Lecture 3
The Laplace transform

- definition & examples
- properties & formulas
  - linearity
  - the inverse Laplace transform
  - time scaling
  - exponential scaling
  - time delay
  - derivative
  - integral
  - multiplication by $t$
  - convolution
the Laplace transform converts *integral* and *differential* equations into *algebraic* equations

- applies to general signals, not just sinusoids
- handles non-steady-state conditions

allows us to analyze

- LCCODEs
- complicated circuits with sources, Ls, Rs, and Cs
- complicated systems with integrators, differentiators, gains

The Laplace transform
Complex numbers

complex number in Cartesian form: \( z = x + jy \)

- \( x = \Re z \), the real part of \( z \)
- \( y = \Im z \), the imaginary part of \( z \)
- \( j = \sqrt{-1} \) (engineering notation); \( i = \sqrt{-1} \) is polite term in mixed company

complex number in polar form: \( z = re^{j\phi} \)

- \( r \) is the modulus or magnitude of \( z \)
- \( \phi \) is the angle or phase of \( z \)
- \( \exp(j\phi) = \cos \phi + j \sin \phi \)

complex exponential of \( z = x + jy \):
\[
e^z = e^{x+jy} = e^xe^{jy} = e^x(\cos y + j \sin y)
\]
The Laplace transform

we’ll be interested in signals defined for $t \geq 0$

the **Laplace transform** of a signal (function) $f$ is the function $F = \mathcal{L}(f)$ defined by

$$F(s) = \int_0^\infty f(t)e^{-st} \, dt$$

for those $s \in \mathbb{C}$ for which the integral makes sense

- $F$ is a complex-valued function of complex numbers
- $s$ is called the (complex) **frequency variable**, with units sec$^{-1}$; $t$ is called the **time variable** (in sec); $st$ is unitless
- for now, we assume $f$ contains no impulses at $t = 0$

**common notation convention:** lower case letter denotes signal; capital letter denotes its Laplace transform, e.g., $U$ denotes $\mathcal{L}(u)$, $V_{\text{in}}$ denotes $\mathcal{L}(v_{\text{in}})$, etc.
Example

let’s find Laplace transform of \( f(t) = e^t \):

\[
F(s) = \int_0^\infty e^t e^{-st} \, dt = \int_0^\infty e^{(1-s)t} \, dt = \left. \frac{1}{1-s} e^{(1-s)t} \right|_0^\infty = \frac{1}{s-1}
\]

provided we can say \( e^{(1-s)t} \to 0 \) as \( t \to \infty \), which is true for \( \Re s > 1 \):

\[
\left| e^{(1-s)t} \right| = \left| e^{-j(\Im s)t} \right| \left| e^{(1-\Re s)t} \right| = e^{(1-\Re s)t}
\]

\[\begin{align*}
\text{\( \left| e^{(1-s)t} \right| = 1 \) } & \text{ \( \left| e^{-(\Im s)t} \right| \left| e^{(1-\Re s)t} \right| = e^{(1-\Re s)t} \)}
\end{align*}\]

- the integral defining \( F \) makes sense for all \( s \in \mathbb{C} \) with \( \Re s > 1 \) (the ‘region of convergence’ of \( F \))
- but the resulting formula for \( F \) makes sense for all \( s \in \mathbb{C} \) except \( s = 1 \)

we’ll ignore these (sometimes important) details and just say that

\[
\mathcal{L}(e^t) = \frac{1}{s-1}
\]
More examples

**constant:** (or unit step) $f(t) = 1$ (for $t \geq 0$)

$$F(s) = \int_{0}^{\infty} e^{-st} \, dt = \left[-\frac{1}{s}e^{-st}\right]_{0}^{\infty} = \frac{1}{s}$$

provided we can say $e^{-st} \to 0$ as $t \to \infty$, which is true for $\Re s > 0$ since

$$\left|e^{-st}\right| = e^{-j(\Im s)t} \quad \left|e^{-(\Re s)t}\right| = e^{-(\Re s)t}$$

