Natural logic: Sánches

- Typed lambda calculus
  - Types
  - Lambek terms
  - Lambek Grammar
- Polarity
- Semantics
  - Fregean Universe
  - Ordering of denotations
  - Model
- Monotonicity

Types

The set of types consists of

e, p, and t

if α and β are types, so is $\alpha \rightarrow \beta$.

Any variable X^n_α is a type

Application:

If N_α and $M_{\alpha\rightarrow\beta}$ are types then also $(M_{\alpha\rightarrow\beta}N_\alpha)$.

Abstraction:

M_β is a type and X_α is a variable then also $[\lambda X_\alpha . M_\beta]$.

Free Variables

The set of free variables in N_α is given by

$FV(X) = X$

$FV(MN) = FV(M) \cup FV(N)$

$FV(\lambda X.M) = FV(M) - \{X\}$

Substitution

If N is a term and X is a variable of the same type as M, then the result of substituting M for the free occurrences of X in N (Notation: N[X:=M]) is given by

$X[X:=M] \equiv N$

$Y[X:=M] \equiv Y$, if $Y \not\equiv X$

$(NP)[X:=M] \equiv N[X:=M]\ P[X:=M]$

$(\lambda Y.N)[X:=M] \equiv \lambda Y.N[X:=M]$

Lambek Term

The set of Lambek Terms (LT) is the set of all the terms P which satisfy:

- P is a variable
- $P \equiv M_{\alpha \rightarrow \beta} N_{\alpha}$ is in LT iff $FV(M) \cap FV(N) = \emptyset$
- $P \equiv \lambda X_{\alpha}.N$ is in LT iff $X_{\alpha} \varepsilon FV(N)$, $FV(N) - \{X\} \neq \emptyset$

(The last condition entails that a Lambek term has at least one free variable. Why?)

β-Reduction

$[\lambda X_{\alpha}.N]\ M_{\alpha}$ contracts to $N[X:=M]$

Lambek Grammar

Lambek Grammar is a proof system where a conclusion β is derived from some set of "open assumptions" $[\alpha^n]$ by rules of deduction.

$$\begin{array}{c} [\alpha^n] \\ D \\ \beta \end{array}$$

There are two rules of inference, *Elimination* and *Introduction*, corresponding to Application and Abstraction in the derivation of Lambek terms. The open assumptions are uniquely indexed by numbers (or by lexical items, as we will see shortly).

Elimination

$$\dfrac{\begin{matrix}[A] \\ D_1 \\ \alpha \to \beta\end{matrix} \qquad \begin{matrix}[B] \\ D_2 \\ \alpha\end{matrix}}{\beta}\ E$$

Elimination does not consume any assumptions. The set of open assumption of the new derivation is $A \cup B$

Introduction

$$\cfrac{\begin{matrix}[\alpha^n] \\ D \\ \beta\end{matrix}}{\alpha \to \beta}\ (n) \to I$$

Introduction consumes one assumption. The introduction step is marked with the index of the assumption that is "discharged."

Example 1

$$\frac{\dfrac{e\rightarrow t^{\,3} \quad e^{2}}{t}\,E}{(e\rightarrow t)\rightarrow t}\,I^{(3)}$$

This is a derivation of Type Raising from two open assumptions of which one, e→t, is discharged, leaving e as the remaining open assumption.

Example 2

$$\cfrac{\cfrac{\cfrac{(e\to t)\to t^{3} \qquad \cfrac{e\to e\to t^{1} \qquad e^{2}}{e\to t}\,E}{t}\,E}{e\to t}\,I\ (2)}{(e\to e\to t)\to (e\to t)}\,I\ (1)$$

This is a derivation of Division (“Geach Rule”) from three open assumptions of which one remains open.

