# Mathematics for Public Policy

**Avidit Acharya**

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# Contents

**Disclaimer:** I am not claiming originality for these notes. Almost all of the theorems and proofs have been taken from various other sources, and in some cases (especially where the exposition is particularly elegant) they have been copied verbatim from these sources. I compiled these notes only to organize my lectures in teaching mathematics in the to D-track MPA students at the Woodrow Wilson School, Princeton University. I taught this class for three consecutive summers, 2008-2010.

While it is impossible for me to list all of the sources for this material, I list some of the major sources as follows. The proof of the Muller-Satterthwaite Theorem in Section 2.3 is the proof that appears in Philip Reny's "Arrow's Theorem and the Gibbard Satterthwaite Theorem: A Unified Approach," *Economics Letters*, 70, 2001, 91-105. Most of Section 3, almost all of Section 4, and Sections 6.1-6.8 are transplanted here from Richard Beals's *Analysis: An Introduction*, Cambridge: Cambridge University Press, 2004 – a beautiful book that I used as an undergraduate. The exposition on the implicit function theorem in Section 4.6 was inspired by Steven Krantz and Harold Parks's *Implicit Function Theorem: History, Theory and Applications*, Boston: Birkhauser, 2002. The continuous-time Solow model that appears in Section 4.8 is taken from Robert Barro and Xavier Sala-i-Martin's *Economic Growth*, Camrbridge, MA: The MIT Press, 1995. Section 5 on Linear Algebra is taken from Serge Lang's *Introduction to Linear Algebra*, New York: Springer-Verlag, 1987, and Rangarajan Sundaram's *A First Course in Optimization Theory*, Cambridge: Cambridge University Press, 1996. Sundaram's book is also the source for the part on Taylor's theorem in $\mathbb{R}^n$ appearing in Section 6.9 and the exposition on Lagrange's theorem in Sections 7.3, 7.5 and 7.6. Section 6.9 on Taylor's theorem in $\mathbb{R}$ is from Serge Lang's *Undergraduate Analysis*, New York: Springer-Verlag, 1997. The remainder of Section 7 with the exception of Section 7.9 is from Geoffrey Jehle and Philip Reny's *Advanced Microeconomic Theory*, Boston: Addison-Wesley, 2001. Section 7.9 on the Agricultural Household Model is from Pranab Bardhan and Chris Udry's *Development Microeconomics*, Oxford: Oxford University Press, 1999.

Lastly, every year when I teach the course I discover a few more typos, sometimes even errors. Although this is the latest version of the notes, I am sure that there are still many typos and errors left. If you discover any, please email them to me at aacharya@princeton.edu.

# Chapter 1

# Introduction

## 1.1 Mathematical Statements

The table at the bottom of the page lists some common mathematical symbols and their abbreviations. Mathematical statements in this course will seldom involve abbreviations or symbols other than the ones listed in the table (except ones that you surely already know such as $=$, $\geq$ etc.). When new symbols arise, I will explain them. As an example, a typical statement is

$$\forall x \in X \text{ and } \forall y \in Y,\ \exists z \in Z \text{ s.t. } x + y = z,$$

which you will read

For every $x$ in the set $X$ and every $y$ in the set $Y$,
there is an element $z$ in the set $Z$ such that $x$ plus $y$ equals $z$.

Thus a mathematical statement is nothing more than a statement in the English language (or any other language for that matter), where the vocabulary is limited to

| Symbol | How to read it |
|---|---|
| $\in$ | "in the set" or "is in the set" depending on context |
| $\exists$ | "there is a(n)," |
| $\forall$ | "for all" or "for every" |
| s.t. | "such that"[1] |
| w.l.o.g. | "without loss of generality" |

words like "for all," "there is," and "such that." Our objective will be to determine which statements are true, and which are not.

In mathematics, no statement is true in an absolute sense. That is, every statement must be derived from other statements. Put another way, the claim that statement $S_1$ is true is always meaningless. Only claims of the form "$S_1$ is true if $S_2$ and $S_3$ are both true" are meaningful. For the claim to be right, we must be able to derive $S_1$ when we *assume* that $S_2$ and $S_3$ are true. At the end of the day, something must be assumed. We cannot derive mathematical statements from nothing.

In light of that, it is useful to note the difference between "if" and "only if." $S_1$ if $S_2$ means that you can derive $S_1$ from $S_2$, but it may not be the case that you can derive $S_2$ from $S_1$. On the other hand, $S_1$ only if $S_2$ means that you can derive $S_2$ from $S_1$ but that you may not be able to derive $S_1$ from $S_2$. $S_1$ if and only if $S_2$ means that $S_1$ can be derived from $S_2$ and $S_2$ can be derived from $S_1$. When this happens, $S_1$ and $S_2$ are **equivalent**: one statement does not say any more or any less than the other statement. Think about that.

Every mathematical statement has a **negation**. The negation of statement $S_1$ is written $\neg S_1$. For example, you can negate the statement

"Every left-handed man in Princeton has a beard,"

by presenting a left-handed man in Princeton who does not have a beard. Sometimes in the policy world we meet people who would try to negate this statement by presenting a right-handed man in Princeton who has a beard. (You should not take any policy advice from such people.) Another common error is to present a beardless left-handed woman in Princeton. You may find it hard to believe that people make these errors, but trust me you will encounter them in life. The negation of the above statement is only the statement

"There is a left-handed man in Princeton who does not have a beard."

The statement "$S_1$ is true if $S_2$ is true" is equivalent to the statement "$\neg S_2$ is true if $\neg S_1$ is true," or "$S_2$ is not true if $S_1$ is not true." The latter two statements are the **contrapositive** of the first statement. A statement is always equivalent to its contrapositive.

A statement is not equivalent to its **converse**. The converse of the statement "$S_1$ is true if $S_2$ is true" is the statement "$S_2$ is true if $S_1$ is true." To see this, let $S_1$ be the statement "U2 rocks," and $S_2$ be the statement "All Irish bands rock."

Mathematicians organize their lives by assigning different names for similar things. Lemma, Claim, Proposition and Theorem all refer to statements that are to be proven, while Axiom, Postulate and Assumption refer to statements that are assumed to be true. A Corollary is an immediate consequence of a Theorem, which is a statement that is very useful to know. A Lemma is not useful per se, except to prove a Theorem. Propositions are interesting results that may or may not be useful, while Claims differ from Propositions in that they are not that interesting. This is all my own rough understanding of the taxonomy of results, but as you may rightly think, the lines between these concepts can be very blurry.

## 1.2 Proving Mathematical Statements

In this course, we will be doing proofs. Often we will prove a statement directly from a set of other statements. But this may not always be convenient. If we assume that $S_1$ is true and would like to prove that $S_2$ is true, then one way of doing this is to begin the proof by assuming that $S_2$ is not true. Then we show that $\neg S_2$ implies $\neg S_1$. But this cannot be, since we assumed $S_1$ is true. Therefore, $\neg S_2$ must be false, i.e. $S_2$ must be true, and that completes the proof. You may have already noticed that this is simply proving the statement "$S_1$ implies $S_2$" by proving the contrapositive. In any case, this kind of proof is called a proof by **contradiction**. It is closely related to the method of proof called **reductio ad absurdum**, which allows us to conclude that the statement $S_1$ is false if $S_1$ implies a statement $S_2$ and its negation $\neg S_2$. (Both $S_2$ and $\neg S_2$ cannot simultaneously be true, so $S_1$ must be false.) I recommend reading the Wikipedia article on reductio ad absurdum.

The third common method of proof is proof by **induction**. Suppose you wanted to prove that a sequence of statements $S_1$, $S_2$, $S_3$ ... are all true if $S_0$ is true. If the sequence never terminates, you have no hope of doing this in your lifetime if you try to prove each statement one at a time. But there is a shortcut. First you show that $S_1$ is true if $S_0$ is true. Then you show that for any positive integer $k$, $S_k$ implies $S_{k+1}$ when $S_0$ is true. That completes the proof. The reason this works is because you can substitute 1 for $k$, and since you showed $S_1$ is true, $S_2$ must be true. Then substitute 2 for $k$, and you get the result that $S_3$ is true, etc. In **strong induction** you show that $S_1$ is true. Then you show that for every positive integer $k$, $S_1, ..., S_k$ implies $S_{k+1}$. This is similar to induction.

How do you feel about induction?

There are also other methods of proof. You can google search "methods of proof" to find out what they are if you are so enthusiastic about math camp that you can't sit still. The reason I think proofs are important for public policy is that proofs are simply arguments, and making an argument is an important skill to have in the policy world.

## 1.3 Numbers and Shorthands

When I say "number" in this course I always mean a **real number** (except in some cases where I mean a positive integer – you will know this by context). The set of real numbers will be denoted $\mathbb{R}$, which consists of all the numbers you know, except the imaginary numbers (e.g. $32, -0.73, 0, 19/7, 4\pi$, and $e^{23}$ are all in $\mathbb{R}$, but $5i$ is not). Beyond this, I do not want to go into much detail about what a real number is. When the symbols $\geq$, $<$, $\leq$, $>$ are used, two numbers are being compared. The two sides of the equal sign, $=$, may have numbers or other kinds of objects such as sets. The context will make that clear. One common shorthand that I will use is "$\forall i = 1, 2, 3...$" which you will read as "for every positive integer $i$."

# Chapter 2

# Preliminaries

## 2.1 Sets, Relations and Functions

A **set** is a "well-defined" collection of elements. "Well-defined" means that I can describe to you what kinds of things are in the set, and you will be able to know exactly whether something is in the set or not. For example, if I ask you to consider the set of even numbers, you know exactly whether 591 is in the set or not. Sometimes, we can describe a set by simply listing out its elements:

$$A = \{a_1, a_2, a_3, ...\};$$

but only as long as this does not take us forever. Whenever we use curly brackets, that is $\{\ \}$, those are sets and inside the brackets is a list of elements in the set or a mathematical statement describing the common property satisfied by the elements belonging to the set.

The set $A$ is a **subset** of the set $B$, written $A \subset B$, if every element of $A$ is also an element of $B$. Two sets are equal if they are subsets of each other. If $A$ is a set, then the set of all subsets of $A$ is called the **power set** of $A$ and is denoted $\mathcal{P}(A)$. If $A$ is a set with a finite number of elements then $|A|$ denotes the number of elements in $A$. The set $B$ where $|B| = 0$ is unique, it is called the **empty set**, and it is denoted $\emptyset$. You should realize that for every set $A$, we have $\emptyset \in \mathcal{P}(A)$ and $A \in \mathcal{P}(A)$. If $A$ is a set and $B$ is a subset of $A$, then the set $A \backslash B$ is the set of all elements that are in $A$ but not in $B$. It is called the **complement** of $B$ in $A$.

A **binary relation** is a set of **pairs** from $A$ and $B$, i.e. elements of the form $(a, b)$, where $a \in A$ and $b \in B$. Since we never work with ternary or quaternary relations, we refer to binary relations simply as relations. The **cartesian product** of $A$ and $B$, denoted $A \times B$, is the set of *all* pairs $(a, b)$. Thus, a relation $R$, over $A \times B$, is a subset of $A \times B$. We often write $aRb$ to mean the same thing as $(a, b) \in R$.

A **function** $f$, over $A \times B$ (often denoted $f : A \to B$), is a relation that has the following property: if $(a, b) \in f$ and $(a, b') \in f$ then $b = b'$. $A$ is called the **domain** while $B$ is called the **range** of $f$. The statement $(a, b) \in f$ is often written $f(a) = b$, which you are probably more familiar with. We say $f$ is **surjective** if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$. We say that $f$ is **injective** if $f(a) = f(a')$ implies $a = a'$. A function that is both injective and surjective is called **bijective**. Bijective functions are **invertible**, that is, given $b \in B$, there is a unique $a \in A$ such that $f(a) = b$. Functions that are not bijective are not invertible. (Please convince yourself that this is true.) If $f$ is invertible, then there is a function $g : B \to A$ such that for all $a \in A$, $g(f(a)) = a$; $g$ is called the inverse of $f$ and is usually denoted $f^{-1}$ instead of $g$. Notice that $f(g(b)) = b$ for all $b \in B$. (Please also convince yourself that this is true.)

Exercise 1: Consider the functions $f : A \to B$ and $g : B \to C$. The set

$$\{(a, g(f(a))) \mid a \in A\}$$

is read "the set of pairs $(a, g(f(a)))$ such that $a \in A$". Verify that this set is also a function. (*Hint:* Is it a relation? Over what? Does it satisfy the property that relations must satisfy to be functions? After you verify that it is a function, you should know that we call such a function the **composition** of $f$ and $g$.)

