a2dr: Anderson Accelerated Douglas-Rachford Splitting
Open-sourced Python Solver for Prox-Affine Distributed Convex Optimization

https://github.com/cvxgrp/a2dr

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Joint work with Anqi Fu and Stephen P. Boyd

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Overview

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2. Douglas-Rachford Splitting
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1 Motivation and Problem Statement

2 Douglas-Rachford Splitting

3 Anderson Acceleration & A2DR

4 Numerical experiments

5 Conclusion
We consider the following **prox-affine** representation/formulation of a **generic** convex optimization problem:

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\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{N} f_i(x_i) \\
\text{subject to} & \quad \sum_{i=1}^{N} A_i x_i = b.
\end{align*}
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with variable \( x = (x_1, \ldots, x_N) \in \mathbb{R}^{n_1 + \cdots + n_N} \), \( A_i \in \mathbb{R}^{m \times n_i} \), \( b \in \mathbb{R}^m \).
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- \( f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R} \cup \{+\infty\} \) is closed, convex and proper (CCP).
- Each \( f_i \) can **only** be accessed through its proximal operator:

\[\text{prox}_{tf_i}(v_i) = \arg\min_{x_i} \left( f_i(x_i) + \frac{1}{2t} \|x_i - v_i\|_2^2 \right).\]
Prox-affine form of generic convex optimization

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Interface of a2dr:

\[ x_{\text{vals}}, \text{primal}, \text{dual}, \text{num\_iters}, \text{solve\_time} = \text{a2dr}(p\_\text{list}, A\_\text{list}, b) \]

Try it out! Simply provide a list of proximal functions \( \text{prox}_{t_i}(v_i) \) (p_list), list of \( A_i \)'s (A_list), and \( b \) (b), and you are done!
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- Existing good ones: CoCoA(+), TMAC, etc.
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  - But hard to extend and use for general purposes.
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**Finally:** CVXPY + a2dr – Expression tree complier exists: Epsilon (Wytock et al., 2015).
Most common approaches for prox-affine formulation (sometimes goes by the name ”distributed optimization”):

- Alternating direction method of multipliers (ADMM).
- Douglas-Rachford splitting (DRS).
- Augmented Lagrangian method (ALM).
Previous Work

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These are typically slow to converge – acceleration techniques:

- Adaptive penalty parameters.
- Momentum methods.
- Quasi-Newton or Newton-type method with line search.
Our Method

**A2DR**: Anderson acceleration (AA) applied to DRS
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- Why AA?

**Why A2DR?**
- Fast and cheap: As fast as (quasi-)Newton acceleration, but as memory efficient as adaptive penalty and momentum, and line-search free
- Flexibility: Applicable to general non-expansive fixed-point (NEFP) iterations (Zhang et al., 2018): projected/proximal gradient descent, DRS, value iteration, etc.
- Globalized type-I AA proposed in (Zhang et al., 2018) used in SCS 2.x.

**Why DRS?**
- Allows for a natural NEFP representation (ADMM not), and amenable to proximal evaluation (ALM not).
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Challenges and contribution

Challenges:

Instability:
AA is unstable without modifications (Scieur et al., 2016, Zhang et al., 2018).

Need for globalized type-II AA:
globalized type-I AA for SCS 2.x (Zhang et al., 2018) does not work that well with DRS + prox-affine.

Non-smoothness and pathology:
DRS is non-smooth, and does not always have a fixed-point solution (unlike SCS).

Theory:
First globally convergent type-II AA variant in non-smooth, potentially pathological settings.

Practice:
An open-source Python solver a2dr based on A2DR:
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Douglas-Rachford Splitting

Anderson Acceleration & A2DR

Numerical experiments

Conclusion
Rewrite problem as ($\mathcal{I}_S$ is the indicator of set $S$)

$$\text{minimize} \quad \sum_{i=1}^{N} f_i(x_i) + \mathcal{I}_{Ax=b}(x).$$
DRS Algorithm

- Rewrite problem as ($\mathcal{I}_S$ is the indicator of set $S$)

  
  \[
  \text{minimize} \sum_{i=1}^{N} f_i(x_i) + g(x) + \mathcal{I}_{Ax=b}(x).
  \]

- DRS iterates for $k = 1, 2, \ldots$, 

  \[
  x_i^{k+1/2} = \text{prox}_{t f_i}(v^k), \quad i = 1, \ldots, N
  \]

  \[
  v^{k+1/2} = 2x^{k+1/2} - v^k
  \]

  \[
  x^{k+1} = \Pi_{Av=b}(v^{k+1/2})
  \]

  \[
  v^{k+1} = v^k + x^{k+1} - x^{k+1/2}
  \]

  $\Pi_S(v)$ is Euclidean projection of $v$ onto $S$. 