Derivations and Lambek terms

For each derivation in Lambek Grammar there is a correspoding Lambek Term and vice versa

$$
(2)\quad \cfrac{\cfrac{e\rightarrow t^{2} \quad e^{1}}{t}}{(e\rightarrow t)\rightarrow t}
\quad \Leftrightarrow \quad
\cfrac{\cfrac{X_{e\rightarrow t} \quad Y_{e}}{XY}}{\lambda X.XY}
$$

Lexical Indices

Lexical items enter proofs as open assumptions
but they are never discharged.

$$\dfrac{\begin{matrix}\text{a theorem}\\(e\to t)\to t\end{matrix} \qquad {\scriptstyle (2)}\dfrac{\dfrac{\begin{matrix}\text{every logician}\\(e\to t)\to t\end{matrix} \qquad \dfrac{\begin{matrix}\text{proves}\\e\to(e\to t)\end{matrix} \quad e^2}{e\to t}}{t}}{e\to t}}{t}$$

“...lexical assumptions are never eliminated. On the other hand, numerical assumptions are not present in the ***analyses***. Numerical assumptions may be seen as empty elements, not realized phonetically.” Sánches, p. 90.

Polarity and Monotonicity

Pierce (1885) came up with the idea that we can compute whether a term in a formula is positive or negative and use that as a basis for monotonicity reasoning. Pierce's *System of Existential Graphs* notation is the starting point for Sánches. Pierce's original formulation covers only a subset of cases treated by Sánches.

Polarity for FOL

Definition:

(i) R occurs positively in $R(t_1, t_2, \ldots, tn)$.

(ii) If R occurs positively (negatively) in ϕ, then R occurs positively (negatively) in $\phi \wedge \psi$, $\phi \vee \psi$, $\psi \rightarrow \phi$, $\forall \phi$, $\exists \phi$.

(iii) If R occurs positively (negatively) in ϕ, then R occurs negatively (positively) in $\neg \phi$, $\phi \rightarrow \psi$.

Polarity generalized

If *a* is a formula and A(ϕ) and B(ϕ) are formulas where A denotes an upward monotone function and B denotes a downward monotone function, then

(i) ϕ is positive in ϕ.
(ii) If ϕ is positive in F(ϕ), then it is positive in A(F(ϕ)) and negative in B(F(ϕ)),
(iii) If ϕ is negative in F(ϕ), then it is negative in A(F(ϕ)) and positive in B(F(ϕ)).

This incorporates Pierce's observation that two negations yield a positive polarity,

Fregean Universe

Let D be a non-empty set. Then D_α is given for all types of α by the following recursion:

(i) $D_e = D$

(ii) $D_t = \{0, 1\}$

(iii) $D_{\alpha\to\beta} = {D_\beta}^{D_\alpha}$, the set of set theoretic functions from D_α to D_β.

$D_{e \to t}$

$D = \{j,b,m\}$

$D_{e \to t} =$

{j,b,m}	{j,b}	{j,m}	{b,m}
$j \to 1$, $b \to 1$, $m \to 1$	$j \to 1$, $b \to 1$, $m \to 0$	$j \to 1$, $b \to 0$, $m \to 1$	$j \to 0$, $b \to 1$, $m \to 1$
$j \to 1$, $b \to 0$, $m \to 0$	$j \to 0$, $b \to 1$, $m \to 0$	$j \to 0$, $b \to 0$, $m \to 1$	$j \to 0$, $b \to 0$, $m \to 0$
{j}	{b}	{m}	{}