- the *integral* defining $F$ makes sense for all $s$ with $\Re s > 0$
- but the resulting *formula* for $F$ makes sense for all $s$ except $s = 0$
**sinusoid:** first express \( f(t) = \cos \omega t \) as

\[
f(t) = \frac{1}{2} e^{j \omega t} + \frac{1}{2} e^{-j \omega t}
\]

now we can find \( F \) as

\[
F(s) = \int_{0}^{\infty} e^{-st} \left( \frac{1}{2} e^{j \omega t} + \frac{1}{2} e^{-j \omega t} \right) \, dt
\]

\[
= (1/2) \int_{0}^{\infty} e^{(-s+j \omega)t} \, dt + (1/2) \int_{0}^{\infty} e^{(-s-j \omega)t} \, dt
\]

\[
= (1/2) \frac{1}{s-j \omega} + (1/2) \frac{1}{s+j \omega}
\]

\[
= \frac{s}{s^2 + \omega^2}
\]

(valid for \( \Re s > 0 \); final formula OK for \( s \neq \pm j \omega \))
powers of $t$: $f(t) = t^n \quad (n \geq 1)$

we’ll integrate by parts, i.e., use

$$\int_a^b u(t)v'(t) \, dt = u(t)v(t) \bigg|_a^b - \int_a^b v(t)u'(t) \, dt$$

with $u(t) = t^n$, $v'(t) = e^{-st}$, $a = 0$, $b = \infty$

$$F(s) = \int_0^\infty t^n e^{-st} \, dt = t^n \left( \frac{-e^{-st}}{s} \right) \bigg|_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} \, dt$$

$$= \frac{n}{s} \mathcal{L}(t^{n-1})$$

provided $t^n e^{-st} \to 0$ if $t \to \infty$, which is true for $\Re s > 0$

applying the formula recusively, we obtain

$$F(s) = \frac{n!}{s^{n+1}}$$

valid for $\Re s > 0$; final formula OK for all $s \neq 0$
Impulses at $t = 0$

if $f$ contains impulses at $t = 0$ we choose to include them in the integral defining $F$:

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st} \, dt$$

(you can also choose to not include them, but this changes some formulas we’ll see & use)

**example:** impulse function, $f = \delta$

$$F(s) = \int_{0^-}^{\infty} \delta(t)e^{-st} \, dt = e^{-st}\bigg|_{t=0} = 1$$

similarly for $f = \delta^{(k)}$ we have

$$F(s) = \int_{0^-}^{\infty} \delta^{(k)}(t)e^{-st} \, dt = (-1)^k \frac{d^k}{dt^k}e^{-st}\bigg|_{t=0} = s^k e^{-st}\bigg|_{t=0} = s^k$$
the Laplace transform is *linear*: if \( f \) and \( g \) are any signals, and \( a \) is any scalar, we have

\[
\mathcal{L}(af) = aF, \quad \mathcal{L}(f + g) = F + G
\]

i.e., homogeneity & superposition hold

example:

\[
\mathcal{L} \left( 3\delta(t) - 2e^{t} \right) = 3\mathcal{L}(\delta(t)) - 2\mathcal{L}(e^{t}) = 3\frac{2}{s - 1} = 3s - 5 \quad \frac{s - 5}{s - 1}
\]
One-to-one property

the Laplace transform is \textit{one-to-one}: if \( \mathcal{L}(f) = \mathcal{L}(g) \) then \( f = g \)
(well, almost; see below)

\begin{itemize}
  \item \( F \) determines \( f \)
  \item inverse Laplace transform \( \mathcal{L}^{-1} \) is well defined
\end{itemize}

(not easy to show)

\textbf{example} (previous page):

\[
\mathcal{L}^{-1} \left( \frac{3s - 5}{s - 1} \right) = 3\delta(t) - 2e^t
\]

in other words, the only function \( f \) such that

\[
F(s) = \frac{3s - 5}{s - 1}
\]

is \( f(t) = 3\delta(t) - 2e^t \)
**what ‘almost’ means:** if $f$ and $g$ differ only at a finite number of points (where there aren’t impulses) then $F = G$

examples:

- $f$ defined as

$$f(t) = \begin{cases} 
1 & t = 2 \\
0 & t \neq 2 
\end{cases}$$

has $F = 0$

- $f$ defined as

$$f(t) = \begin{cases} 
1/2 & t = 0 \\
1 & t > 0 
\end{cases}$$

has $F = 1/s$ (same as unit step)
In principle we can recover \( f \) from \( F \) via

\[
f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s) e^{st} \, ds
\]

where \( \sigma \) is large enough that \( F(s) \) is defined for \( \Re s \geq \sigma \)

surprisingly, this formula isn’t really useful!
Time scaling

define signal $g$ by $g(t) = f(at)$, where $a > 0$; then

$$G(s) = \frac{1}{a} F(s/a)$$

makes sense: times are scaled by $a$, frequencies by $1/a$

let’s check:

$$G(s) = \int_0^\infty f(at)e^{-st} \, dt = \frac{1}{a} \int_0^\infty f(\tau)e^{-(s/a)\tau} \, d\tau = \frac{1}{a} F(s/a)$$

where $\tau = at$

example: $\mathcal{L}(e^t) = 1/(s - 1)$ so

$$\mathcal{L}(e^{at}) = \left(\frac{1}{a}\right) \frac{1}{(s/a) - 1} = \frac{1}{s - a}$$
Exponential scaling

let $f$ be a signal and $a$ a scalar, and define $g(t) = e^{at}f(t)$; then

$$G(s) = F(s - a)$$

let’s check:

$$G(s) = \int_0^\infty e^{-st}e^{at} f(t) \, dt = \int_0^\infty e^{-(s-a)t} f(t) \, dt = F(s - a)$$

example: $\mathcal{L}(\cos t) = s/(s^2 + 1)$, and hence

$$\mathcal{L}(e^{-t} \cos t) = \frac{s + 1}{(s + 1)^2 + 1} = \frac{s + 1}{s^2 + 2s + 2}$$
let $f$ be a signal and $T > 0$; define the signal $g$ as

$$g(t) = \begin{cases} 0 & 0 \leq t < T \\ f(t - T) & t \geq T \end{cases}$$

($g$ is $f$, delayed by $T$ seconds & ‘zero-padded’ up to $T$)
then we have \( G(s) = e^{-sT} F(s) \)

derivation:

\[
G(s) = \int_0^\infty e^{-st} g(t) \, dt = \int_T^\infty e^{-st} f(t - T) \, dt
\]

\[
= \int_0^\infty e^{-s(\tau + T)} f(\tau) \, d\tau
\]

\[
= e^{-sT} F(s)
\]
example: let’s find the Laplace transform of a rectangular pulse signal

\[
f(t) = \begin{cases} 1 & \text{if } a \leq t \leq b \\ 0 & \text{otherwise} \end{cases}
\]

where \(0 < a < b\)

we can write \(f\) as \(f = f_1 - f_2\) where

\[
f_1(t) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases} \quad f_2(t) = \begin{cases} 1 & t \geq b \\ 0 & t < b \end{cases}
\]

i.e., \(f\) is a unit step delayed \(a\) seconds, minus a unit step delayed \(b\) seconds

hence

\[
F(s) = \mathcal{L}(f_1) - \mathcal{L}(f_2) = \frac{e^{-as} - e^{-bs}}{s}
\]

(can check by direct integration)
Derivative

if signal $f$ is continuous at $t = 0$, then

$$\mathcal{L}(f') = sF(s) - f(0)$$

- time-domain differentiation becomes multiplication by frequency variable $s$ (as with phasors)
- \textit{plus} a term that includes initial condition (\textit{i.e.}, $-f(0)$)

higher-order derivatives: applying derivative formula twice yields

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0)$$

$$= s(sF(s) - f(0)) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

similar formulas hold for $\mathcal{L}(f^{(k)})$
Examples

- $f(t) = e^t$, so $f'(t) = e^t$ and

$$\mathcal{L}(f) = \mathcal{L}(f') = \frac{1}{s - 1}$$

using the formula, $\mathcal{L}(f') = s\left(\frac{1}{s - 1}\right) - 1$, which is the same

