Lecture 11
Feedback: static analysis

- feedback: overview, standard configuration, terms
- static linear case
- sensitivity
- static nonlinear case
- linearizing effect of feedback
Feedback: general

a portion of the output signal is ‘fed back’ to the input

standard block diagram:

\[ u \rightarrow e \rightarrow A \rightarrow y \]

\[ F \]

\[ u \] is the input signal; \( y \) is the output signal; \( e \) is called the error signal

\[ A \] is called the forward or open-loop system or plant

\[ F \] is called the feedback system

in equations: \( y = Ae, \ e = u - Fy \)
• feedback ‘loop’: \( e \) affects \( y \), which affects \( e \) . . .
• overall system is called *closed-loop* system
• signals can be analog electrical (voltages, currents), mechanical, digital electrical, . . .
• the – sign is a tradition only

Feedback is very widely used

• in amplifiers
• in automatic control (flight control, hard disk & CD player mechanics)
• in communications (oscillators, phase-lock loop)
when properly designed, feedback systems are

- less sensitive to component variation
- less sensitive to some interferences and noises
- more linear
- faster

(when compared to similar open-loop systems)

we will also see some disadvantages, *e.g.*

- smaller gain
- possibility of instability
Other feedback configurations

we will also see other feedback configurations, \textit{e.g.}

\begin{center}
\begin{tikzpicture}
  \node[coordinate] (input) at (0,0) {};
  \node[coordinate] (error) at (1,0) {};
  \node[coordinate] (controller) at (2,0) {};
  \node[coordinate] (plant) at (4,0) {};
  \node[coordinate] (output) at (5,0) {};
  \node[coordinate] (feedback) at (2,-1) {};

  \draw[->] (input) -- node[above] {$r$} (error);
  \draw[<->] (error) -- node[above] {$e$} (controller);
  \draw[->] (controller) -- node[above] {$C$} (plant);
  \draw[->] (plant) -- node[above] {$P$} (output);
  \draw[->] (feedback) -- node[below] {} (controller);
  \draw[->] (feedback) -- node[below] {} (plant);
  \draw[<->] (feedback) -- node[below] {} (output);
\end{tikzpicture}
\end{center}

which is often used in automatic control

for now we stick to the ‘standard configuration’ (p.11–2)
sometimes the ‘feedback loop’ is not clear (e.g., in amplifier circuits)

here we have

\[ V_{\text{out}} = R_l f(V_{\text{GS}}), \quad V_{\text{GS}} = V_{\text{in}} - (R_s/R_l)V_{\text{out}}, \]

where \( I_d = f(V_{\text{GS}}) \)
Static linear case

static case: signals do not vary with time, \textit{i.e.}, signals \( u, e, y \) are (constant) real numbers

(dynamic analysis of feedback is \textit{very} important — we’ll do it later)

suppose forward and feedback systems are linear, \textit{i.e.}, \( A \) and \( F \) are numbers (‘gains’)

eliminate \( e \) from \( y = Ae, e = u - Fy \) to get \( y = Gu \) where

\[
G = \frac{A}{1 + AF}
\]

is called the \textit{closed-loop system gain} (\( A \) is called open-loop system gain)

\( L = AF \) is called the \textit{loop gain} — it is the gain around the feedback loop, cut at the summing junction
observation: if $L = AF$ is large (positive or negative!) then $G \approx 1/F$ and is relatively independent of $A$.

how close is $G$ to $1/F$?

consider relative error: $\frac{1/F - G}{1/F} = \frac{1}{1 + AF}$ (after some algebra)

$$S = \frac{1}{1 + AF} = \frac{1}{1 + L}$$

is called the sensitivity (and will come up many times)

for large loop gain, sensitivity $\approx 1$/loop gain

thus:

for 20dB loop gain, $G \approx 1/F$ within about 10%

for 40dB loop gain, $G \approx 1/F$ within about 1%

etc.
Example: feedback amplifier

\[ v_{\text{out}} = A v, \quad v = v_{\text{in}} - \left( \frac{R_1}{R_1 + R_2} \right) v_{\text{out}} \]

- \( v_{\text{in}} \) is the input \( u \); \( v_{\text{out}} \) is the output \( y \)
- \( v \) is the ‘error signal’ \( e \)
- open-loop gain is \( A \)
- feedback gain is \( F = \frac{R_1}{R_1 + R_2} \)

\[ v_{\text{out}} = G v_{\text{in}}, \text{ where closed-loop gain is } G = \frac{A}{1 + AF} \]
example: for $F = 0.1$ and $A \geq 100$, $G \approx 10$ within $10\%$

as $A$ varies from, say, 100 to 1000 (20dB variation), $G$ varies about $10\%$ (around 1dB variation)

in this example, large variations in open-loop gain lead to much smaller variations in closed-loop gain
Sensitivity to small changes in $A$

how do small changes in the open-loop gain $A$ affect closed-loop gain $G$?