For any two sets, $A$ and $B$, define their **union** by $A \cup B = \{x : x \in A \text{ or } x \in B\}$ and their **intersection** by $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Let $y > x$. Then $[x, y]$ is the set of all numbers between $x$ and $y$, including both $x$ and $y$. Alternatively, we can write $(x, y]$ to exclude $x$ or $[x, y)$ to exclude $y$ or $(x, y)$ to exclude both $x$ and $y$. Remember that you should not confuse the interval $(x, y)$ with the pair $(x, y)$: when this notation is used the context will make it clear which of these we refer to. All of these sets, $[x, y]$, $(x, y]$, $[x, y)$ and $(x, y)$, which happen to be subsets of $\mathbb{R}$, are called **intervals**. $[x, y]$ is a **closed interval**, while $(x, y)$ is an **open interval**; $[x, y)$ and $(x, y]$ are **half open** intervals.

Let us look at the special case of $f : X \to \mathbb{R}$, where $X$ is a closed interval. We say that $f$ is **concave** if $\forall x \in X$ and $\forall y \in X$ such that $y \neq x$ we have,

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y) \qquad \forall \alpha \in [0, 1]. \tag{2.1}$$

$f$ is **strictly concave** if in statement (2.1) above, you replace $\geq$ with just $>$ and the interval $[0, 1]$ with the interval $(0, 1)$. It is **convex** if you replace $\geq$ with $\leq$, and it is **strictly convex** if you replace $\geq$ with $<$, and the interval $[0, 1]$ with $(0, 1)$.

Exercise 2. Which of the following functions are (a) concave, (b) strictly concave, (c) convex, (d) strictly convex, or (e) neither: (i) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x + 9$, (ii) $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^3$, (iii) $f : [0, \infty) \to \mathbb{R}$ where $f(x) = (\alpha - 1)x^\alpha$ and $\alpha \in (0, 1)$, (iv) the same function as in (iii) but where $\alpha > 1$.

## 2.2 Economic Preference Theory

A binary relation, $R$ over $A \times B$, is said to be a **preference** relation over $A$ if $A = B$ and the following two properties both hold: (i) $aRa'$ and $a'Ra''$ imply $aRa''$ and (ii) for all $a \in A$ and $a' \in A$, either $aRa'$ or $a'Ra$ (or both). The first property is called **transitivity**, and the second is called **completeness**. Suppose Jim has a preference relation $R$ over the set $C = \{h, l, n\}$, where $h$ stands for higher taxes, $l$ for lower taxes and $n$ for no change in the tax rate. Then you can read the statement $nRh$ to say "Jim prefers no change in the tax rate to higher taxes."

Exercise 3. Define a binary relation over the set $C = \{h, l, n\}$ that satisfies completeness but not transitivity, another one that satisfies transitivity but not completeness, and finally one that satisfies both completeness and transitivity and is thus a preference relation.

The exercise demonstrates the awkwardness in needing $nRn$, for instance, for $R$ to be a preference relation. We read $nRn$ as "Jim prefers $n$ to $n$," which sounds absurd.

In order to deal with this absurdity, economists instead read statements like $nRh$ as "Jim thinks $n$ is at least as good as $h$," which seems to make more sense. However, since saying "is at least as good as" all the time is cumbersome we simply say "Jim prefers..." to mean the same thing, though we know this is semantically inaccurate. (It is similar to reading $\geq$ as $>$ because saying "bigger than" is easier than saying "bigger than or equal to.")

The fundamental assumption of economics is the assumption of **rationality**, which says that all individuals have a preference relation $R$, over a set of choices, $A$. The function $u : A \to \mathbb{R}$ is said to **represent** $R$ if $(a, a') \in R$ if and only if $u(a) \geq u(a')$.

Exercise 4. Show that if the function $u$ represents $R$, then so does the function $v : A \to \mathbb{R}$ defined by $v(a) = (u(a))^3$, but not necessarily the function $w : A \to \mathbb{R}$ defined by $w(a) = (u(a))^2$. (A function that represents a preference relation is called a **utility** function.)

The exercise is meant to show you that there is nothing special about utility functions. An economic "agent" is posited to possess a preference relation over a set of choices $A$, but not a utility function. It is a result, not an assumption, that for every preference relation $R$ over (a finite set) $A$, there is a function $u : A \to \mathbb{R}$ that represents $R$. But there is a large class of utility functions that represent $R$, as you can imagine after having done exercise 4. Therefore, when an economist says that $u$ is someone's utility function, what she really means to say is that the person has a preference relation that can be represented by $u$, among other functions. Utility is *not* an absolute measure of happiness. Think about this.

Exercise 5. For any preference relation, $R$, its **strict preference** subset $P$, and **indifference** subset, $I$, are binary relations defined by

$$aPb \text{ if and only if } aRb \text{ and it is not the case that } bRa\text{; and}$$
$$aIb \text{ if and only if both } aRb \text{ and } bRa.$$

Think about what $P$ and $I$ mean and show that $P$ and $I$ also obey transitivity. Remember that from now on whenever $P$ and $I$ are mentioned in the context of a preference relation $R$ the refer to the strict preference and indifference relations associated with $R$ respectively.

Now suppose there are three policy-makers in government, Laura, Jim and Mark, who must decide which of the following projects to spend government money on: a new security measure $s$, health-care $h$, and education, $e$. The set of choices is $C = \{s, h, e\}$, and only one project can get chosen. Laura has a strict preference relation over $C$, denoted $P_L$. Jim's strict preference is $P_J$, and Mark's is $P_M$. Assume $eP_Lh$ and $hP_Ls$: Laura prefers to spend on education than to spend on health-care, and she also prefers spending on health-care to spending on security. Next, assume $hP_Js$ and $sP_Je$. Finally, let $sP_Me$ and $eP_Mh$. Find a strict preference relation for the government, $P_G$, such that if the majority of policy-makers (at least two out of the three) strictly prefers alternative $a$ to $b$ then government prefers $a$ to $b$ (i.e. $aP_Gb$). If you can't find one, what is the problem? And does the problem necessarily go away if you have more policy-makers?

## 2.3 Application: Social Choice Theory

There is a society of individuals $1, ..., n$ and a set of alternatives, $A$. Let $\wp$ be the set of all strict preference relations over $A$. Each individual $i$ has a strict preference relation $P_i$ over $A$. A **social choice function** is a function $f : \wp^n \to A$, where $\wp^n$ denotes the set $\wp \times ... \times \wp$ ($n$ times) of which $\mathbf{P} = (P_1, ..., P_n)$ is a typical element. The social choice rule $f$ chooses an alternative for each profile of strict preferences.

Now consider the following restrictions on $f$.

**Monotonicity (MON)** If $f(P_1, ...P_n) = a$ and $aP_ib$ implies $aP_i'b$ $\forall i = 1, ..., n$ and $\forall b \in A$, then $f(P_1', ..., P_n') = a$.

**Unanimity (UNA)** If $\forall i = 1, ..., n$, $aP_ib$ $\forall b \in A \backslash \{a\}$, then $f(P_1, ..., P_n) = a$.

How do you feel about these restrictions?

**Muller-Satterthwaite Theorem.** Suppose $|A| \geq 3$ and the social choice rule $f$ satisfies MON and UNA. Then there is an individual $j$ such that $f(P_1, ..., P_n) = a$ if and only if $aP_jb$ for all $b \in A \backslash \{a\}$.

*Proof.* The theorem states that there is an individual $j$ that is a dictator. To prove the theorem we proceed in two steps. We first use the **Geanakoplos algorithm** to find the individual that we would like to accuse of being the dictator. We then prove that this individual is in fact a dictator.

**Step 1.** (Geanakoplos algorithm): Consider a profile of strict preferences where $a$ is ranked highest and $b$ lowest for all $i = 1, ..., n$. UNA implies that $f$ gives $a$ at this profile. Raise $b$ one spot at a time in individual 1's ranking until it rises above $a$. MON implies that either $b$ is the social choice or $a$ is. (Why can't it be another alternative, $c$?) If $a$ is still the social choice then move on to individual 2 and do the same thing: raise $b$ from the bottom until it rises above $a$. As you do this, MON says that $a$ is the social choice except possibly just as when $b$ rises above it. Keep doing this and move across individuals until you hit individual $k \leq n$ for whom raising $b$ above $a$ makes $b$ the social choice for that particular configuration of strict preference relations. (We know $k \leq n$ because by the time $k = n$, UNA tells us that $b$ would have to be chosen.) Next, we will show that this individual $k$ is a dictator.

**Step 2.** ($k$ is the dictator): Consider the following two preference profiles generated by the Geanakoplos algorithm: $\mathbf{P}^1$, where $b$ is at the top of the ranking for $i < k$, just below $a$ for $i = k$ and at the bottom for $i > k$; and $\mathbf{P}^2$ which is otherwise the same as $\mathbf{P}^1$ except that $b$ is at the top for $i = k$ as well. These are supposed to depict the "just before" and "just after" situations where the social choice switches from $a$ to $b$. To construct $\mathbf{P}^3$, take $\mathbf{P}^2$ and lower $a$ to the very bottom for $i < k$ and to only just above $b$ for $i > k$, leaving it unmoved for $i = k$. By MON, $b$ is still the social choice in $\mathbf{P}^3$. If we constructed $\mathbf{P}^4$ from $\mathbf{P}^1$ in just the same way, the social choice would still be either $b$ or $a$ since $\mathbf{P}^3$ and $\mathbf{P}^4$ differ only in how $k$ ranks $a$ and $b$, which in his ranking are adjacent to each other. But if the social choice in $\mathbf{P}^4$ was $b$, then the social choice in $\mathbf{P}^1$ would have to be $b$, by MON. But the social choice in $\mathbf{P}^1$ is $a$. So the social choice in $\mathbf{P}^4$ must be $a$.

Now consider $\mathbf{P}^5$ constructed from $\mathbf{P}^4$ first lowering $b$ to just above $a$ for all $i < k$, then taking a third alternative $c \neq a$ or $b$ and lowering it just above $b$ for $i < k$, placing it between $a$ and $b$ for $i = k$ and just above $a$ for $i > k$. Since the social choice in $\mathbf{P}^4$ was $a$ and the relative ranking of $a$ against any other alternative was not changed when we constructed $\mathbf{P}^5$, the social choice here must, by MON, also be $a$. Now construct $\mathbf{P}^6$ from $\mathbf{P}^5$ by interchanging the spots of $a$ and $b$ for all individuals $i > k$. By MON, the social choice is either $a$ or $b$. But the social choice cannot be $b$ since $c$ is higher than $b$ in every individual's ranking and MON would imply that $b$ is still the choice if $c$ is raised to the top of everyone's ranking. This would contradict UNA. Thus, the social choice in $\mathbf{P}^6$ must be $a$.

Now, any profile of rankings with $a$ at the top of individual $k$'s ranking can be constructed from $\mathbf{P}^6$ with MON requiring that $a$ is the social choice in each of these arbitrarily constructed profiles. Thus, $a$ is the social choice whenever it is at the top of individual $k$'s ranking. Since $a$ was arbitrary we have shown that for any alternative, there is an individual such that whenever that alternative is at the top of the said individual's ranking, that alternative is the unique social choice. But it would be a contradiction of $f$ being a function if there was any such individual besides $k$ for any other alternative. Thus $k$ is a dictator. $\square$

# Chapter 3

# Differential Calculus

## 3.1 Limits, Continuity and the Derivative

A real sequence, or simply **sequence**, is a collection of numbers $a_1, a_2, a_3, ...$ that can be indexed $1, 2, 3, ....$ The sequence in the previous sentence can be abbreviated $\{a_n\}_{n=1}^{\infty}$, and is said to **converge** if there is a number $a$ such that

$$\forall \epsilon > 0,\ \exists N \ (N \text{ is an } \textbf{integer}\text{, e.g. 1, 2, 3...) such that } \forall n \geq N,\ |a_n - a| < \epsilon.$$

The number $a$, if it exists, is unique and is called the **limit** of the sequence $\{a_n\}_{n=1}^{\infty}$. To see why it is unique, suppose both $a$ and $a'$ were limits of the convergent sequence $\{a_n\}_{n=1}^{\infty}$. Then that would mean that for all $\epsilon > 0$, $\exists N$ and $N'$ such that $|a_n - a| < \epsilon$ for all $n \geq N$ and $|a_n - a'| < \epsilon$ for all $n \geq N'$. Then for $n \geq M = \max\{N, N'\}$,

$$|a - a'| = |(a - a_n) + (a_n - a')| \leq |a_n - a| + |a_n - a'| < \epsilon + \epsilon = 2\epsilon.$$

The first inequality in the centered statement is called the **triangle inequality**:

$$|a + b| \leq |a| + |b|,$$

which is true for all numbers $a$ and $b$. The fact that $|a - b| = |b - a|$ is also put in use here. Since you can pick an $\epsilon$ arbitrarily small, this concludes the argument that $a = a'$. Therefore, the limit of a convergent sequence is unique. We often abbreviate the statement "the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the limit $a$" as

$$\lim_{n \to \infty} a_n = a. \tag{3.1}$$

Exercise 6. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be convergent sequences with limits $a$ and $b$ respectively, and let $c$ be a number. Convince yourselves that the following statements are true: (a) $\lim_{n\to\infty} ca_n = ca$, (b) $\lim_{n\to\infty} a_n + b_n = a + b$, (c) $\lim_{n\to\infty} a_n - b_n = a - b$, (d) $\lim_{n\to\infty} a_n b_n = ab$, and (e) if $\forall n$, $b_n \neq 0$ and $b \neq 0$, then $\lim_{n\to\infty} a_n/b_n = a/b$.