DRS iterations can be conceived as a fixed point (FP) mapping

\[ v^{k+1} = F(v^k) \]

- \( F \) is \textbf{firmly non-expansive}. \\
- \( v^k \) converges to a fixed point of \( F \) (if it exists). \\
- \( x^k \) and \( x^{k+1/2} \) converge to a solution of our problem.
Convergence of DRS

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In practice, this convergence is often rather slow.
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  - **Multi-secant** quasi-Newton method (Fang & Saad, 2009).
  - Type-I AA: approximate the Jacobian of the FP mapping
  - Type-II AA: approximate the inverse Jacobian of the FP mapping
  - Also has an intuitive extrapolation formulation (used later).

- Why AA (but not other quasi-newton methods)?
  - Successful applications in SCS 2.x (type-I) and SuperSCS (type-II).
  - AA is more memory-efficient (AA with $M = 5 \sim 10$ beats LBFGS/restarted Broyden with $M = 200 \sim 500$).
  - AA is line-search free: just accept or reject is the best practice.

- Why type-II AA?
  - Work better with DRS + prox-affine than type-I AA
  - Better stability for general purpose solvers and distributed settings.
  - prox operators have much larger diversity than solvable cones in SCS.
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Extrapolation perspective of type-II AA:

- *Extrapolates* next iterate using $M + 1$ most recent iterates

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- Let $G(v) = v - F(v)$ (FP residual), then $\alpha^k \in \mathbb{R}^{M+1}$ is solution to

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- Minimizing the FP residual of extrapolated point $\sum_{j=0}^{M} \alpha_j^k v^{k-M+j}$ when $F$ is affine.
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- **Insufficient** for global convergence both in theory and practice.
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- Change variables to $\gamma^k \in \mathbb{R}^M$ (unconstrained LS):

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- Adaptive quadratic regularization: (adaptive LS)
  \[
  \text{minimize} \quad \|g^k - Y_k \gamma^k\|_2^2 + \eta \left( \|S_k\|_F^2 + \|Y_k\|_F^2 \right) \|\gamma^k\|_2^2, \\
  \text{where } \eta \geq 0 \text{ is a regularization parameter and}
  \]
  \[
  g^k = G(v^k), \quad y^k = g^{k+1} - g^k, \quad Y_k = [y^{k-M} \ldots y^{k-1}] \\
  s^k = v^{k+1} - v^k, \quad S_k = [s^{k-M} \ldots s^{k-1}] 
  \]
A2DR iterates for $k = 1, 2, \ldots$, ($\epsilon > 0$, $M$ positive integer)

1. Compute $v^{k+1}_{\text{DRS}} = F(v^k)$, $g^k = v^k - v^{k+1}_{\text{DRS}}$.

2. Update $Y_k$ and $S_k$ to include the new information & Compute $\alpha^k$ by solving the adaptive LS w.r.t. $\gamma^k$.

3. Compute $v^{k+1}_{\text{AA}} = \sum_{j=0}^{M} \alpha^k_j v^{k-M+j+1}_{\text{DRS}}$.

4. If the residual $\|G(v^k)\|_2 = O(1/n_{\text{AA}}^{1+\epsilon})$: (safeguard)
   - Adopt $v^{k+i} = v^{k+i}_{\text{AA}}$ for $i = 1, \ldots, M$.
   - ($n_{\text{AA}}$: # of adopted AA candidates)

5. Otherwise, take $v^{k+1} = v^{k+1}_{\text{DRS}}$. 
Stopping Criterion of A2DR

- Stop and output $x^{k+1/2}$ when $\|r^k\|_2 \leq \epsilon_{\text{tol}} = \epsilon_{\text{abs}} + \epsilon_{\text{rel}}\|r^0\|_2$:

  $r^k_{\text{prim}} = Ax^{k+1/2} - b,$

  $r^k_{\text{dual}} = \frac{1}{t}(v^k - x^{k+1/2}) + A^T\lambda^k,$

  $r^k = (r^k_{\text{prim}}, r^k_{\text{dual}}).$
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  $$

- Remark:
  - Just KKT conditions. Notice that $(v^k - x^{k+1/2})/t \in \partial f(x^{k+1/2})$.
  - $\text{prox}_f$ is enough, and no need for access to $f$ or its sub-gradient.