$$\frac{\partial G}{\partial A} = \frac{\partial}{\partial A} \frac{A}{1 + AF} = \frac{1}{(1 + AF)^2}$$

so for small change $\delta A$, we have

$$\delta G \approx \frac{1}{(1 + AF)^2} \delta A$$

express in terms of *relative* or *fractional* gain changes:

$$(\delta G/G) \approx \frac{1}{1 + AF} (\delta A/A) = S(\delta A/A)$$

hence the name ‘sensitivity’ for $S$
for small fractional changes in open-loop gain,

\[ S \approx \frac{\text{fractional change in closed-loop gain}}{\text{fractional change in open-loop gain}} \]

(so ‘sensitivity ratio’ is perhaps a better term for \( S \))

for large loop gain (positive or negative), \( |S| \ll 1 \), so small fractional changes in \( A \) yield \textit{much smaller} fractional changes in \( G \):

feedback has \textit{reduced} the sensitivity of the gain \( G \) w.r.t. changes in the gain \( A \)
we can relate (small) relative changes to changes in dB:

$$\delta(20 \log_{10} X) = \frac{20}{\log 10} \delta \log X \approx \frac{20}{\log 10} (\delta X/X)$$

($20/\log 10 \approx 9$, i.e., 10% relative change $\approx 0.9$dB)

hence we have (for small changes in $A$),

$$\delta(20 \log_{10} G) \approx S \delta(20 \log_{10} A)$$

thus (for small changes in open-loop gain),

$$S \approx \frac{\text{dB change in closed-loop gain}}{\text{dB change in open-loop gain}}$$

**Example:** $\pm 2$dB variation in $A$, with $L \approx 10$, yields approximately $\pm 0.2$dB variation in $G$
Summary:

for loop gain $|L| \gg 1$,

- gain is reduced by about $|L|$
- sensitivity of gain w.r.t. $A$ is reduced by about $|L|$

thus, feedback allows us to trade gain for reduced sensitivity

e.g., convert amplifier with gain $30 \pm 2\text{dB}$ to one with gain $20 \pm 0.7\text{dB}$ or $10 \pm 0.2\text{dB}$
Remarks:

- feedback critical with vacuum tube amplifiers (gains varied substantially with age . . .
- get benefits for ‘negative’ \( AF > 0 \) or ‘positive’ \( AF < 0 \) feedback — makes little difference in static case
- sensitivity w.r.t. \( F \) is \( not \) small — need accurate, reliable feedback components
- can also trade sensitivity for more gain, by setting \( AF \approx -1 \)
Nonlinear static feedback

We suppose now that the forward system is nonlinear static, i.e., $A$ is a function from $\mathbb{R}$ into $\mathbb{R}$, e.g.,

![Graph showing a curve that represents $A(e)$, with $e$ on the horizontal axis and $y$ on the vertical axis, ranging from $-1.5$ to $1.5$.]

very common for amplifiers, transducers, etc. to be at least a bit nonlinear $A$ is called the *nonlinear transfer characteristic* of the forward system (never to be confused with transfer function!)
we’ll keep the feedback system $F$ linear for now

\[
\begin{array}{c}
  \text{\textbf{Feedback system is described by}} \\
  y = A(e), e = u - F y \\
  \text{these are coupled \textit{nonlinear} equations:} \\
  \text{\bullet maybe \textit{multiple} solutions; maybe \textit{no} solutions} \\
  \text{\bullet usually impossible to solve analytically} \\
  \text{\bullet can be solved graphically, or by computer} \\
  \text{usually for each } u \in \mathbb{R} \text{ there is one solution } y, \text{ so we can express the} \\
  \textit{closed-loop transfer characteristic} \text{ as a function: } y = G(u)
\end{array}
\]
Example: open-loop characteristic $A$:
with feedback gain $F = 0.2$, yields closed-loop characteristic

(you should check a few points!)
Observations: with feedback

- ‘gain’ is lower (note different horizontal scales)
- characteristic is more linear (for $|y| < 1$)

these phenomena are general . . .

closed-loop transfer characteristic function $G$ satisfies

$$G(u) = y = A(e), \quad e = u - FG(u)$$

differentiate w.r.t. $u$:

$$G'(u) = A'(e)\frac{de}{du}, \quad \frac{de}{du} = 1 - FG'(u)$$
eliminate $de/du$ to get

$$G'(u) = \frac{A'(e)}{1 + A'(e)F}$$

conclusions: for $u$ s.t. $|A'F| \gg 1$,

- $G'' \approx 1/F$ (independent of $u$) i.e., $G$ is nearly linear!