If $\{a_n\}_{n=1}^{\infty}$ is a sequence then let $s_n = \sum_{k=1}^{n} a_k$. This gives rise to the sequence $\{s_n\}_{n=1}^{\infty}$ of **partial sums**. If this sequence converges to a limit $s$, then we say that the **series** $\sum_{n=1}^{\infty} a_n$ converges to the sum $s$. If the sequence of partial sums does not converge, then we say that the series diverges.

Exercise 7. If $|r| < 1$ and $a$ is a number then the series $a + ar + ar^2 + ... + ar^n + ...$ converges. Show that the sum of such a series is given by

$$s = \frac{a}{1-r}$$

*Hint:* Write $s = a + ar + ar^2 + ...$, then $rs = ar + ar^2 + ar^3 + ...$, then subtract $rs$ from $s$, and solve for $s$.

Now find an expression that does not use ellipses ("...") for the sum

$$a + ar + ar^2 + ... + ar^N,$$

where $a$ is a number, $N$ is an integer and $|r| < 1$.

Exercise 8. There are $m$ shop-owners in Mali. A tourist enters Mali and spends \$10 at Mr 1's shop. Mr 1 takes 80% of his profit and spends it at Mr 2's shop; Mr 2 spends 80% of his profit at Mr 3's; ... and so on; Mr $m$ spends 80% of his profit at Mr 1's, and this continues in a loop. For every dollar transaction at a Malian shop, 70 cents is the cost of the goods sold. What Malians do not spend at each others shops, they save at the Timbuktu Bank. What fraction of the \$10 spent by the Mongolian tourist gets saved at the bank? What is the value of total purchases by Malians resulting from the Mongolian tourist spending \$10 at Mr 1's shop?

This exercise is the basis of the much talked about "multiplier" effect of government spending in the macroeconomy. Can you see why?

Now, we want to capture the idea that a function $f : S \to \mathbb{R}$ (where $S$ is an interval, could be $(-\infty, \infty)$) is "**continuous** at $x \in S$" if for all sequences $\{x_i\}_{i=1}^{\infty}$ that converge to $x$, the sequence $\{f(x_i)\}_{i=1}^{\infty}$ converges to $f(x)$. By the definition of convergence, this means that if $\{x_i\}_{i=1}^{n}$ converges to $x$, then

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq N, |f(x_n) - f(x)| < \epsilon.$$

But by definition of $\{x_i\}_{i=1}^{\infty}$ converging to $x$, this is none other than saying

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } y \in S \text{ and } |x - y| < \delta \text{ implies } |f(x) - f(y)| < \epsilon.$$

We say that $f$ is "a continuous function" if it is continuous at every point in $S$.

Exercise 9. Note that the sum and product of two continuous functions are also continuous. Prove that the composition of two continuous functions is continuous.

The function $f : S \to \mathbb{R}$ is "**differentiable** at $x \in S$" if $S$ is an open interval and $\exists a \in \mathbb{R}$ such that

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } y \in S \text{ and } |x - y| < \delta \text{ implies } \left| \tfrac{f(x)-f(y)}{x-y} - a \right| < \epsilon.$$

It is a "differentiable function" if it is differentiable at every point in $S$.

Typically, the number $a$ will depend on $x$, so we may as well write $a(x)$. If $a(x)$ is unique (which it is, and you can verify this), then $\{(x, a(x)) : x \in S\}$ is a function over $S \times \mathbb{R}$ whenever $f$ if a differentiable function. In that case, we define the function $a : S \to \mathbb{R}$, which we call the (first) **derivative** of $f$. This function is denoted $f'$ instead of $a$. The derivative of $f'$, if it exists, is denoted $f''$, and is called the second derivative of $f$, and so on.

It is also important to know that we can define differentiability another way. If for all sequences $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = y$ and $x_n \neq y$ for all $n$ we have

$$\lim_{n\to\infty} \frac{f(x_n) - f(y)}{x_n - y} = f'(y) \tag{3.2}$$

then we say that $f$ is differentiable at $y$, where its derivative is $f'(y)$. Often, we abuse notation to write this statement as

$$\lim_{x\to y} \frac{f(x) - f(y)}{x - y} = f'(y). \tag{3.3}$$

In fact, I'll call this limit the "abusive limit," to be read as "limit as $x$ reaches $y$..."

Exercise 10. Convince yourself that the two definitions of differentiability are equivalent. That is, derive the second from the first, and the first from the second. (*Hint:* Write down the $\epsilon, \delta$ definition of the limit in (3.3).) Also convince yourself that if a function is differentiable, then it is continuous. (*Hint:* Multiply the last expression in the $\epsilon, \delta$ definition of differentiability by $|x - y|$.)

## 3.2 Properties of the Derivative

If $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are differentiable at $y \in S$ and $c$ is a number, then $cf$, $f + g$, $f - g$ and $fg$ are all differentiable at $y$. Here, $cf$ is the function defined by multiplying $f(x)$ by $c$ at all $x \in S$, $f + g$ is the function defined by adding $f(x)$ to $g(x)$ at all $x \in S$. Instead of adding, we subtract to define $f - g$ and multiply to define $fg$. If $g(x) \neq 0$ for all $x \in S$, then $f/g$, which is the function defined by dividing $f(x)$ by $g(x)$, is also differentiable. In fact, it is easy to show that

$$\begin{gathered}[cf]'(y) = cf'(y) \\ [f+g]'(y) = f'(y) + g'(y) \text{ and} \\ [f-g]'(y) = f'(y) - g'(y).\end{gathered}$$

Now notice that

$$\frac{f(x)g(x) - f(y)g(y)}{x - y} = f(x)\frac{g(x) - g(y)}{x - y} + \frac{f(x) - f(y)}{x - y}g(y), \tag{3.4}$$

which is the main step in the proof of the **product rule**:

$$[fg]'(y) = f(y)g'(y) + f'(y)g(y). \tag{3.5}$$

In fact, all that one has to do is take the abusive limit on both sides of (3.4) and then use the fact that differentiable functions are continuous. Similarly, notice that

$$\frac{1/g(x) - 1/g(y)}{x - y} = -\frac{1}{g(x)g(y)}\frac{g(x) - g(y)}{x - y} \tag{3.6}$$

helps prove that $\left[\frac{1}{g}\right]'(y) = -\frac{g'(y)}{(g(y))^2}$. Take the abusive limit on both sides of (3.6) and combine this result with the product rule to get the beloved **quotient rule**:

$$\left[\frac{f}{g}\right]'(y) = \frac{g(y)f'(y) - g'(y)f(y)}{(g(y))^2}. \tag{3.7}$$

Dwell on why it is we are allowed to take the abusive limit on both sides.

Exercise 11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = ax^n$ where $a \in \mathbb{R}$ and $n \in \mathbb{R}$. Find its first, second and third derivatives using the limits definition of the derivative.

Finally, let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ and be two functions and assume that the composition $f(g)$ is defined on an open interval, $S$. Suppose that $g$ is differentiable at $x \in S$ and that $f$ is differentiable at $g(x)$. Then $f(g)$ is differentiable at $x$ and

$$[f(g)]'(x) = f'(g(x))g'(x).$$

This is the **chain rule**. Why is it true? Since $f$ is differentiable at $g(x)$, then there is an error term $r(y)$, implicitly defined for any $y \in S$ by

$$f(g(y)) - f(g(x)) = [f'(g(x)) + r(y)][g(y) - g(x)]; \tag{3.8}$$

this error term has limit 0 as $g(y) \to g(x)$. But by the definition of continuity, it has limit 0 as $y \to x$ as well. Now divide both sides of (3.8) by $y - x$ and take the abusive limit on both sides. On the left you will get $[f(g)]'(x)$. On the right, the $r(y)$ term will vanish, and voila, you have what you need.

## 3.3 Some Matrix Algebra

An $n \times m$ **matrix** $A$ is an array of numbers with $n$ rows and $m$ columns. $A_i$ denotes the $i$th row and is itself a $1 \times m$ matrix. $A^j$ denotes the $j$th column and is an $n \times 1$ matrix. Any $n \times 1$ matrix is also called a **vector** of size $n$. $\mathbb{R}^n$ denotes the set of all vectors of size $n$ and $\mathbb{R}^{n \times m}$ denotes the set of all matrices that are $n \times m$.

Often we write $[a_{ij}]_{i=1,...,n}^{j=1,...,m}$ (or simply $[a_{ij}]$ when it is clear what $n$ and $m$ are) to denote the matrix $A$; and $[a_i]_{i=1,...,n}$ (or simply $[a_i]$) to denote the $n \times 1$ matrix (i.e. vector) $a$. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $n \times m$ matrices then $A + B$ is defined as the $n \times m$ matrix $[a_{ij} + b_{ij}]$. The **transpose** of the matrix $A = [a_{ij}]$ is the matrix $A' = [a_{ji}]$. The **dot product** of two vectors $a = [a_i]$ and $b = [b_i]$ is defined as the sum $\sum_{i=1,...,n} a_i b_i$ and is denoted $a'b$ or $b'a$ or $a \cdot b$. The **length** of a vector $a$ of size $n$ is $(a \cdot a)^{0.5}$ and is denoted $||a||$. If $a_i = 0$ for all $i = 1, ..., n$ then the vector $a$ is called the zero vector of size $n$ and is denoted $0_n$ or just 0 when it is clear what $n$ should be. If, on the other hand, $a_i = 1$ for all $i = 1, ..., n$ then $a$ is called the one-vector of size $n$ and is denoted $1_n$.

The product $AB$ of an $n \times m$ matrix,$A$ and an $l \times k$ matrix $B$ is not defined unless $l = m$, in which case it is the $n \times k$ matrix $[(A_i \cdot B^j)_{ij}]$. If $c$ is a number then $c[a_{ij}] = [ca_{ij}]$. A **square matrix** is an $n \times n$ matrix, where $n$ is called the **order** of the matrix. A **symmetric matrix** is one that is equal to its transpose. A **lower triangular matrix** of order $n$ is a square matrix of order $n$ where $a_{ij} = 0$ for all $j > i$. An **upper triangular matrix** of order $n$ is a square matrix of order $n$ whose transpose is a lower triangular matrix of order $n$. A **diagonal matrix** of order $n$ is a lower triangular matrix of order $n$ that is also an upper triangular matrix of order $n$. The **identity** matrix of order $n$ is a diagonal matrix of order $n$ where $a_{ij} = 1$ for all $i = j$. It is denoted $I_n$ or just $I$ when it is clear what $n$ should be.

Exercise 12. Verify that (i) $A + B = B + A$, (ii) $(A + B) + C = A + (B + C)$, (iii) $(AB)C = A(BC)$, (iv) $A(B + C) = AB + AC$, (v) $(A + B)' = A' + B'$, (vi) $(AB)' = B'A'$, and (vii) $AI = A$ and $BI = B$ for any $n \times m$ matrices $A$ and $B$ (note that $I$ does not denote the same matrix in the two equations: the two $I$s differ by their order so that the products are defined), and (viii) $I = I^2 = I^3 = ....$

## 3.4 The Derivative in Multiple Dimensions

Let $f : S \to \mathbb{R}$ be a function and $S \subset \mathbb{R}^n$. Suppose that for any $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in S$, $||x - y|| < \delta$ implies that $|f(x) - f(y)| < \epsilon$ then $f$ is said to be **continuous** at $x$. If the statement is true for every $x \in S$ then $f$ is said to be a continuous function. Similarly, let $S_1$, $S_2$, ..., $S_n$ be open intervals; we can allow some or all of them to be $(-\infty, \infty)$. Now define $S \subset \mathbb{R}^n$ to be the set of all vectors such that the first entry is an element of $S_1$, the second an element of $S_2$, and so on:

$$S = \{b \in \mathbb{R}^n | b_i \in S_i \text{ for all } i = 1, ..., n\}.$$

We call $S$ an **open box**. A function $f : S \to \mathbb{R}$ is said to be **differentiable** at $x \in S$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that $y \in S$ and $||x - y|| < \delta$ implies

$$|f(x) - f(y) - a(x) \cdot (x - y)| < \epsilon ||x - y||,$$

for some vector $a(x)$ of size $n$. Akin to the one-dimensional case, the vector $a(x)$ is called the **derivative** of $f$ at $x \in S$ and is unique for each $x$ whenever it exists. If $f$ is differentiable at all points in $S$ then it is a differentiable function, and we can define the derivative of $f$ to be the function $\nabla f : S \to \mathbb{R}^n$ such that $\nabla f(x) = a(x)$.