$= \mathcal{P}(D)$

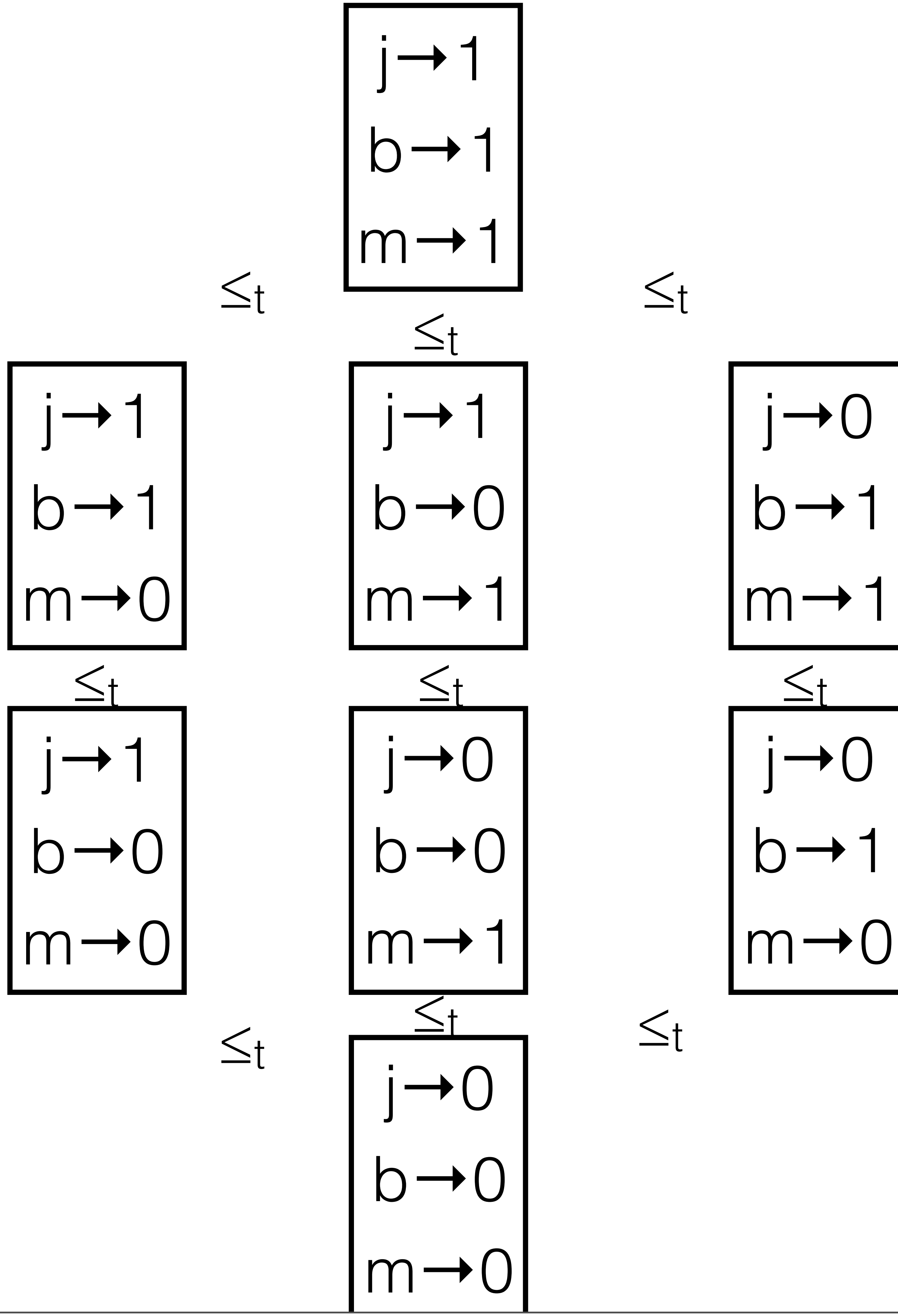
Ordering

We partially order the sets D_α of the Fregean universe by a relation $\leq_\alpha$ as follows:

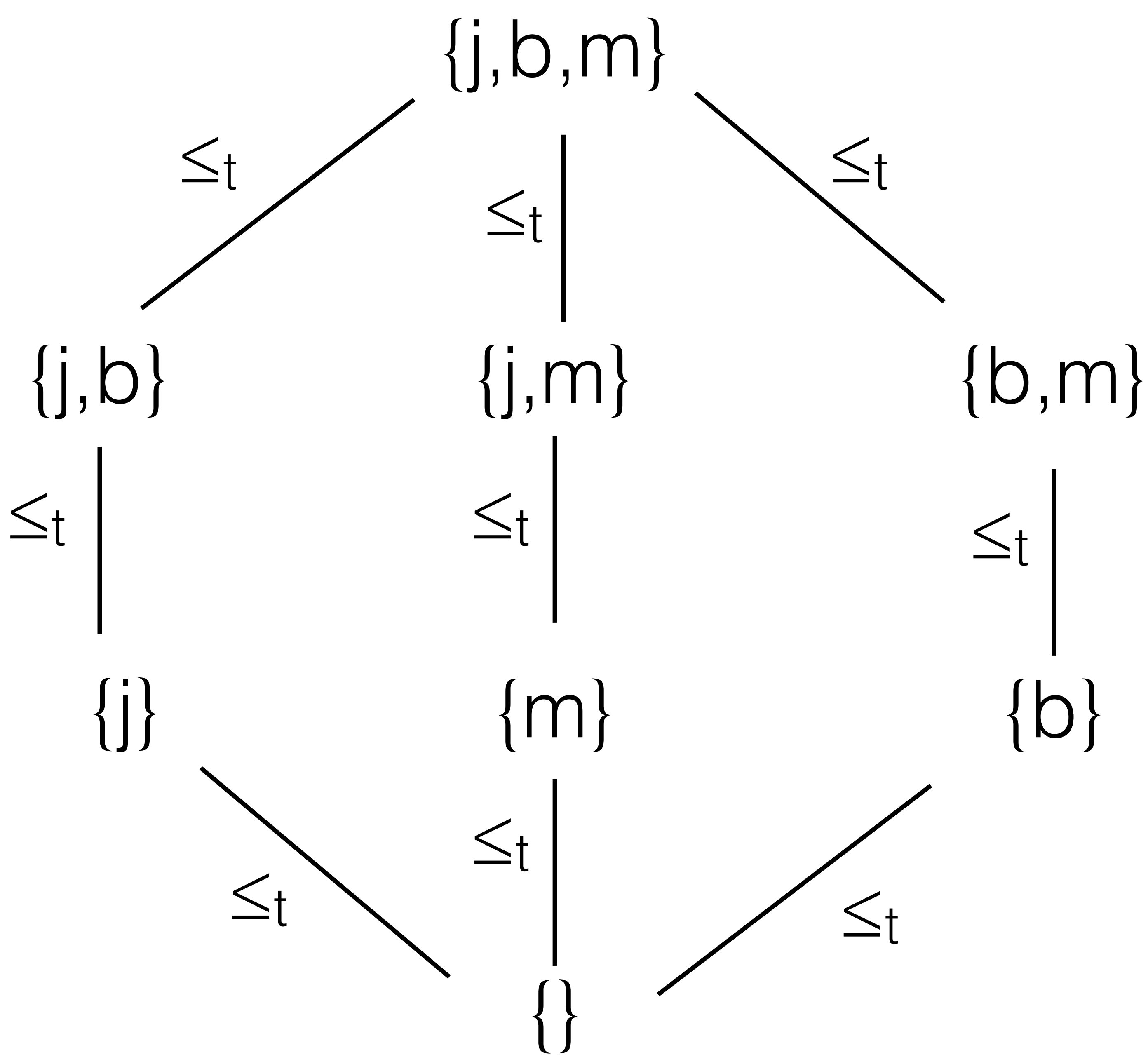
(i) If c, d ε D_e then c $\leq_\alpha$ d iff c = d.

(ii) If c, d ε D_t then c $\leq_t$ d iff c = 0 or d = 1.

(iii) If c, d ε $D_{\alpha\rightarrow\beta}$ then c $\leq_{\alpha\rightarrow\beta}$ d iff for each a ε D_α c(a) $\leq_\beta$ d(a).


j→1
b→1
m→1
≤t
≤t
≤t
j→1
b→1
m→0
j→1
b→0
m→1
j→0
b→1
m→1
≤t
≤t
≤t
j→1
b→0
m→0
j→0
b→0
m→1
j→0
b→1
m→0
≤t
≤t
≤t
j→0
b→0
m→0

Example



Next to Come

Interpretation of a typed language.

Using syntactic types for polarity marking

Sánchez’ system of Natural logic (ChapterVI).

Dowty’s take on Sánchez

Moss’ take on Dowty