  - Dual variable is solution to least-squares problem

  $$
  \lambda^k = \text{argmin}_{\lambda} \|r^k_{\text{dual}}\|_2
  $$
Key lemmas to the proof

Lemma (Bounded approximate inverse Jacobian)

We have $v^{k+1}_{AA} = v^{k} - H_{k}g^{k}$, where $g^{k} = G(v^{k})$ is the FP residual at $v^{k}$, and $\|H_{k}\|_2 \leq 1 + 2/\eta$, where $\eta > 0$ is the regularization parameter in the adaptive LS subproblem.

Lemma (Connecting FP residuals with OPT residuals)

Suppose that $\lim \inf_{j \to \infty} \|G(v^{j})\|_2 \leq \epsilon$ for some $\epsilon > 0$, then

$$\lim \inf_{j \to \infty} \|r^{j}_{\text{prim}}\|_2 \leq \|A\|_2 \epsilon, \quad \lim \inf_{j \to \infty} \|r^{j}_{\text{dual}}\|_2 \leq \frac{1}{t} \epsilon.$$
Convergence of A2DR

Theorem (Solvable Case)

If the problem is solvable (e.g., feasible and bounded), then

$$\liminf_{k \to \infty} \| r^k \|_2 = 0$$

and the AA candidates are adopted infinitely often. Furthermore, if $F$ has a fixed point, then

$$\lim_{k \to \infty} v^k = v^\star \quad \text{and} \quad \lim_{k \to \infty} x^{k+1/2} = x^\star,$$

where $v^\star$ is a fixed-point of $F$ and $x^\star$ is a solution to our problem.

Remark. when the proximal operators and projections are evaluated with errors bounded by $\epsilon$, then $\liminf_{k \to \infty} \| r^k \|_2 = O(\sqrt{\epsilon})$. 
Convergence of A2DR

Theorem (Pathological Case)

If the problem is pathological (strongly primal infeasible or strongly dual infeasible), then

\[
\lim_{k \to \infty} (v^k - v^{k+1}) = \delta v \neq 0.
\]

Furthermore, if \( \lim_{k \to \infty} Ax^{k+1/2} = b \), then the problem is unbounded and

\[
\|\delta v\|_2 = \text{dist}(\text{dom } f^*, \text{range}(A^T))
\]

Otherwise, it is infeasible and

\[
\|\delta v\|_2 \geq \text{dist}(\text{dom } f, \{x : Ax = b\})
\]

with equality when the dual problem is feasible.
**Pre-conditioning** (convergence greatly improved by rescaling problem):

- Replace original $A$, $b$, $f_i$ with

\[ \hat{A} = DAE, \quad \hat{b} = Db, \quad \hat{f}_i(\hat{x}_i) = f_i(e_i\hat{x}_i) \]

- $D$ and $E$ are diagonal positive, $e_i > 0$ corresponds to $i$th block diagonal entry of $E$, and chosen by equilibrating $A$

- Proximal operator of $\hat{f}_i$ can be evaluated using proximal operator of $f_i$

\[
\text{prox}_{t\hat{f}_i}(\hat{v}_i) = \frac{1}{e_i} \text{prox}_{(e_i^2 t)f_i}(e_i\hat{v}_i)
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Choice of $t$ (in DRS, $\text{prox}_{tf_i}$): $t = \frac{1}{10} \left( \prod_{j=1}^{N} e_j \right)^{-2/N}$. 
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Parallelization: multiprocessing package in Python.
1. Motivation and Problem Statement
2. Douglas-Rachford Splitting
3. Anderson Acceleration & A2DR
4. Numerical experiments
5. Conclusion
Nonnegative Least Squares (NNLS)

minimize $\|Fz - g\|_2^2$
subject to $z \geq 0$

with respect to $z \in \mathbb{R}^q$

- Problem data: $F \in \mathbb{R}^{p \times q}$ and $g \in \mathbb{R}^p$
- Can be written in standard form with

$$f_1(x_1) = \|Fx_1 - g\|_2^2, \quad f_2(x_2) = \mathcal{I}_{\mathbb{R}_+^n}(x_2)$$

$$A_1 = I, \quad A_2 = -I, \quad b = 0$$

- We evaluate proximal operator of $f_1$ using LSQR
NNLS: Convergence of $\|r^k\|_2$

$p = 10^4$, $q = 8000$, $F$ has 0.1% nonzeros

OSQP and SCS took respectively 349 and 327 seconds, while A2DR only took 55 seconds.
NNLS: Effect of regularization

\( p = 300, \ q = 500, \ F \) has 0.1% nonzeros

\begin{figure}
\centering
\includegraphics[width=\textwidth]{residuals_plot}
\caption{Residuals for different regularization methods}
\end{figure}
Samples $z_1, \ldots, z_p$ IID from $\mathcal{N}(0, \Sigma)$