- $\sin \omega t = -\frac{1}{\omega} \frac{d}{dt} \cos \omega t$, so

$$\mathcal{L}(\sin \omega t) = -\frac{1}{\omega} \left( s \frac{s}{s^2 + \omega^2} - 1 \right) = \frac{\omega}{s^2 + \omega^2}$$

- $f$ is unit ramp, so $f'$ is unit step

$$\mathcal{L}(f') = s \left( \frac{1}{s^2} \right) - 0 = 1/s$$
derivation of derivative formula: start from the defining integral

\[ G(s) = \int_0^\infty f'(t)e^{-st} dt \]

integration by parts yields

\[
G(s) = e^{-st} f(t) \Big|_0^\infty - \int_0^\infty f(t)(-se^{-st}) \, dt
\]

\[
= \lim_{t \to \infty} f(t)e^{-st} - f(0) + sF(s)
\]

for \( \Re s \) large enough the limit is zero, and we recover the formula

\[ G(s) = sF'(s) - f(0) \]
derivative formula for discontinuous functions

if signal $f$ is discontinuous at $t = 0$, then

$$\mathcal{L}(f'(t)) = sF(s) - f(0-)$$

example: $f$ is unit step, so $f'(t) = \delta(t)$

$$\mathcal{L}(f'(t)) = s \left( \frac{1}{s} \right) - 0 = 1$$
Example: RC circuit

- capacitor is uncharged at $t = 0$, i.e., $y(0) = 0$
- $u(t)$ is a unit step

from last lecture,

$$y'(t) + y(t) = u(t)$$

take Laplace transform, term by term:

$$sY(s) + Y(s) = 1/s$$

(using $y(0) = 0$ and $U(s) = 1/s$)
solve for $Y(s)$ (just algebra!) to get

$$Y(s) = \frac{1/s}{s + 1} = \frac{1}{s(s + 1)}$$

to find $y$, we first express $Y$ as

$$Y(s) = \frac{1}{s} - \frac{1}{s + 1}$$

(check!)

therefore we have

$$y(t) = \mathcal{L}^{-1}(1/s) - \mathcal{L}^{-1}(1/(s + 1)) = 1 - e^{-t}$$

Laplace transform turned a differential equation into an algebraic equation (more on this later)
let $g$ be the running integral of a signal $f$, i.e.,

$$g(t) = \int_{0}^{t} f(\tau) \, d\tau$$

then

$$G(s) = \frac{1}{s} F(s)$$

i.e., time-domain integral becomes division by frequency variable $s$

**example:** $f = \delta$, so $F(s) = 1$; $g$ is the unit step function

$$G(s) = \frac{1}{s}$$

**example:** $f$ is unit step function, so $F(s) = 1/s$; $g$ is the unit ramp function ($g(t) = t$ for $t \geq 0$),

$$G(s) = \frac{1}{s^2}$$
derivation of integral formula:

\[
G(s) = \int_{t=0}^{\infty} \left( \int_{\tau=0}^{t} f(\tau) \, d\tau \right) e^{-st} \, dt = \int_{t=0}^{\infty} \int_{\tau=0}^{t} f(\tau)e^{-st} \, d\tau \, dt
\]

here we integrate horizontally first over the triangle \(0 \leq \tau \leq t\)

let's switch the order, \text{i.e.}, integrate vertically first:

\[
G(s) = \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} f(\tau)e^{-st} \, dt \, d\tau = \int_{\tau=0}^{\infty} f(\tau) \left( \int_{t=\tau}^{\infty} e^{-st} \, dt \right) \, d\tau
\]

\[
= \int_{\tau=0}^{\infty} f(\tau)(1/s)e^{-s\tau} \, d\tau
\]

\[
= F(s)/s
\]
Multiplication by $t$

Let $f$ be a signal and define

$$g(t) = tf(t)$$

then we have

$$G(s) = -F'(s)$$

to verify formula, just differentiate both sides of

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt$$

with respect to $s$ to get

$$F'(s) = \int_0^\infty (-t)e^{-st} f(t) \, dt$$
examples

- \( f(t) = e^{-t}, \ g(t) = te^{-t} \)

\[
\mathcal{L}(te^{-t}) = -\frac{d}{ds} \frac{1}{s + 1} = \frac{1}{(s + 1)^2}
\]