It is not hard to show that if both $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $x \in \mathbb{R}^n$ then so is $c_1 f + c_2 g$, where $c_1$ and $c_2$ are numbers. Fortunately,

$$\nabla(c_1 f + c_2 g)(x) = c_1 \nabla f(x) + c_2 \nabla g(x).$$

In fact, the chain rule also applies: if $h : \mathbb{R} \to \mathbb{R}$, then

$$\nabla[h(f)](x) = h'(f(x)) \nabla f(x). \tag{3.9}$$

Let $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$ is an open box. Let $e_j \in \mathbb{R}^n$ be the vector with 0s in every entry except the $j$th, where the entry there is a 1. Then the $j$th **partial derivative** of $f$ at the point $x \in S$ exists if for all $\epsilon > 0$ there is a $\delta > 0$ such that for any number $t$ for which $x + te_j \in S$, $t < \delta$ implies

$$\left| \frac{f(x) - f(x + te_j)}{t} - a \right| < \epsilon \tag{3.10}$$

The number $a$, if it exists, is unique for each $x$ and is the $j$th partial derivative. It defines the partial derivative function, $\frac{\partial f}{\partial x_j} : S \to \mathbb{R}$, a function defined by $\frac{\partial f(x)}{\partial x_j} = a$.

Similarly if we replace every occurrence of $e_j$ in the definition of partial derivative by $h$, where $h \in \mathbb{R}^n$ and restrict $t$ to be positive, then we have the definition of "the **directional derivative** of $f$ at $x$ in the direction $h$."

Now, the following are some **true facts**. Let $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}^n$ is an open box. Then (i) if $f$ is differentiable then it is continuous; (ii) if $f$ is differentiable at $x$ then $\partial f(x)/\partial x_j$ exist for all $j = 1, ..., n$ and $\nabla f(x) = [\partial f(x)/\partial x_1, ..., \partial f(x)/\partial x_n]'$; (iii) If $\partial f(x)/\partial x_j$ exist for all $j = 1, ..., n$ and are all continuous at $x$ then $\nabla f(x)$ exists and is given by $\nabla f(x) = [\partial f(x)/\partial x_1, ..., \partial f(x)/\partial x_n]'$; (iv) If $f$ is differentiable at $x$ then the directional derivative of $f$ exists for any $h$ and is equal to $\nabla f(x) \cdot h$.

Exercise 13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0, 0) = 0$ and for $(x, y) \neq 0$,

$$f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}.$$

Is $f$ differentiable at $(0, 0)$?

Let $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}^n$ is an open box. Suppose $f$ is differentiable at $x \in S$, and suppose that each partial derivative function of $f$ is differentiable at $x$. Denote the $j$th partial of $\partial f(x)/\partial x_i$ (also called the "$(i, j)$-cross partial") by $\partial^2 f(x)/\partial x_j x_i$ if $j \neq i$ and $\partial^2 f(x)/\partial x_i^2$ if $j = i$. Then the **Hessian** of $f$ at $x$ is the matrix

$$Hf(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & ... & ... & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ ... & \frac{\partial^2 f(x)}{\partial x_2^2} & ... & ... \\ ... & ... & ... & ... \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & ... & ... & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix} \tag{3.11}$$

If every partial derivative of $f$ is a continuous function, then we say that $f$ is **continuously differentiable** or $C^1$. If every $(i, j)$-cross partial of $f$ is a continuous function then we say that $f$ is $C^2$, and when $f$ is $C^2$, it turns out that the Hessian is a symmetric matrix with

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \tag{3.12}$$

for all $i = 1, ..., n$ and $j = 1, ..., n$. This is called **Young's theorem**, and you will demonstrate it through an example momentarily.

Let $f : S \to \mathbb{R}$ be a function, where $S \subset \mathbb{R}^n$ is an open box. Now let us treat $x_j$, $j \neq i$ as constants and define the function $g : S_i \to \mathbb{R}$ to be

$$g(x_i) \equiv f(x_i; x_1, ..., x_{i-1}, x_{i+1}, ..., x_n),$$

where the semicolon simply divides the free and fixed variables. Then you will be relieved to know that

$$\frac{\partial f}{\partial x_i} \equiv \frac{dg}{dx_i}. \tag{3.13}$$

So go ahead and use the chain rule, product rule, quotient rule etc. that we described in the one variable case to calculate partial derivatives.

Exercise 14. Provide arguments for (3.12) and (3.13)

Exercise 15. Let $f(x_1, x_2) = \ln[x_1(x_2)^2] + x_1 x_2$ and assume that $f$ is $C^2$. Demonstrate Young's theorem.

# Chapter 4

# Real Analysis

*Most of the material in this chapter is straight out of Richard Beals' *Analysis: An Introduction*, Chapter 8. This is a beautiful book and it was the first book I used to learn analysis.

Let $\{a_{1k}\}$, $\{a_{2k}\}$, ... and $\{a_{nk}\}$ be sequences that converge to $a_1, a_2, ...$ and $a_n$ respectively. Then the sequence of vectors, $\{[a_{1k}, a_{2k}, ..., a_{nk}]'\}_{k=1}^{\infty}$ converges to the vector $[a_1, a_2, ..., a_n]'$. This is the **convergence** of vectors. A **closed set** is a set of vectors $X \subset \mathbb{R}^n$ where the limit of every convergent sequence $\{x_k\} \subset X$ also lies in $X$. If for all $x \in X \subset \mathbb{R}^n$, there exists an open box $S \subset X$ such that $x \in S$, then $X$ is said to be an **open set**. A **bounded set** is a set $X$ for which there is an open box $S = S_1 \times S_2 \times ... \times S_n$ such that $X \subset S$ and $S_i = (-z, z)$ for all $i$, where $z > 0$. A subsequence $\{x_{m(k)}\}$ of a sequence $\{x_k\}$ is a sequence of some (or all) of the elements of $\{x_k\}$ appearing in the order in which they appear in $\{x_k\}$. A **compact set** is a set is a set $X \subset \mathbb{R}^n$ such that every sequence in $X$ has a convergent subsequence whose limit is in $X$. A **convex set** is a set $X \subset \mathbb{R}^n$ where if $x \in X$ and $y \in X$ then $\alpha x + (1-\alpha)y \in X$ for all $\alpha \in (0, 1)$. The **supremum** of a set $X \subset \mathbb{R}$ is the lowest number $\sup X$ such that every number greater than $\sup X$ is greater than every number in $X$. This is also called the *lowest upper bound* of $X$ for obvious reasons. The **infimum** is the symmetric concept that is the *greatest lower bound.*

Exercise 16. Show that if $A$ and $B$ are both convex sets, their intersection is convex but not necessarily their union.

## 4.1 Intermediate Value Theorem

Let $S = [a, b]$ and $f : S \to \mathbb{R}$ be a continuous function. Let $f(x) = p$ and $f(y) = q$ with $q > p$. The **intermediate value theorem** says that for any $c \in (p, q)$,

$$\exists\ z \in (\min\{x, y\}, \max\{x, y\}) \text{ such that } f(z) = c.$$

To prove this, define $g(x) = f(x) - c$. Construct a sequence of intervals $\{S_i\}_{i=0}^{\infty}$ beginning with $S_0 = [a, b]$. If $g(x) = 0$ at the midpoint of this interval, then we're done. If it is not, then $g$ changes sign between the endpoints on either the right half or the left. Pick the half that it changes sign on and call the interval $S_1$. If it is 0 at the midpoint, then again we're done. If not, again pick the half on which it changes sign and call it $S_2$, and so on. Either we reach a point were $g(x)$ takes a value of 0, or we obtain an infinite sequence of intervals. In the latter case, the sequence of left endpoints and the sequence of right endpoints both converge to the same limit, $p$. By *continuity* and the change of sign condition, $g(p) = 0$, and we're done.

The **generalized intermediate value theorem** says the following. Let $X \subset \mathbb{R}^n$ be a convex set and let $f : X \to \mathbb{R}$ be a continuous function. Let $x \in X$ and $y \in X$ be points such that $f(x) < f(y)$. Then for any $c$ such that $f(a) < c < f(b)$ there is an $\alpha \in (0, 1)$ such that $f((1 - \alpha)x + \alpha y) = c$.

The proof of this is simple. Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(\beta) = f((1-\beta)x+\beta y)$ for $\beta \in [0, 1]$. Since $f$ is continuous, $g$ is continuous and $g(0) = f(x)$, $g(1) = f(y)$, and $g(0) < c < g(1)$. By the intermediate value theorem there is $\alpha \in (0, 1)$ such that $g(\alpha) = c$. But $g(\alpha) = f((1 - \alpha)x + \alpha y)$, and this completes the argument.

## 4.2 Heine-Borel Theorem

The Heine-Borel Theorem says that a set is compact if and only if it is closed and bounded. (Note that this is only true because we are working with $X \subset \mathbb{R}^n$.)

First let us show that a compact set, $X$, is closed and bounded. To show that it is closed, take any convergent sequence $\{x_k\} \subset X$. Since $X$ is compact, this sequence has a convergent subsequence $\{x_{m(k)}\}$ whose limit is in $X$. By the uniqueness of the limit, this is also the limit of $\{x_k\}$. Hence $X$ is closed.

If $X$ is not bounded, then for each $n$, there is $x_n \in X$ such that $||x_n|| > n$. You will argue in Exercise 17 that the sequence $\{x_n\}$ does not have a convergent subsequence, which contradicts the fact that $X$ is compact. Therefore, $X$ must be bounded.

Exercise 17. Make the argument that the sequence $\{x_n\}$ defined above does not have a convergent subsequence. (*Hint:* Suppose there was a convergent subsequence with limit $y$ and note that $||x_m - y|| \geq ||x_m|| - ||y||$ by the triangle inequality.)

Now we show that a closed and bounded set, $X$, is compact. By boundedness, there is a number $z > 0$ such that $|x_i| \leq z$ for all $x \in X$ and all $i$, where $x_i$ is the $i$th component of the vector $x$. Then in Exercise 18 you show that $Z \equiv [-z, z] \times ... \times [-z, z]$ is compact. Obviously, $X \subset Z$. If we can show that a closed subset of a compact set is also compact, then we are done.

To do this last step, take any sequence in $X$. Since $X \subset Z$, this is also a sequence in $Z$, which is a compact set. So it must have a convergent subsequence with limit in $Z$. But since $X$ is closed, and this subsequence lies in $X$, the limit must also lie in $X$. Therefore, $X$ is compact.

Exercise 18. First argue that $[-z, z]$ is compact; then it is an obvious step to show that $Z$ is compact. (*Hint:* The idea is to divide $[-z, z]$ into two equal halves. Since a sequence has an infinite number of elements, there must be an infinite number of elements in one half or the other, or both. Pick any that has an infinite number of elements. Divide that half into two halves again, and continue the process. It should be fairly obvious now.)

## 4.3 Weierstrass Theorem

Let $S \subset \mathbb{R}^n$ be a compact set and $f : S \to \mathbb{R}$ be a continuous function on $S$. Then the Weierstrass Theorem says that $f$ attains a minimum and maximum on $S$.

To see why, define

$$f(S) = \{y \in \mathbb{R} : \exists x \in S \text{ such that } f(x) = y\}.$$

The first step is to show that $f(S)$ is compact. Let $\{y_k\} \subset f(S)$ be a sequence. For each $k$ pick $x_k \in S$ such that $f(x_k) = y_k$ (which you can do by construction). This gives us a sequence $\{x_k\} \subset S$. Since $S$ is compact you can pick an infinite subsequence $\{x_{m(k)}\} \subset \{x_k\}$ that converges to some $x \in S$. Let $y = f(x)$ and $y_{m(k)} = f(x_{m(k)})$. Since $\{x_{m(k)}\}$ converges to $x$ and $f$ is continuous, the infinite sequence $\{f(x_{m(k)})\}$ converges to $f(x)$. But $f(x) \in f(S)$ so $f(S)$ is compact.

The second step is to show that because $f(S)$ is compact $\sup f(S) \in f(S)$ and $\inf f(S) \in f(S)$, and these are the maximum and minimum we need. First of all, boundedness (from Heine-Borel) tells us that $\sup f(S) < \infty$ and $\inf f(S) > -\infty$. Now, let $N_k$ be the interval $(\sup f(S) - 1/k, \sup f(S)]$ where $k = 1, 2, ....$ Let $f(S)_k = f(S) \cap N_k$. Then $f(S)_k$ is not empty for each $k$, otherwise we would have an upper bound strictly smaller than $\sup f(S)$. Now for each $f(S)_k$ pick any $y_k \in f(S)_k$. The sequence $\{y_k\}$ must converge to $\sup f(S)$. Since $f(S)$ is closed (again, Heine-Borel) $\sup f(S) \in f(S)$. The argument for inf is almost identical.

