leq ln(1+e) 2^{-1}
\]

\[
ln \left( \prod_{i=1}^{j+1} \left( 1 + \frac{1}{x_i^*} \right) \right) = \sum_{i=1}^{j+1} ln(1 + \frac{1}{x_i^*}) \leq \sum_{i=1}^{j+1} ln(1+e) 2^{-1} \leq ln(1+e)
\]
Thus \( h(x_{j+1}) \leq 1 + e \). Since \( \varepsilon \) is increasing and \( x^{*} \in \mathbb{R} \),

\[
\frac{b(x_{j+1}^{*})}{h(x_{j+1})} \leq \frac{b(x_{j+1}^{*})}{h(x_{j+1})} (1 + \varepsilon) \geq \varepsilon (1 + \frac{1}{j+1})
\]

establishing claim 2.

We now construct \( S \), using the sequence \( \langle x_j \rangle \). Let \( S_0 = [0, 1] \cdot B(0) = (-\infty, 0), B(1) = (1, \infty) \)

Let \( S_1 = [0, x_1] \cup [x_1 + y_1, 2x_1 + y_1] \cup \ldots \cup [1-x_1, 1] \)

consist of the closed intervals \( J_j \), \( j = 1, 2, \ldots, K_1 \) got by deleting the \( K_1 - 1 \) open intervals \( B(j) \), \( j = 1, 2, \ldots, K_1 - 1 \) (ordered left to right) of length \( y_1 \) from \( S_0 \). By claim 2, \( K_1 \geq 2 \) and \( x_1 \leq 1 \), so that none of the \( B \) intervals is empty. Proceeding inductively we let \( S_{n+1} \) be the

\[
\prod_{j=1}^{n+1} J_j
\]

closed intervals \( J_{n+1} \) got by deleting the open intervals \( B(k_1, \ldots, k_{n+1}) \), \( i = 1, 2, \ldots, K_{n+1} - 1 \) (ordered left to right) of length \( y_{n+1} \) from the \( J_n \) to the right of \( B(k_1, \ldots, k_n) \). As above we note that \( K_{n+1} \geq 2 \), and \( x_{n+1} + x_{n+1} < 1 \), so this slicing is really possible.

\[
S = \bigcap_{n=1}^{\infty} S_n
\]

is the desired set.

Claim 3: \( n(S) \leq 1 \)

Proof: Given \( \varepsilon > 0 \) we choose \( n \) to be large enough that \( x_n < \varepsilon \).

Consider the

\[
\prod_{j=1}^{n} J_j
\]

closed intervals \( J_n \) which make up \( S_n \).
They cover $S$, since $S \subseteq \mathcal{B}_n$, and they are each of length $x_n$.

Hence
\[ m_d(S) \leq \sum_{i=1}^{n} h(x_n) = \sum_{j=1}^{n} K_j h(x_n) = 1 \]

As $d$ was arbitrary, we conclude $m_d(S) \leq 1$.

The converse is true, but the proof is more involved. It is easy to see that the $\mathcal{B}'s$ are disjoint and lexicographically ordered left to right. We let
\[ |\mathcal{B}[k_1, \ldots, k_n]| = \sum_{j=1}^{B} k_j h(x_j) \]

and rank $\mathcal{B}[k_1, \ldots, k_n] = n$ (we assume $k > 0$). For technical convenience, we let $|\mathcal{B}(0)| = n$, rank $\mathcal{B}(0) = \text{rank } \mathcal{B}(0) = 0$. Say $\mathcal{B} = (u_2, v_2)$.

Claim 4: If $0 < r < s$, then $h(x_n + (r-1)y_n) \geq rh(x_n)$

Proof by induction on $r$. The inequality is clear for $r = 1$.

Suppose $h(x_n + (r-1)y_n) \geq rh(x_n)$ and $r+1 < K_n$. Then
\[ x_n + (r-1)y_n \leq (r+1)x_n + ry_n \leq x_n + (K-1)y_n \]

So by convexity of $h$ and inductive hypothesis,
\[ h((r+1)x_n + ry_n) \geq \frac{rh(x_n) + h((K-r)x_n + (r+1)y_n)}{K} \]

and
\[ h(x_n + (K-r)x_n + y_n) \geq \frac{h((K-r)x_n + (r+1)y_n)}{K-r} \]
Claim 5: If \(|B_2| > |B_1|\), \((B_2, B_1)\) lies to the right of \(B_1\), then
\(h(uB_2, vB_1) > |B_2| - |B_1|\)

Proof by induction on the ranks of the \(B_i's\). When the ranks are 0, we must have \(B_1 = 3[0]\) and \(B_2 = 5(1)\), then
\(h(uB_2, vB_1) = 1 = |B_2| - |B_1|\). Now suppose Claim 5 holds for \(B_i's\) of ranks \(\leq n\), and \(B_1, B_2\) have ranks \(\leq n+1\). There are three nontrivial cases:

Case I: rank\(B_1 = n+1\), rank\(B_2 \leq n\)

Say \(B_1 = 5[k_1, \ldots, k_n, r]\)

Let \(L = 3[k_1, \ldots, k_n]\)

and \(R = 5[k_1, \ldots, k_n, r + 1]\)

and \(S(l)\) if \(k_j = k_n\) for \(j = 1, 2, \ldots, n\)

\(L\) and \(R\) are merely the left and right nearest neighbors of \(B_1\) which have rank \(\leq n\).