Know covariance $\Sigma \in S^q_+$ has **sparse** inverse $S = \Sigma^{-1}$

One way to estimate $S$ is by solving the penalized log-likelihood problem

$$
\text{minimize} \quad -\log \det(S) + \text{tr}(SQ) + \alpha \|S\|_1,
$$

where $Q$ is the sample covariance, $\alpha \geq 0$ is a parameter

Note $\log \det(S) = -\infty$ when $S \not\succ 0$
Problem can be written in standard form with

\[ f_1(S_1) = -\log \det(S_1) + \text{tr}(S_1 Q), \quad f_2(S_2) = \alpha \|S_2\|_1, \]
\[ A_1 = I, \quad A_2 = -I, \quad b = 0. \]

Both proximal operators have closed-form solutions.
Covariance Estimation: Convergence of $\|r^k\|_2$

$p = 1000, q = 100, S$ has 10% nonzeros

Residuals (DRS) vs. Residuals (A2DR)
Ran A2DR on instances with $q = 1200$ and $q = 2000$ (vectorizations on the order of $10^6$) and compared its performance to SCS:

In the former case, A2DR took 1 hour to converge to a tolerance of $10^{-3}$, while SCS took 11 hours to achieve a tolerance of $10^{-1}$ and yielded a much worse objective value.

In the latter case, A2DR converged in 2.6 hours to a tolerance of $10^{-3}$, while SCS failed immediately with an out-of-memory error.
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- In the latter case, A2DR converged in 2.6 hours to a tolerance of $10^{-3}$, while SCS failed immediately with an out-of-memory error.
Minimize \( \phi(W\theta, Y) + \alpha \sum_{l=1}^{L} \|\theta_l\|_2 + \beta \|\theta\|_\ast \)

with respect to \( \theta = [\theta_1 \cdots \theta_L] \in \mathbb{R}^{s \times L} \)

- Problem data: \( W \in \mathbb{R}^{p \times s} \) and \( Y = [y_1 \cdots y_L] \in \mathbb{R}^{p \times L} \)
- Regularization parameters: \( \alpha \geq 0, \beta \geq 0 \)
- Logistic loss function

\[
\phi(Z, Y) = \sum_{l=1}^{L} \sum_{i=1}^{p} \log (1 + \exp(-Y_{il}Z_{il}))
\]
Multi-Task Logistic Regression

- Rewrite problem in standard form with:

\[ f_1(Z) = \phi(Z, Y), \quad f_2(\theta) = \alpha \sum_{l=1}^{L} \| \theta_l \|_2, \quad f_3(\tilde{\theta}) = \beta \| \tilde{\theta} \|_*, \]

\[ A = \begin{bmatrix} I & -W & 0 \\ 0 & I & -I \end{bmatrix}, \quad x = \begin{bmatrix} Z \\ \theta \\ \tilde{\theta} \end{bmatrix}, \quad b = 0 \]

- We evaluate proximal operator of \( f_1 \) using Newton-CG method, and the rest with closed-form formulae.
Multi-Task Logistic: Convergence of $\|r^k\|_2$

$p = 300, s = 500, L = 10, \alpha = \beta = 0.1$
A (very) brief summary of other examples (see the paper for more details):

- $l_1$ trend filtering.
- Stratified models.
- Single commodity flow optimization (match the performance of OSQP, and largely outperform SCS).
- Optimal control (largely outperform both SCS and OSQP).
- Coupled quadratic program (match the performance of OSQP and SCS).
Other examples

A (very) brief summary of other examples (see the paper for more details):

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- Optimal control (largely outperform both SCS and OSQP).
- Coupled quadratic program (match the performance of OSQP and SCS).

Remark. The advantage compared to OSQP probably comes from the inclusion of AA, while the advantage compared to SCS (which includes type-I AA) is probably due to the more compact standard form representation.
A2DR is a fast, robust algorithm for solving generic (non-smooth) convex optimization problems in the prox-affine form.
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- Parallelized, scalable and memory-efficient.
- Consistent and fast convergence with no parameter tuning, and beat SOTA open source solvers like SCS (2.x) and OSQP in many cases.

Python library: [https://github.com/cvxgrp/a2dr](https://github.com/cvxgrp/a2dr)
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Future Work

- More work on feasibility detection.
- Expand library of proximal operators (non-convex proximal).
- User-friendly interface with CVXPY (with the help of Epsilon).
- GPU parallelization and cloud computing,
Acknowledgment

- Thanks to Brendan O'Donoghue for his advice on pre-conditioning and his inspirational ideas of developing solvers with Anderson acceleration, pioneered by SCS 2.x:
- Thanks to Anqi Fu for the input to the slides.
Thanks for listening!

Any questions?