## 4.4 Mean Value Theorem

If $f$ is continuous on the interval $[a, b]$, differentiable everywhere on $(a, b)$, and $f(a) = f(b)$, then **Rolle's Theorem** says that $\exists\, c \in (a, b)$ such that $f'(c) = 0$.

The proof goes like this. If $f$ is constant then $f' = 0$ everywhere, and we'd be done. So for challenge's sake, let $f$ not be constant. By the Weierstrass Theorem, $f$ attains maximum and minimum values on $[a, b]$. Now since $f$ is not constant, either the maximum is greater than $f(a)$ or the minimum is less than $f(a)$ (or both). If the maximum value is greater than $f(a)$ then any point $x$ at which it is attained lies in $(a, b)$ (it can't be $b$ because $f(a) = f(b)$ by assumption). The numerator of

$$\frac{f(y) - f(x)}{y - x} \tag{4.1}$$

where $y \neq x$ is an element in $[a, b]$ is always non-positive and the denominator can have either sign depending on which side of $x$ the $y$ is on. Now take the abusive limit on the centered expression above. Due to different signs on different sides, the limit cannot be positive or negative. But we know it exists by the assumption that $f$ is differentiable on $(a,b)$. So it must be 0. The argument is similar if $f$'s minimum is less than $f(a)$.

Now the **mean value theorem** says that if $f$ is continuous on the interval $[a,b]$ and differentiable everywhere on $(a,b)$, then $\exists\ c \in (a,b)$ such that

$$(b-a)f'(c) = f(b) - f(a).$$

To prove this, note that we have the same assumptions as in Rolle's theorem, except we drop the assumption that $f(a) = f(b)$. We need to show that there is a point $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c). \tag{4.2}$$

But this is easy. Let $g$ be a function on $[a,b]$ defined by

$$g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) \tag{4.3}$$

and notice that $g(a) = g(b) = f(a)$. By Rolle's theorem, there is a point $c \in (a,b)$ such that

$$0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}, \tag{4.4}$$

and we are done.

Exercise 19. Suppose that $f : S \to \mathbb{R}$ is a differentiable function, and so is $f' : S \to \mathbb{R}$. ($f$ is said to be "twice differentiable" if its derivative is a differentiable function.) Suppose also that $f''(x) < 0$ for all $x \in S$. Show that $f$ must be strictly concave. If instead $f$ is convex and twice differentiable, show that $f''(x) \geq 0$ for all $x \in S$.

Exercise 20. Let $X \in \mathbb{R}^n$ be a convex and open set and let $f : X \to \mathbb{R}$ be a differentiable function. Then the **generalized mean value theorem** states that for any $x \in X$ and $y \in X$ there is an $\alpha \in (0,1)$ such that

$$f(x) - f(y) = \nabla f((1-\alpha)x + \alpha y)(x-y)$$

Prove this by defining $g$ exactly the same as in the proof of the generalized intermediate value theorem. *Hint:* Notice that $g'(\alpha) = \nabla f((1-\alpha)x + \alpha y) \cdot (b-a)$.

Suppose that there are continuous functions $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$, where $S = [a, b]$. Suppose that these functions are differentiable at every point in $(a, b)$ and that $g'(x) \neq 0$ for all $x \in (a, b)$. With these assumptions, the mean value theorem implies that $g(b) - g(a) \neq 0$. Now define $h : [a, b] \to \mathbb{R}$ by

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]. \tag{4.5}$$

I'll give you \$10 if $h(a)$ is not equal to $h(b)$. Now again, by the mean value theorem, there is $c \in (a, b)$ such that

$$h'(c) = f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] = 0. \tag{4.6}$$

Since in a previous paragraph I argued that $g(b) - g(a) \neq 0$, you can divide both sides of this by $[g(b) - g(a)]g'(c)$, to prove a much celebrated result: that with the above assumptions, there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \tag{4.7}$$

This is the **glorious mean value theorem**, which we use to provide an argument for L'Hopital's rule in the next section.

## 4.5 L'Hopital's Rule

Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be differentiable everywhere on $(a, b)$ and that $g(x) \neq 0$ and $g'(x) \neq 0$ for $x \in (a, b)$. Then **L'Hopital's Rule** says it is not so unfortunate that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, for it is the case that:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{4.8}$$

so long as the limit on the right hand side exists. (*Note*: This statement of L'Hopital's rule is not exact. To be exact, you should understand that all of the abusive limits in the statement of L'Hopital's rule refer to only the subset of all sequences converging to $a$ where every element of the sequence is greater than $a$. This is also called "approaching $a$ from the right.")

Let's try to derive (4.8) from the assumptions. To make sure $f$ and $g$ are continuous at $a$, we need $f(a) = g(a) = 0$. This does not have to be the case, but we can just redefine $f$ and $g$ to be so if it isn't. Call the limit on the right side of (4.8), $L$. By the properties of limits, for any $\epsilon > 0$, we can find an interval $T = (a, a+\delta)$ such that

$$\left| \frac{f'(c)}{g'(c)} - L \right| < \epsilon \tag{4.9}$$

for $c \in T$. Invoking the glorious mean value theorem, we can then argue

$$\left| \frac{f(x) - f(a)}{g(x) - g(a)} - L \right| < \epsilon \tag{4.10}$$

for all $x \in T$. But what did we say $f(a)$ and $g(a)$ were? That's right. Once you plug these values in, you've derived (4.8) from the assumptions of L'Hopital's rule.

Even though we didn't prove it, L'Hopital's rule would still be true if $a = -\infty$ or $a = \infty$, and/or if $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$ instead of 0. Furthermore, $a$ does not have to be approached from the right (which is not possible in the case of $a = \infty$ anyway).

Exercise 21. Use L'Hopital's rule to calculate $\lim_{x \to 0^+} x^x$, where the superscripted $+$ means that you are "approaching 0 from the right." (*Hint:* Do some easier L'Hospital's rule problems from the textbook first, remember the properties of log and $e$, and then think about continuity.)

## 4.6 Implicit Function Theorem

Given $n \geq 1$, let a typical point of the set $\mathbb{R}^{n+1}$ be denoted by $(x, y)$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Let $S \subset \mathbb{R}^{n+1}$ be an open box, and let $F : S \to \mathbb{R}$ be a differentiable function with continuous partial derivatives. Let $(x^*, y^*)$ be a point in $S$ such that

$$\frac{\partial F(x^*, y^*)}{\partial y} \neq 0 \tag{4.11}$$

and let $F(x^*, y^*) = 0$. Then there is an open box $B \subset \mathbb{R}^n$ such that $x^* \in B$, and a differentiable function $g : B \to \mathbb{R}$ whose partial derivatives are continuous, such that

$g(x^*) = y^*$, and $F(x, g(x)) \equiv 0$ for all $x \in B$. The derivative of $g$ at any $x \in B$ is:

$$\frac{\partial g}{\partial x_j} = -\frac{\partial F/\partial x_j}{\partial F/\partial y}. \tag{4.12}$$

We only sketch the proof of this very important theorem. Due to (4.11) we can assume without loss of generality that $\partial F(x^*, y^*)/\partial y > 0$. By continuity of $\partial F/\partial y$, there is a small open box $A \subset \mathbb{R}^{n+1}$ containing $(x^*, y^*)$ such that

$$\frac{\partial F(x, y)}{\partial y} > 0 \tag{4.13}$$

for all $(x, y) \in A$. Thus $F(x^*, \cdot)$ is increasing in $y$ in a neighborhood of $y^*$, which means we can find $y_1, y_2$ satisfying $F(x^*, y_1) < 0 < F(x^*, y_2)$ and $y_1 < y^* < y^2$. Again by the continuity of $F$ we can find an open box $B \in \mathbb{R}^n$ containing $x^*$ such that $B \times [y_1, y_2] \subset A$ and $F(x, y_1) < 0 < F(x, y_2)$ for all $x \in B$. By the intermediate value theorem, for each $x \in B$, there is $y \in (y_1, y_2)$ such that $F(x, y) = 0$. Uniqueness of this $y$ is guaranteed by (4.13). This uniqueness allows us to define the continuous function $g(x) = y$ having the properties described in the theorem, except that we have yet to show (4.12) and the fact that that the partial derivatives are continuous.

To that end, fix $x \in B$, and let $y = g(x)$. Then by definition of the derivative,

$$F(x + se_j, y + t) - F(x, y) = s\frac{\partial F(x, y)}{\partial x_j} + t\frac{\partial F(x, y)}{\partial y} + \epsilon\sqrt{s^2 + t^2}$$

where $\epsilon \to 0$ as $\sqrt{s^2 + t^2} \to 0$. Now pick $s$ small enough so that $x + se_j \in B$ and set $t = g(x + se_j) - g(x)$ to get

$$t\frac{\partial F(x, y)}{\partial y} = -s\frac{\partial F(x, y)}{\partial x_j} - \epsilon\sqrt{s^2 + t^2},$$

which rearranges to

$$\frac{g(x + se_j) - g(x)}{s} = -\frac{\partial F(x, y)/\partial x_j}{\partial F(x, y)/\partial y} - \frac{\epsilon}{\partial F(x, y)/\partial y}\frac{\sqrt{s^2 + t^2}}{s} \tag{4.14}$$

Keeping in mind that our choice of $t \to 0$ as $s \to 0$, take the limit as $s \to 0$ on both sides. The only question is: Can we kill the right hand term by doing this? We hand-waive here and say yes, roughly because $\sqrt{s^2 + t^2}/s$ is bounded and therefore cannot move fast enough to overwhelm the convergence of $\epsilon$ to 0. So this gives us the partial derivative of $g$ with respect to $x_j$. We know that these partials are continuous since $\partial F(x, y)/\partial y$ is non-vanishing, and the partials of $F$ are continuous. Thus $g$ is differentiable by the third true fact.

Exercise 22. Use the implicit function theorem to find $dy/dx$ along the circle, $x^2 + y^2 = 1$. Where does $dy/dx$ not exist?

## 4.7 Inverse Function Theorem

Let $f$ be differentiable at every point on an open interval, $S$, and let $f'(x) \neq 0$ for all $x \in S$. Assume that $f$ is invertible and let its inverse be $g$. Then $g$ is differentiable at $f(x)$ and

$$g'(f(x)) = \frac{1}{f'(x)}. \tag{4.15}$$

To see why this is true, let $f(x) = y$ and $f(x') = y' \neq y$ for some $x \in S$ and $x' \in S$. In order to make this assumption you must understand that there is an open interval that contains $x$, and $y' \neq y$ for any $x'$ that we choose in this interval. This is because $f'(x) \neq 0$. Now since $g$ and $f$ are inverses of each other,

$$\frac{g(y') - g(y)}{y' - y} = \frac{x' - x}{f(x') - f(x)}. \tag{4.16}$$

Now recall that the inverse of a continuous function is continuous, take the abusive limits on both sides (in this case $\lim_{x' \to x}$), then invoke the two useful facts to arrive at (4.15). This is the **inverse function theorem**. We can use it to find the derivative of $\ln x$. But first we define this function.

Exercise 23. Show that a continuous strictly increasing function $f$ defined on an interval $[a, b]$ has a continuous strictly increasing inverse.

Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n \neq 0$ for all $n$. If the limit

$$L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$$

exists, then the series converges if $L < 1$, diverges if $L > 1$, and no conclusions can be made if $L = 1$. I am not going to prove this fact, but I am going to use it to show that the series

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \tag{4.17}$$

converges for every finite value of $x$. To see this, note that for this series,

$$L = \lim_{n\to\infty} \frac{x^n/n!}{x^{n-1}/(n-1)!} = \lim_{n\to\infty} \frac{x}{n} = 0 < 1, \tag{4.18}$$

so that the series must converge. The series has what's called an infinite radius of convergence, i.e. it converges for any finite value of $x$. There is a theorem on power series that tells us that we can find the derivative of such a function, $f(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$, by differentiating each of its terms. Therefore,

$$f'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \tag{4.19}$$

That's not strange. In fact, $f(x) = f'(x) = f''(x) = ...$ for this function, and we have a special name for it. We call such a function $e^x$. This function is strictly increasing and bijective if its range is defined to be only the positive numbers. (In fact, you can draw a graph of it to verify this.) Recall that bijective functions are invertible. The associated inverse function is called $\ln x$.

Now let the functions in the inverse functions theorem be $f(x) = \ln x$ and $g(x) = e^x$. There is no way to get $e^x = \infty$ from a finite $x$ so the slope of $e^x$ is nowhere infinite. This means that the slope of $f(x)$ is nowhere 0. Check. Now $g'(f(x)) = g(f(x)) = x$ in this special case; so by the inverse function theorem above, $x = \frac{1}{f'(x)}$, i.e. the derivative of $\ln x$ is $\frac{1}{x}$.

Exercise 24. Is the inverse of $f(x) = x^3$ differentiable everywhere?

Exercise 25. Find the derivative of $f(x) = \frac{\ln(3x^2+2)}{e^{6x}+1}$.