Then \(|L| < |B_1| < |R| \leq |B_2|\). If \(n-3 \geq S(l)\),

\(h(uB_2, vB_1) = h((5, n+1, r)x + (5, n+1, r-1)y_{n+1})\)

\(= (k_n+1-r)h(x_{n+1}) = |B_2| - |B_1|\)
using claim 4. If $|\gamma| \leq |\beta(1)|$ we note

$$ u \beta_2 - v \gamma \leq u \beta_2 - v \gamma \leq u \beta_2 - v \gamma $$

Therefore by inductive hypothesis and convexity of $h$,

$$ h(u \beta_2 - v \gamma) \geq \frac{|\beta_2| - |\gamma(1)| (v \beta_2 - v \gamma) + (|\beta_2| - |\gamma(1)| (v \beta_2 - v \gamma))}{v \gamma - v \gamma} $$

Now $|\gamma| > |\beta(1)|$, and $|\gamma| > |\beta(1)| (v \gamma + (\gamma - \gamma)n + 1)$,

$$ v \gamma = v \gamma + r \gamma n + 1 + y n + 1 $$

$$ v \gamma = v \gamma + r \gamma n + 1 + y n + 1 $$

Hence $h(u \beta_2 - v \gamma) \geq$

$$ \geq \frac{rh(x(n+1)) (y(n+1) - y(n))}{y n + 1 + y n + 1} \geq |\beta_2| - |\beta_1| $$

since $y(n) \leq y(n+1)$

Case 2: $\text{rank} \beta_0 = n$, $\text{rank} \beta_0 = n+1$

Again, let $1$, $\pi$ be the the rank $n+1$ left and right nearest neighbors of $\beta_2$. Then $|\beta_1| \leq |\beta_2| < |\pi|$. Say $\beta_2 = \beta [k_1, \ldots, k_r]$. If $n = 0$.

$$ h(u \beta_2 - v \beta_1) - h(r \gamma n + 1 + (n-1)y_{n+1}) \geq rh(x(n)) $$
\( -1\leq |B_{1}\| \) using claim 4. If \( 0<|L| \), we note
\[
\nu L - \nu B_{1} \leq \nu B_{2} - \nu B_{1} \leq \nu R - \nu B_{1}
\]
Hence by inductive hypothesis and convexity of \( B_{1} \),
\[
h(uB_{2} - \nu B_{1}) \geq \frac{(|L| - |B_{1}|)(\nu R - \nu B_{1}) + (|R| - |B_{1}|)(\nu B_{1} - \nu R)}{\nu L - \nu R}
\]
Again, \( |L| - |B_{1}| \leq |x_{n+1}| \), \( |R| - |B_{1}| = (\nu L - \nu B_{1})h(x_{n+1}) \),
\[
u L = uB_{2} - (K_{n+1} - \nu)(x_{n+1} + \nu L)
\]
\[
u L = uB_{2} - y_{n+1} - (\nu L - y_{n+1})
\]
\[
h(uB_{2} - \nu B_{1}) \geq |B_{2}| - |B_{1}| + \frac{h(x_{n+1})(K_{n+1} - \nu)(y_{n+1} - y_{n+1})}{\nu L - \nu L}
\]
\( \geq |B_{2}| - |B_{1}| \) since \( r \leq K_{n+1} - \nu L \).

Case 3: \( \text{rank } B_{1} = \text{rank } B_{2} = n+1 \).

Let L, R be the rank \( n+1 \) left and right nearest neighbors of \( B_{2} \). Then \( 0<|L| < |B_{1}| < |R| < 1 \), and
\[
\nu L - \nu B_{1} \leq \nu B_{2} - \nu B_{1} \leq \nu R - \nu B_{1}
\]
Now using case 1 and convexity,
\[
h(uB_{2} - \nu B_{1}) \geq \frac{(|L| - |B_{1}|)(\nu R - \nu B_{1}) + (|R| - |B_{1}|)(\nu B_{1} - \nu R)}{\nu L - \nu L}
\]
\[ h(x_{n+1}) \frac{(K+y_2)}{n+y} \frac{(K-y_2)}{n+y} \geq |B_2| - |B_1| \]

Proving Claim 5.

Claim 6: \( m^h(S) = 1 \). 