- \( f(t) = te^{-t}, \ g(t) = t^2e^{-t} \)

\[
\mathcal{L}(t^2e^{-t}) = -\frac{d}{ds} \frac{1}{(s + 1)^2} = \frac{2}{(s + 1)^3}
\]

- in general,

\[
\mathcal{L}(t^k e^{-t}) = \frac{(k - 1)!}{(s + 1)^{k+1}}
\]
Convolution

the convolution of signals \( f \) and \( g \), denoted \( h = f \ast g \), is the signal

\[
h(t) = \int_0^t f(\tau)g(t - \tau) \, d\tau
\]

• same as \( h(t) = \int_0^t f(t - \tau)g(\tau) \, d\tau \); in other words,

\[
f \ast g = g \ast f
\]

• (very great) importance will soon become clear

in terms of Laplace transforms:

\[
H(s) = F(s)G(s)
\]

Laplace transform turns convolution into multiplication
let’s show that $\mathcal{L}(f \ast g) = F(s)G(s)$:

$$H(s) = \int_{t=0}^{\infty} e^{-st} \left( \int_{\tau=0}^{t} f(\tau)g(t - \tau) \, d\tau \right) \, dt$$

$$= \int_{t=0}^{\infty} \int_{\tau=0}^{t} e^{-st} f(\tau)g(t - \tau) \, d\tau \, dt$$

where we integrate over the triangle $0 \leq \tau \leq t$

• change order of integration: $H(s) = \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} e^{-st} f(\tau)g(t - \tau) \, dt \, d\tau$

• change variable $t$ to $\bar{t} = t - \tau; \, d\bar{t} = dt$; region of integration becomes $\tau \geq 0, \, \bar{t} \geq 0$

$$H(s) = \int_{\tau=0}^{\infty} \int_{\bar{t}=0}^{\infty} e^{-s(\bar{t}+\tau)} f(\tau)g(\bar{t}) \, d\bar{t} \, d\tau$$

$$= \left( \int_{\tau=0}^{\infty} e^{-s\tau} f(\tau) \, d\tau \right) \left( \int_{\bar{t}=0}^{\infty} e^{-s\bar{t}} g(\bar{t}) \, d\bar{t} \right)$$

$$= F(s)G(s)$$
examples

- $f = \delta$, $F(s) = 1$, gives

$$H(s) = G(s),$$

which is consistent with

$$\int_0^t \delta(\tau) g(t - \tau) d\tau = g(t)$$

- $f(t) = 1$, $F(s) = e^{-sT}/s$, gives

$$H(s) = G(s)/s$$

which is consistent with

$$h(t) = \int_0^t g(\tau) d\tau$$

- more interesting examples later in the course . . .
Finding the Laplace transform

you should \textit{know} the Laplace transforms of some basic signals, \textit{e.g.},

- unit step ($F(s) = 1/s$), impulse function ($F(s) = 1$)
- exponential: $\mathcal{L}(e^{at}) = 1/(s - a)$
- sinusoids $\mathcal{L} \left( \cos \omega t \right) = s/(s^2 + \omega^2)$, $\mathcal{L} \left( \sin \omega t \right) = \omega/(s^2 + \omega^2)$

these, combined with a table of Laplace transforms and the properties given above (linearity, scaling, \ldots) will get you pretty far

and of course you can always integrate, using the defining formula

$$F(s) = \int_{0}^{\infty} f(t) e^{-st} \, dt \quad \ldots$$
Patterns

while the details differ, you can see some interesting symmetric patterns between

- the time domain (i.e., signals), and
- the frequency domain (i.e., their Laplace transforms)

- differentiation in one domain corresponds to multiplication by the variable in the other
- multiplication by an exponential in one domain corresponds to a shift (or delay) in the other

we’ll see these patterns (and others) throughout the course