## 4.8 Application: The Swan-Solow Model

To analyze production in an economy, assume that there are only two inputs: capital, $K(t)$ and labor, $L(t)$ where $t$ denotes time. The amount of output produced is a function of these inputs and at any time $t$, it is given by

$$Y(t) = F(K(t), L(t)), \tag{4.20}$$

$F$ being the production function. Some of the output is consumed, $C(t)$ and the remainder, $I(t)$ is invested to generate capital for future production. We assume that the economy is "closed" (there is no interaction with other foreign economies) and output and capital input are the same single type of goods. Let $s$ be the fraction of output saved at any time – the saving rate – which we assume is constant. All savings are invested. Capital is not permanent. It depreciates at the rate of $\delta > 0$ so that the rate of increase of capital with time is given by

$$\frac{dK(t)}{dt} = I(t) - \delta K(t) = sF(K(t), L(t)) - \delta K(t), \tag{4.21}$$

which is called the *flow equation*. Population grows over time; that's why $L$ depends on $t$. Assume that the labor force grows at a constant rate

$$\frac{1}{L(t)} \frac{dL(t)}{dt} = n \geq 0$$

and each worker has equal productivity for a given amount of capital. Assume that at the beginning of time, $t = 0$, there is only 1 worker in the economy.

Exercise 26. Find an expression for the number of workers at any time, $t$. (*Hint:* Try to find it first for $n = 1$.)

Momentarily forget about the dependence on time of each of the variables, $K$, $L$, $Y$ etc. Assume that the production function, $F$, is **neoclassical**, which means:

1. for all $K > 0$ and $L > 0$, $\frac{\partial F}{\partial K} > 0$, $\frac{\partial^2 F}{\partial K^2} < 0$, $\frac{\partial F}{\partial L} > 0$, and $\frac{\partial^2 F}{\partial L^2} < 0$,
2. $F(\lambda K, \lambda L) = \lambda F(K, L)$ for all $\lambda > 0$, and
3. $\lim_{K \to 0} \frac{\partial F}{\partial K} = \lim_{L \to 0} \frac{\partial F}{\partial L} = \infty$ and $\lim_{K \to \infty} \frac{\partial F}{\partial K} = \lim_{L \to \infty} \frac{\partial F}{\partial L} = 0$

Notice that the second condition lets us write

$$Y = F(K, L) = L \cdot F(K/L, 1) = Lf(k), \text{ or } y = f(k)$$

where $y = Y/L$ is the output per capita, and $k = K/L$ the capital-labor ratio.

Exercise 27. Find expressions for $\partial Y/\partial K$ and $\partial Y/\partial L$ in terms of $f$, $f'$ and $k$.

Exercise 28. Argue that $f(0) = 0$.

Exercise 29. Use the information provided so far to show that

$$\frac{dk}{dt} = sf(k) - (n + \delta)k. \tag{4.22}$$

A **steady state** is defined as a point where the growth rate of per capita captial $\frac{dk}{dt}/k$ is constant. We can show that at any steady state $\frac{dk}{dt} = 0$, i.e. per capita capital does not grow. Divide both sides of (4.22) by $k$ to get

$$\frac{dk/dt}{k} = sf(k)/k - (n + \delta). \tag{4.23}$$

At a steady state, the left hand side is constant. That means that $f(k)/k$ should be constant on the right hand side. This implies

$$\frac{d(f(k)/k)}{dt} = \frac{dk/dt}{k}\left(f'(k) - \frac{f(k)}{k}\right) = 0. \tag{4.24}$$

The term in the parenthesis between the two equal signs is negative, so that if $k$ is finite then $dk/dt = 0$.

Exercise 30. Why is the term in parenthesis on the right hand side of (4.24) negative?

Plugging this result into (4.23), it must be the case that $sf(k^*) = (n + \delta)k^*$, where $k^*$ is the steady state level of capital. (Since $dk/dt = 0$, capital doesn't grow in the steady state so it is a constant amount, $k^*$.) Since $k$ is constant in the steady state, so is $y$, and since the saving rate is constant, the level of consumption is also constant.

Exercise 31. At what rate do $K$ and $Y$ grow in the steady state?

The growth rate of $k$, $\gamma_k$ is given by (4.23). The first term of the expression has derivative

$$\frac{d}{dk}\left[\frac{sf(k)}{k}\right] = \frac{s[kf'(k) - f(k)]}{k^2}, \tag{4.25}$$

which is negative because the term in the square brackets is negative for the same reason as you gave in Exercise 29. Therefore, $f(k)/k$ has a downward sloping graph, which cuts $n + \delta$ at the steady state level of capital. We are sure it cuts $n + \delta$, i.e. the steady state exists, for the following reason:

$$\lim_{k \to 0} \frac{sf(k)}{k} = \lim_{k \to 0} sf'(k) = \infty \tag{4.26}$$

from neoclassical condition 3 and L'Hopital's rule, and similarly $\lim_{k\to\infty}[sf(k)/k] = 0$. Then we can apply the intermediate value theorem. In fact, because of this and the downward sloping property of $sf(k)/k$, there is one and only one steady state level of capital, $k^*$. If $k < k^*$ then the growth rate of capital is positive and it grows (at progressively slower rates) as it approaches the steady state. If $k > k^*$ the growth of capital is negative and it shrinks (again at progressively slower rates) as it reaches the steady state. Therefore, the steady state is globally stable. In the long run, economies are supposed to be at their steady states, but unfortunately, there is no growth in $y$ at this point. Therefore, *capital accumulation cannot be the reason for long run economic growth.* Yet we know that economies have been growing at a significantly positive rate on average for a very long time.

Now ponder this: is it possible that maybe all economies start at levels of capital to the left of their steady state levels, and that economic growth is simply just convergence to the steady state that has not yet been completed? Could the growth rates in the years after the industrial revolution being higher than those today be due to the added effect of capital accumulation? Or similarly, are poorer countries like China and India growing rapidly because they are not yet at their steady states, and are experiencing the added kick of capital accumulation? What are the limitations of the Solow model? How would you make it better?

Exercise 32. Look at the graph depicted on the blackboard. How many steady states are there? Which are stable? How could such a graph arise? Qualitatively, what is the situation depicted?

# Chapter 5

# Linear Algebra

## 5.1 Cauchy-Schwartz Inequality

Let $a$ and $b$ be two vectors each of size $n$. Then $|a \cdot b| \leq ||a||||b||$. To understand why this is true, follow this argument: If $b = 0_n$ then the two sides of the inequality are equal, so we have no problem. If, on the other hand, $b \neq 0_n$ then we can let $x = \frac{a \cdot b}{b \cdot b}$ and write $a = a - xb + xb$. Then it can be shown that

$$||a||^2 = ||a - xb||^2 + ||xb||^2 = ||a - xb||^2 + x^2||b||^2. \tag{5.1}$$

Therefore, $x^2||b||^2 \leq ||a||^2$. But then

$$x^2||b||^2 = \frac{(a \cdot b)^2}{(b \cdot b)^2}||b||^2 = \frac{(a \cdot b)^2}{||b||^4}||b||^2 = \frac{|a \cdot b|^2}{||b||^2}. \tag{5.2}$$

Plugging this into the most recent inequality, we get $|a \cdot b|^2 \leq ||a||^2||b||^2$. Take the square root of both sides of this inequality.

Exercise 33. Provide an argument for why (5.1) is true. (*Hint:* The first equality is a simple consequence of what is called the generalized pythagorean theorem.)

## 5.2 The Rank of a Matrix

Consider $m$ vectors each of length $n$. Call the set of these vectors $V = \{a_1, ..., a_m\}$. A linear combination of $V$ is an expression of the form $x_1a_1 + x_2a_2 + ... + x_ma_m$ where

$x_1, ..., x_m$ are all numbers. $V$ is said to be **linearly independent** if

$$x_1 a_1 + x_2 a_2 + ... + x_m a_m = 0_n \tag{5.3}$$

implies that $x_i = 0$ for all $i = 1, ..., m$. $V$ is said to be **linearly dependent** if there are numbers $x_1, ..., x_m$, not all of which are equal to 0, such that

$$x_1 a_1 + x_2 a_2 + ... + x_m a_m = 0_n. \tag{5.4}$$

Any set like $V$ is either linearly independent or linearly dependent.

Exercise 34. Let $a_1 = 3$, $a_2 = 7$, $b_1 = 2$, $b_2 = 4$, $c_1 = 0$ and $c_2 = 2$. Is the set $\{[a_i], [b_i], [c_i]\}$ linearly dependent or independent?

Let $A$ be an $n \times m$ matrix. Take $\mathcal{A}^C = \{A^1, ..., A^m\}$, which is the set of columns of $A$, and let $\phi^C : \mathcal{P}(\mathcal{A}^C) \to \mathbb{R}$ be the function defined by

$$\phi^C(Z) = \begin{cases} 0 \text{ if } Z \text{ is linearly dependent} \\ |Z| \text{ if } Z \text{ is linearly independent} \end{cases} \tag{5.5}$$

Similarly take $\mathcal{A}_R = \{A_1, ..., A_n\}$, which is the set of rows of $A$, and let $\phi_R : \mathcal{P}(\mathcal{A}_R) \to \mathbb{R}$ be the function defined in exactly the same way as $\phi^C$. Since $n$ and $m$ are both finite, $\phi^C$ and $\phi_R$ both achieve maximums on their domains. The **column rank** of $A$ is then defined as

$$c = \max_{Z \in \mathcal{P}(\mathcal{A}^C)} \phi^C(Z).$$

Similarly, the **row rank** of $A$ is defined as $r = \max_{Z \in \mathcal{P}(\mathcal{A}_R)} \phi_R(Z)$.

The row rank (and column rank) of a matrix does not change when any of the following three operations are applied to the matrix:

1. interchanging any two rows (or columns)

2. multiplying each entry in a given row (or column) by a nonzero number

3. replacing any row (or column) by itself plus a number $k$ times another row (or column)

These changes are called row (or column) operations and it is easy to prove that these must be true.

Exercise 35. Argue that the column and row ranks of a matrix are invariant to row and column operations respectively.

The main result of this section is if $A$ is an $n \times m$ matrix with $r > 0$, then $r = c$. The argument goes like this. Since $r > 0$ the matrix is not one where all of the entries are 0. Pick one nonzero component and through a series of successive row and column operations convert it to a matrix $B$ where $b_{11} \neq 0$. This $b_{11} \neq 0$ is called the pivot entry. Now multiply the first row of this matrix by $b_{21}/b_{11}$ and subtract it from the second row. Then multiply it by $b_{31}/b_{11}$ and subtract it from the third row. Continue doing so down the rows. Then go across the columns doing the same thing until you get a matrix that has 0s in every row except the first, and in every column except the first. If there are any other entries that are nonzero, then you can pick any nonzero entry and after a series of column and row interchanges you can convert it to a matrix $C$ where $c_{22} \neq 0$. Taking $c_{22}$ to be the pivot entry, after a series of operations like those performed on $B$, you arrive at a matrix, $D$ that has nothing but zeros in the second column and second row except in the the $d_{22}$ position. Continue this process until you run out of candidates for pivot entries or you run out of spaces for pivot entries. Either way, you have a matrix of 0s except along a diagonal. Therefore, the column rank is equal to the row rank since the row and column ranks of this final matrix are equal to that of the matrix you started with. This concludes the argument.

In light of this result, the column rank and row rank of a matrix are referred to simply as the **rank** of the matrix. An $n \times m$ matrix $A$ is said to have **full rank** if the rank of the matrix is equal to $\min\{m, n\}$.

Exercise 36. Use row and column operations to calculate the rank of the matrix:

$$M = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & 2 \end{bmatrix} \tag{5.6}$$

## 5.3 The Determinant

Square matrices are special because they are the only kinds of matrices for which we can calculate what is called the *determinant*. Consider the square matrix $A$ of order $n$. Consider the $(n-1)\times(n-1)$ submatrix of $A$ created by deleting row $i$ and column $j$. Call that matrix $A(i,j)$. The $(i,j)-$**cofactor** of $A$ is defined as

$$C_{ij}(A) = (-1)^{i+j}\det A(i,j),$$

where $\det A(i,j)$ is the determinant of the matrix $A(i,j)$. Now the **determinant** of a $1\times 1$ matrix is the value of the single entry. For an $n\times n$ matrix $A$, the determinant is defined as

$$\det A = a_{11}C_{11}(A) + ... + a_{1n}C_{1n}(A). \tag{5.7}$$

You may object that this definition is circular since we use the notion determinant to define the cofactor. However, since we defined the determinant of a $1\times 1$ matrix, the above equality helps us to recursively define determinants for any $n\times n$ matrix.

Exercise 37. Find a simple formula for the determinant of any $2\times 2$ matrix. Use this formula and equation (5.7) to calculate the determinant of the matrix $M$ given in (5.6) with the second row deleted.

Exercise 38. Show that the determinant of any lower- or upper- triangular matrix is simply the product of the diagonal entries.