Proof: Suppose \( S \subseteq \bigcup_{i=1}^{\infty} I_i \), \( I_i \) open intervals. Since \( S \) is compact, \( S \) is covered by a finite number of the \( I_i \) which intersect \( S \), say

\[ S \subseteq \bigcup_{i=1}^{N} (a_i, b_i) \]

where \((a_i, b_i)\) are some of the \( I_i \)'s which intersect \( S \) and

\[ a_i < 0 < b_1 < a_2 < \ldots < a_j < 1 < b_N \]

I claim \( \sum_{i=1}^{N} h(b_i-a_i) \geq 1 \). \( b_i \neq 6 \), say \( b_i \notin B_1 \). Now \( v \in S \), \( a_2 < v < 2 \), \( a_2 \notin 3 \). Continuing, we get \( S_2, \ldots, S_{N-1} \) with \( a_j+1 < v_b_j < a_j+1 \). Let \( 8 = \emptyset \), \( 8 = I(1) \). Then \( b_j = a_j \geq v_b_j \). \( v \), therefore by claim 5, \( u(b_i-a_j) \geq |B_i| - |B_i-1| \).

\[ \therefore \sum_{i=1}^{N} h(b_i-a_i) \geq 1 \]

But clearly \( \sum_{i=1}^{\infty} h(diameter) \geq \sum_{i=1}^{N} h(b_i-a_i) \geq 1 \). Thus \( m^h(S) \geq 1 \) for \( d > 0 \), hence \( m^h(S) \geq 1 \), so by Claim 3, \( m^h(S) = 1 \), establishing Claim 6.
Claim 7: \( a \leq m^h(S) \leq (1+\varepsilon)a \)

Proof: As in Claim 3, given \( d > 0 \) choose \( n \) large enough that 
\( x_{n+1} < \varepsilon \). Consider the 
\( \prod_{j=1}^{n+1} \) closed intervals which make up 
\( S_{n+1} \). They cover \( S \) and have length \( \varepsilon d \), hence 
\[
m^h_a(S) \leq \prod_{j=1}^{n+1} x_j h\left( x_j - \varepsilon \right) = \prod_{j=1}^{n+1} x_j \frac{h\left( x_j \right)}{h\left( x_j + \varepsilon \right)} \leq \prod_{j=1}^{n+1} \frac{x_j}{x_j + \varepsilon} \leq (1+\varepsilon) a.
\]

By Claim 2 and Corollary 2.1, \( m^h(S) \geq m^h_a(S) \) \( \Rightarrow \)
\[
a \leq m^h_a(S) \leq (1+\varepsilon)a.
\]

Thus by Claims 5 and 7, 
\[
m^h(S) = 1
\]

establishing Theorem 4.1 in the case \( a < \infty \). If \( a = \infty \), Claim 5 (setting \( g = h \), say) yields a set \( S \) with \( m^h(S) = 1 \). By Corollary 2.1, though, \( m^h(S) = \infty \), establishing Theorem 4.1 when \( a = \infty \).

I don't know whether Theorem 4.1 is true when \( h \) is not concave.
Corollary 4.1: If \( h \) is concave and continuous, there is a set \( S \subset \mathbb{R} \) with \( 0 < m^h(S) < \infty \).

This is Hausdorff's result, slightly weaker than Wróblewski's result, which assumes only \( \lim_{t \to 0} h(t) = 0 \).

Corollary 4.2: If \( h \) and \( \zeta \) are concave and \( \lim_{t \to 0} \frac{\zeta(t)}{h(t)} = \infty \), \( \limsup_{t \to 0^+} \frac{\zeta(t)}{h(t)} = 0 \), then there are sets \( S_1, S_2 \subseteq \mathbb{R} \) such that

\[
0 < m^h(S_1) < \infty
\]

\[
am^h(S_1) \leq m^\zeta(S_1) \leq (1 + \epsilon)am^h(S_1)
\]

\[
h(1 - \epsilon)m^h(S_2) \leq m^\zeta(S_2) \leq bn^h(S_2)
\]

This and corollary 3.2 are the relationship between \( \zeta \) and \( m^h \) referred to in section 2.

Corollary 4.3: Concave functions \( h \) and \( \zeta \) generate the same measures in \( \mathbb{R} \) if and only if \( \lim_{t \to 0^+} \frac{\zeta(t)}{h(t)} = 1 \).

Corollary 4.3 answers another query of section 2, and is by no means obvious.

We have shown that the Hausdorff measures in \( \mathbb{R} \) generated by concave functions are in one to one correspondence with the equivalence classes of concave continuous functions whose ratio tends to one as \( t \) goes to \( 0 \). I close by remarking that this set \( C \) has a very complicated structure. It is not linearly ordered by any of the natural partial orders, for example \( h \preceq g \iff \lim_{t \to 0} \frac{\zeta(t)}{h(t)} = 1 \) or \( h \preceq g \iff \lim_{t \to 0^+} \frac{\zeta(t)}{h(t)} = 0 \).
do either of these orders have a countable basis in $\mathbb{C}$.

Halvorsen constructed the "Logarithmic Scale"

$$h[a_1, \ldots, a_k](t) = t^{a_1[1]} \cdot [1]^{a_2[2]} \cdots [1]^{a_k}$$

(first nonzero $a_j$ is $> \mathbb{C}$)

which is a countably based linear chain in $\mathbb{C}$, but by the preceding remarks is only a (very) small part of $\mathbb{C}$.

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Footnotes


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