- slope of $G$ is smaller than slope of $A$
  (by factor $1 + A'F'$)
A measure of nonlinear distortion

let \( w = H(v) \) be a nonlinear I/O characteristic

assume \( H(0) = 0 \) and look at Taylor series

\[
H(v) = H'(0) v + \frac{1}{2} H''(0) v^2 + \cdots
\]

ratio of quadratic term to first order term is

\[
\frac{H''(0)}{2H'(0)} v,
\]

so \( H''(0)/H'(0) \) gives a measure of distortion
(for a given input \( v \), or a given output \( w \))

now consider feedback system, with \( A(0) = 0 \)

distortion measure for open-loop system is \( A''(0)/A'(0) \)
differentiate $G' = A'/(1 + A'F)$ w.r.t. $u$ to get

$$G''(u) = \frac{A''(e)}{(1 + A'(e)F)^2}$$

distortion measure for closed-loop system is

$$G''(0)/G'(0) = \frac{1}{1 + A'(0)F} \frac{A''(0)}{A'(0)}$$

thus, nonlinear distortion measure is reduced by the sensitivity $S$ of the linearized system!
**Finding the closed-loop characteristic**

**Graphical method** (load line): write feedback equations as \( y = A(e) \), \( e = u - Fy \)

for given \( u \) sketch both equations on \( e-y \) plane; intersection gives solution

easy to visualize what happens as \( u \) or \( F \) changes
Newton’s method to solve \( y = A(e), \ e = u - Fy \) (given \( A, u, \) and \( F \))

1. guess a value \( e_0 \) for \( e \); set \( k = 0 \)
2. set \( y_k := A(e_k) \)
3. if \( e_k = u - Fy_k \), quit
4. replace nonlinear equation \( y = A(e) \) with first-order Taylor expansion near \( e_k \),
\[
y \approx A(e_k) + A'(e_k)(e - e_k)
\]
Then solve the linear equations
\[
\hat{y} = A(e_k) + A'(e_k)(\hat{e} - e_k), \\
\hat{e} = u - F\hat{y}
\]
for \( \hat{e} \) and \( \hat{y} \); set \( e_{k+1} := \hat{e} \)

I.e., set \( e_{k+1} := \frac{u - Fy_k + FA'(e_k)e_k}{1 + FA'(e_k)} \)
5. \( k := k + 1; \) go to 2
works very well when initial guess is good; may not converge for bad initial guess
Graphical interpretation of Newton’s method

\[ e = u - Fy \]

\[ y = A(e_k) + A'(e_k)(e - e_k) \]
Tracing the closed-loop characteristic curve

write feedback equations as

\[
y = A(e), \quad u = e + Fy
\]

given error \(e\), we can easily find associated \(y\) and \(u\)!

can use this to trace the curve, parametrized by \(e\):

1. choose \(e_1, e_2, \ldots, e_n\) that cover an appropriate range for \(e\)
2. for \(i = 1\) to \(n\), set \(y_i := A(e_i), u_i := e_i + Fy_i\)
3. plot \((u_1, y_1), \ldots, (u_n, y_n)\)

note that here we don’t specify the \(u\) values (as in Newton’s method)
**Example:** JFET amplifier (we assume $v_{GS} \leq 0$)

\[ v_{out} = A(v_{GS}), \quad v_{GS} = v_{in} - F v_{out}, \]

with $F = R_s / R_l$ and

\[
A(v_{GS}) = \begin{cases} 
R_l I_{DSS} (1 - v_{GS}/V_P)^2 & V_P \leq v_{GS} \leq 0 \\
0 & v_{GS} < V_P
\end{cases}
\]
we’ll take $R_l = 10k\Omega$, $I_{DSS} = 1\text{mA}$, $V_P = -2\text{V}$

plot shows $v_{\text{out}}$ vs. $v_{\text{in}}$ for $R_s = 0, 1, \ldots, 5k\Omega$
(corresponds to $F = 0, 0.1, \ldots, 0.5$)

as feedback increases, closed-loop ‘gain’ is smaller; closed-loop characteristic is more linear
Summary

- using feedback we can trade raw gain for lower sensitivity, greater linearity

- benefits determined by $S = 1/(1 + AF)$:
  sensitivity and nonlinearity are both reduced by $S$

- large loop gain $L = AF$ (positive or negative) yields small $S$ hence benefits of feedback