After having done Exercise 38, and knowing that you can convert a matrix into a lower or upper triangular matrix using row and column operations, the following properties will be useful to you in calculating the determinant of any matrix.

Let $A$ be any square matrix of order $n$.

1. If the matrix $B$ is obtained from $A$ by interchanging any two rows (or columns) of $A$ then $\det B = -\det A$.

2. If $B$ is obtained from $A$ by multiplying each entry of some given row (or column) of $A$ by a nonzero constant $k$, then $\det B = k\det A$.

3. If $B$ is obtained from $A$ by replacing any row (or column) of $A$ by itself plus $k$ times some other row (or column), where $k$ is any number, then the determinant remains unchanged.

4. $\det A = \det A'$

Exercise 39. Prove the four properties of determinants listed above. Then use property 1 to show that if a matrix has a row (or column) of zeros then its determinant is 0.

Exercise 40. Show that a square matrix has full rank if and only if $\det A \neq 0$.

## 5.4 Cramer's Rule

Let $[A^1, ..., A^n]$ denote a square matrix $A$ of order $n$ with columns $A^1, ..., A^n$. If $\det A \neq 0$ then by Exercise 40, the matrix has full rank. By row operations of the kind described in Section 5.2 the augmented $n \times n + 1$ matrix $[A^1, ..., A^n, v]$, where $v$ is a vector of size $n$ can be reduced to a matrix with zeros above and below the diagonal and 1s on the diagonal, as in

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & c_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & c_n \end{bmatrix}.$$

Therefore, the system of equations $Ax = v$ where $x$ is a vector of $n$ variables has a *unique* solution. Call it $x^*$. Thus

$$\begin{aligned} \det[A^1, ..., A^{i-1}, v, A^{i+1}, ..., A^n] &= \det[A^1, ..., A^{i-1}, Ax^*, A^{i+1}, ..., A^n] \\ &= \sum_{j=1}^{n} x_j^* \det[A^1, ..., A^{i-1}, A^j, A^{i+1}, ..., A^n] \\ &= x_i^* \det A. \end{aligned}$$

which follows from the properties of determinants listed in the previous section. Divide both sides by $\det A$ to find the solution

$$x_i^* = \det[A^1, ..., A^{i-1}, v, A^{i+1}, ..., A^n] / \det A$$

for all $i = 1, ..., n$. This is Cramer's rule.

## 5.5 The Inverse of a Matrix

If $A$ is an $n \times n$ matrix with $\det A \neq 0$ we can find a unique matrix $B$ such that $AB = BA = I_n$. To see why $B$ is unique suppose that there was another matrix $C$ such that $CA = I_n$. Then $CAB = B$, but also $CAB = C(AB) = CI_n = C$. So $B = C$. The same holds if $AC = I_n$. Now we prove that $B$ exists, and we also calculate the entries of $B$.

Let $e_{jn}$ be the size $n$ vector such that there is a 1 in the $j$th position and 0 everywhere else. Then for any $n \times n$ matrix $X = [x_{ij}]$ solving $AX = I$ we have $e_{jn} = AX^j$ where $X^j$ is the $j$th column of $X$. Since $\det A \neq 0$, the matrix $A$ has full rank (by Exercise 40), and thus the solution exists and is unique (by row reduction). We have left to show that $XA = I_n$. By the properties of matrix multiplication and determinants, we can find a matrix $Y$ such that $A'Y = I_n$, which is equivalent to $Y'A = I_n$, and we have $I_n = Y'(AX)A = (Y'A)XA = XA$. We are done.

The unique matrix $X = B$, which is the **inverse** of $A$ is denoted $A^{-1}$ can be calculated by Cramer's rule. Note that

$$\begin{aligned} x_{ij} &= \det[A^1, ..., A^{i-1}, e_{jn}, A^{i+1}, ..., A^n]/\det A \\ &= \det[A^1, ..., A^{i-1}, e_{jn}, A^{i+1}, ..., A^n]'/\det A \\ &= C_{ji}(A)/\det A \end{aligned}$$

Therefore, $A^{-1} = [C_{ij}(A)/\det A]'$. Now, the following properties are useful. Whenever inverses exist,

1. $(A')^{-1} = (A^{-1})'$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $\det A^{-1} = 1/\det A$
4. The inverse of a lower (or upper) triangular matrix is a lower (or upper) triangular matrix.

Exercise 41. Prove the four properties above and find the inverse of the matrix in Exercise 36 (if it exists).

## 5.6 Eigenvectors and eigenvalues

Let $A$ be a square matrix of order $n$. A vector of size $n$ is an **eigenvector** of $A$ if there is a number $\lambda$ such that $Av = \lambda v$. If $v \neq 0_n$ then $\lambda$ is unique because $\lambda_1 v = \lambda_2 v$ implies $\lambda_1 = \lambda_2$. In that case, $\lambda$ is said to be an **eigenvalue** of $A$ belonging to $v$.

Now here is an important result concerning determinants, eigenvectors and eigenvalues. Let $A$ be a square matrix of order $n$. Then $\lambda$ is an eigenvalue of $A$ belonging to some nonzero vector if and only if $\det(A - \lambda I) = 0$.

To see why assume that $\lambda$ is an eigenvalue of $A$. Then by definition, there is a vector $v \neq 0$ such that $Av = \lambda v$. In other words, $Av - \lambda v = 0_n$. This implies $(A - \lambda I_n)$ is a matrix with linearly dependent columns (since $v \neq 0$), so that the rank of the matrix is less than $n$. Therefore, by exercise 40 it must be that $\det(A - \lambda I) = 0$.

Now interpret $(A - \lambda I_n)v = 0$ as a set of $n$ equations. If $(A - \lambda I_n)$ does not have full rank then there is at least one equation that is a linear combination of the others. Eliminate one of the redundant equations. Now you are left with a system of equations with more variables than unknowns, which means that at least one variable can be set freely. That is equivalent to setting one entry of $v$ freely. Make that one entry nonzero. Therefore, there is a vector $v \neq 0$ such that $Av = \lambda v$.

**Determinants, eigenvectors and eigenvalues?** Suppose $A$ is a $2 \times 2$ matrix and its columns are $A_1$ and $A_2$. The determinant of $A$ is the area of the parallelogram spanned by the vectors $A_1$ and $A_2$. If $A$ is a $3 \times 3$ matrix then the determinant of $A$ is the volume of the parallelepiped spanned by the vectors $A_1$, $A_2$ and $A_3$. If $A$ is a $4 \times 4$ matrix, then the determinant of $A$ is ...

Now eigenvalues and eigenvectors. Suppose $A$ is an $n \times n$ matrix and the solution to $\det(A - \lambda I) = 0$ yields $n$ eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. When plugged back into $Av = \lambda v$, one can find corresponding eigenvectors associated with these eigenvalues. Since $Akv = \lambda kv$ is equivalent to $Av = \lambda v$ for any nonzero $k$, these eigenvectors are not unique. We may take ones that are normalized, i.e. ones whose lengths are set to 1. For each eigenvalue, we therefore have one eigenvector. Then create the diagonal matrix

$$
D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \tag{5.8}
$$

and the matrix $P$ whose first column is $v_1$, second is $v_2$, third is $v_3$ etc. all the way up to $v_n$. Now it turns out that $A = PDP^{-1}$. You may look at a proof in any advanced linear algebra textbook or try to come up with one yourself. (*Hint:* If you try to come up with your own proof, think in terms of matrices shifting axes; e.g., in $\mathbb{R}^2$ the vectors $(0,1)$ and $(1,0)$ define unit movement in the $x$ and $y$ directions. Suppose we wanted to rotate our coordinate system and re-write the vector in the new system. How would we do that?)

Exercise 42. Let

$$
A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \tag{5.9}
$$

Find $A^{19}$.

## 5.7 Application: Central Planning

The formal mathematical approach to central planning began with the Russian mathematician Leonid Kantorovich formulating of the *linear programming* problem. Kantorovich wanted to maximize a linear output function subject to linear production constraints. (We will illustrate this today, so hang on a while if you don't know what that means.) He proposed the problem in his 1939 book *Mathematical Method of Production Planning and Organization* and in his 1959 book, *The Best Uses of Economic Resources* he argued that knowing the price of inputs and outputs is important even when the prices are hidden, as they are in planned economies.

The most celebrated solution to the linear programming problem is the **Simplex Method**, developed by American mathematician George Dantzig in 1947. This is the method we will illustrate to solve a linear programming problem today. The method gave rise to several explorations in the relationship between the planned model and the general equilibrium model (in which prices are determined by the market in a capitalist exchange economy).

Central planning has, with good reason, become an artifact of research and knowledge from a previous era. You will likely never see the context in which I present the Simplex method in a classroom ever again. But had we been born in a different place at a different time, attending a public policy school to study economics might very well have meant focusing our time and attention on the kind of problem that I am about to show you. Since the method requires the drawing of many tables, I illustrate the Simplex method on the blackboard with a background to the problem.

**The Planning Problem.** There are two goods we are interested in producing: agricultural goods (or, for simplicity, wheat), $x$, and national defense goods (missiles) $y$. Suppose we think that the value of a a kilo-bushel of wheat is 90, and the value of a missile is slightly lower at 75. We want to maximize

$$P = 90x + 75y.$$

We face production constraints. Both missiles and wheat must be produced from inputs: land, labor and capital. Producing a missile takes 3 units of labor, 9 units of capital and 2 units of land. Producing a kilo-bushel of wheat takes 2 units of labor, 4 units of capital and 10 units of land. Suppose we have with us only a total of 66 units of labor, 180 units of capital and 200 units of land. The constraints can thus be represented

$$\begin{aligned} 3x + 2y &\leq 66 \\ 9x + 4y &\leq 180 \\ 2x + 10y &\leq 200 \\ x \geq 0, y &\geq 0 \end{aligned}$$

First we draw the constraints on the blackboard (see board) and solve the problem geometrically. Now, it wouldn't be in the spirit of a central planning model if I told you how the simplex method works before demonstrating it, so let me show you how to solve the problem by setting up a matrix algorithm.

The steps to follow in the algorithm are as follows. First, write the inequality constraints as equalities with *slack* variables.

$$3x + 2y + s_1 = 66$$
$$9x + 4y + s_2 = 180$$
$$2x + 10y + s_3 = 200$$
$$x \geq 0, y \geq 0, s_i \geq 0 \forall i$$

(It will be useful for you to remember the concept of a slack variable when we do non-linear programming next week.) Then write the first equation as:

$$-90x + 70y + P = 0.$$

Form the augmented matrix of coefficients:

| $x$ | $y$ | $s_1$ | $s_2$ | $s_3$ | $P$ | |
|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 0 | 0 | 0 | 66 |
| 9 | 4 | 0 | 1 | 0 | 0 | 180 |
| 2 | 10 | 0 | 0 | 1 | 0 | 200 |
| -90 | -75 | 0 | 0 | 0 | 1 | 0 |

Now follow this process:

1. Find the least of the negative numbers in the bottom row (choosing any in case of a tie). The column containing this number is the pivot column.

2. Divide each entry in the last column (except the one in the last row) by the corresponding number in the pivot column as long as the number in the pivot column is positive. The row with the lowest quotient is the pivot row. In case of a tie, pick either one. If the pivot column does not have any positive entries above the last row, then stop; the problem has no solution.

3. The pivot element is the element in the pivot column and pivot row. Perform row operations on the matrix so that there is a 1 where the pivot element is and 0 everywhere else in the pivot column. Look at the bottom row. If all entries are nonnegative then you have found a solution; so stop. Otherwise, go back to step 1.

At a solution, each variable under which there are 0s everywhere except a 1 in some spot take the corresponding value on the right most column. The number on the bottom right is the value of the objective function at the solution.

What is going on in the simplex method? Mostly magic; but each time you pivot around an entry in the matrix you are moving from one point on the constraint boundary to another. It is your job, for homework, to figure out how exactly the row operations do this. Think in terms of equations.

Exercise 43. What is going on in the simplex method? What is the main problem with using linear programming to plan an economy?

# Chapter 6

# Integration

## 6.1 Upper and Lower Sums

Let $S = [a, b]$ be a closed interval, and let $f$ be a function that is bounded on $S$, i.e. there is a number, $C$, such that $|f(x)| \leq C$ for all $x \in S$. A **partition** of $[a, b]$ is a set $P$ of points $x_0, x_1, ..., x_n$ such that

$$a = x_0 < x_1 < ... < x_n = b.$$

A **refinement** of a partition $P$ is another partition $P'$ such that $P \subset P'$. Define

$$L(f, P) = \textstyle\sum_{k=1}^{n} m_k(x_k - x_{k-1}), \text{ where } m_k = \inf\{f(x)|x \in [x_{k-1}, x_k]\},$$
$$U(f, P) = \textstyle\sum_{k=1}^{n} M_k(x_k - x_{k-1}), \text{ where } M_k = \sup\{f(x)|x \in [x_{k-1}, x_k]\}.$$

Now if $P$ and $P'$ are partitions of $S$ and $P'$ is a refinement of $P$, then

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P). \tag{6.1}$$

This is easy to see geometrically. In fact, the coarsest partition bounds the set of all $L(f, P)$ and $U(f, P)$ with respect to partitions, since

$$m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a),$$

where $m = \inf\{f(x)|x \in S\}$ and $M = \sup\{f(x)|x \in S\}$. Now, define

$$\mathcal{L}(f) = \sup_P\{L(f, P)\} \text{ and } \mathcal{U}(f) = \inf_P\{U(f, P)\}.$$

It is easy to see that $\mathcal{L} \leq \mathcal{U}$. The argument is as follows. For a fixed partition $P^*$, $L(f, P^*)$ is a lower bound for the set of $U(f, P)$, because for any two partitions $P$ and $P'$, there is a refinement of both of them, $P''$, which by (6.1), leads to

$$L(f, P) \leq L(f, P'') \leq U(f, P'') \leq U(f, P').$$

Therefore, $L(f, P) \leq \mathcal{U}$, and this is true for every $P$, completing the argument.

Now a bounded function, $f : S \to \mathbb{R}$, is **Riemann integrable** if $\mathcal{L}(f) = \mathcal{U}(f)$. If this is the case, the common value is denoted $\int_S f(x)dx$ or $\int_a^b f(x)dx$.

## 6.2 Integrability of Continuous Functions

We now prove an important result: If $S = [a, b]$ and $f : S \to \mathbb{R}$ is continuous, then it is Riemann integrable. But first I argue the claim that $f$ is Riemann integrable on $S = [a, b]$ if and only if for all $\epsilon > 0$ there is a partition $P$ such that

$$U(f, P) - L(f, P) < \epsilon. \tag{6.2}$$

Start by assuming that $f$ is Riemann integrable on $[a, b]$, and let's try to obtain the implication. Let

$$\mathcal{I} = \int_a^b f(x)dx. \tag{6.3}$$

Take any $\epsilon > 0$. Since $\mathcal{L} = \mathcal{U}$, it follows that there must be partitions $P'$ and $P''$ such that

$$\mathcal{I} - \tfrac{\epsilon}{2} < L(f, P') \text{ and } U(f, P'') < \mathcal{I} + \tfrac{\epsilon}{2}.$$

Now let $P$ be a refinement of both $P'$ and $P''$. Then

$$\mathcal{I} - \tfrac{\epsilon}{2} < L(f, P') \leq L(f, P) \leq U(f, P) < U(f, P'') < \mathcal{I} + \tfrac{\epsilon}{2},$$

which is just a simple application of (6.1). From this we get $\mathcal{I} < L(f, P) + \frac{\epsilon}{2}$ and $U(f, P) - \frac{\epsilon}{2} < \mathcal{I}$, which when combined gives us $U(f, P) - L(f, P) < \epsilon$. Check.

We can go the other way now. Assume that (6.2) is true. By definition, $\mathcal{U} \leq U(f, P)$ for any $P$ and $\mathcal{L} \geq L(f, P)$ for any $P$. $U(f, P) \geq L(f, P)$ and $\mathcal{U} \geq \mathcal{L}$, so it must be that $0 \leq \mathcal{U} - \mathcal{L} < \epsilon$. But $\epsilon$ is arbitrary, so $\mathcal{U} = \mathcal{L}$. Therefore, $f$ is Riemann integrable.

Exercise 44. To continue the proof, we need to introduce a new concept. A function $f : [a, b] \to \mathbb{R}$ is **uniformly continuous** if for all $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. This differs from regular continuity in that here $\delta$ does not depend on $x$, whereas in the regular case it may. Show that if $f : [a, b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.

Now we are going to use (6.2) to prove that continuous functions are integrable. Since $f$ is uniformly continuous (see Exercise 44), we know that for any $\epsilon > 0$ there is $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/(b - a)$, where $x$ and $y$ are elements of $S$. (It doesn't matter that we divided $\epsilon$ by $b - a$; if you don't like that, you could have chosen $(b - a)\epsilon$ instead; it was arbitrary after all.) Now for any $\epsilon$ take the associated $\delta$ and partition the interval $[a, b]$ into $n$ subintervals, each of size $l < \delta$. By definition, the $m_k$ and $M_k$ differ by at most $\epsilon/(b - a)$. Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &< \sum_{k=1}^{n} \frac{\epsilon}{b - a} l = \frac{\epsilon}{b - a} \sum_{k=1}^{n} l = \epsilon, \end{aligned}$$

and we're done. Now one thing that I'm not going to prove but that you should know is that if a bounded function is continuous except at a finite number of points, then it is integrable. There are even stronger results, but for now just relax.

## 6.3 Properties of the Integral

Let $f$ and $g$ be Riemann integrable functions on $[a, c]$ and let $k \in \mathbb{R}$. Then

1. $\int_a^c kf(x)dx = k \int_a^c f(x)dx$
2. $\int_a^c [f(x) + g(x)]dx = \int_a^c f(x)dx + \int_a^c g(x)dx$
3. The same as (2) but with minus signs.
4. $|\int_a^c f(x)dx| \leq \int_a^c |f(x)|dx$
5. $\int_a^c f(x)dx \leq \int_a^c g(x)dx$ if $f(x) \leq g(x)$ for all $x \in [a, c]$

6. If $b \in [a, c]$ then $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
7. $\int_b^b f(x)dx \equiv 0$ for $b \in [a, c]$, and
8. $\int_a^c f(x)dx \equiv -\int_c^a f(x)dx$

The last two are actually definitions rather than properties.

## 6.4 The Fundamental Theorem of Calculus

Suppose that $S = [a, b]$ and $f : S \to \mathbb{R}$ is continuous so that it is Riemann integrable. For $x \in S$ define $F$ by

$$F(x) = \int_a^x f(t)dt. \tag{6.4}$$

Then $F$ is differentiable at every point $x \in (a, b)$ and $F'(x) = f(x)$.

Here is an intuitive justification for the veracity of this claim. Fix $x \in (a, b)$ and choose $y \in (a, b)$ close to $x$. Then

$$\frac{F(y) - F(x)}{y - x} = \frac{1}{y - x}\left[\int_a^y f(t)dt - \int_a^x f(t)dt\right] = \frac{1}{y - x}\int_x^y f(t)dt \tag{6.5}$$

whence

$$\frac{F(y) - F(x)}{y - x} - f(x) = \frac{1}{y - x}\int_x^y [f(t) - f(x)]dt. \tag{6.6}$$

So given $\epsilon > 0$ we can choose $\delta > 0$ such that $|y - x| < \delta$ implies that the absolute value of the right hand side is less than $\epsilon$.

We have now arrived at a very important truth: the Fundamental Theorem of Calculus. Suppose that $f : S \to \mathbb{R}$ and $G : S \to \mathbb{R}$ are continuous functions, $G$ is differentiable everywhere in $(a, b)$, and $G'(x) = f(x)$. Then

$$\int_a^b f(x)dx = G(b) - G(a)$$

To see why this is true, take $F$ from (6.4) and notice that $F - G$ has derivative 0, so it must be constant. Therefore,

$$F(b) - G(b) = F(a) - G(a) = -G(a) \text{ implies } F(b) = G(b) - G(a).$$

But $F(b) = \int_a^b f(x)dx$, completing the argument.

## 6.5 Integration by Parts

As usual, let $S = [a, b]$ and suppose that $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ have continuous first derivatives on $(a, b)$. Then if $x$ and $y$ are elements of $(a, b)$,

$$\int_x^y f(t)g'(t)dt = f(t)g(t)|_x^y - \int_x^y f'(t)g(t)dt \tag{6.7}$$

To get this, you integrate the product rule in the form: $fg' = (fg)' - f'g$.

Exercise 45. Find $\int_0^\pi x \sin x dx$.

## 6.6 Fubini's Theorem

For $X \subset \mathbb{R}^n$ let $f : X \to \mathbb{R}$ be a continuous function. Let $f^1 = f$ and let us view $f^1(x_1; x_2, ..., x_n)$ as a function of only $x_1$, i.e. we are holding $x_2, ..., x_n$ constant. It is good to know that even in the multivariate setting the continuity of $f$ allows us to define the Riemann integral,

$$f^2(x_2; x_3..., x_n) = \int_{a_1(x_2,...,x_n)}^{b_1(x_2,...,x_n)} f(x_1; x_2, ..., x_n)dx_1, \tag{6.8}$$

which is a continuous function only of $x_2, ..., x_n$, but viewed as a function of $x_2$ while holding $x_3, ..., x_n$ constant. Now holding $x_3, ..., x_n$ constant and consider

$$f^3(x_3; x_4, ..., x_n) = \int_{a_2(x_3,...,x_n)}^{b_2(x_3,...,x_n)} f^2(x_2; x_3..., x_n)dx_2. \tag{6.9}$$

Continue this idea until you have integrated with respect to $x_n$. What you have calculated then is the multiple integral

$$\int_{a_n}^{b_n} ... \int_{a_1}^{b_1} f(x_1, ..., x_n)dx_1...dx_n. \tag{6.10}$$

Now it may not surprise you that the integral $\int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, x_3)dx_1 dx_2 dx_3$ is the *volume* under $f$ in the region defined by

$$X = \{(x_1, x_2, x_3) | a_i \leq x_i \leq b_i, i = 1, 2, 3\} \subset \mathbb{R}^3.$$

Since it does not matter how we compute this quantity, **Fubini's theorem** tells us that if $f$ is continuous on the region $X$, then which variables you integrate with respect to first does not matter, you get the same answer at the end. This generalizes to $n$-dimensional volume, i.e. Fubini's theorem is also true if instead $X = \{(x_1, ..., x_n) | a_i \leq x_i \leq b_i, i = 1, ..., n\} \subset \mathbb{R}^n$ and we wanted to calculate the multiple integral in (6.10).

## 6.7 The Change of Variables

Suppose that $g$ is differentiable on the open interval $(a, b)$ and that its derivative is continuous. Let $T$ be an open interval such that $g(x) \in T$ for all $x \in (a, b)$. If a function $f$ is continuous on $T$ then the composition $f(g)$ is continuous on $(a, b)$, and

$$\int_a^b f(g)(x) g'(x) dx = \int_{g(a)}^{g(b)} f(g) dg \tag{6.11}$$

which you get by integrating the chain rule.

Exercise 46. Find $\int_0^1 x(1 - x^2)^{0.5} dx$.

The change of variables theorem generalizes in the following way. Let $x = f(v)$ where $v$ is a vector of size $n$. Suppose that $g : A' \to A$ is an injective function with continuous partial derivatives, where $A' \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^n$. In other words, $g$ is defined by

$$g[(v_1, ..., v_n)'] = (g_1(v_1), ..., g_n(v_n))' = (x_1, ..., x_n)$$

where $g_i$ is a function $g_i : A' \to \mathbb{R}$, such that $g_i(v_i) = x_i$ for all $i = 1, ..., n$. Suppose that the **Jacobian** matrix, $J = [\partial g_i / \partial v_j]_{i=1,...,n}^{j=1,...,n}$, is invertible (i.e. has an inverse) for all $v \in A'$. Then if we wanted to calculate the volume under the function $f(x)$ in the region $A$, we can do so by calculating the right hand side of

$$\int ... \int_A f(x) dx_1 ... dx_n = \int ... \int_{A'} f(g_1(v), ..., g_n(v)) |\det J| dv_1 ... dv_n,$$

which may be easier than calculating the left hand side. (Sometimes it may just not be possible to calculate the left hand side.) The only difficulty is that it is up to us to find the best change of variables, i.e. the function $g$, that makes the calculation possible or easiest.

Exercise 47. Let $A'$ be the region:

$$A' = \{x \in \mathbb{R}^2 | y \geq -x - 1, y \leq -x + 1, y \geq x - 1, y \leq x + 1\}$$

Now calculate

$$\int\int_{A'} \left(\frac{x - y}{x + y + 2}\right)^2 dxdy. \tag{6.12}$$

## 6.8 Improper Integrals

You will be pleased to know that if $f$ is Riemann integrable on the interval $[a, b]$ is integral on $(a, b]$, $[a, b)$ and $(a, b)$, are all defined to be the same as its integral on $[a, b]$. You will also be pleased to know that $a = -\infty$ and/or $b = \infty$ are allowed so long as you remember that

$\int_a^\infty f(x)dx$ is really $\lim_{b\to\infty} \int_a^b f(x)dx$ and $\int_{-\infty}^b f(x)dx$ is really $\lim_{a\to\infty} \int_a^b f(x)dx$.

## 6.9 Taylor's Theorem

Let $S$ be an open interval and let $f : S \to \mathbb{R}$ be a function whose first, second,..., and $m$th derivatives exist and are all continuous. Denote the $j$th derivative of $f$ by $f^{(j)}$. For $a \in S$ and $b \in S$, and $n \leq m$ we have

$$f(b) = f(a) + \frac{f'(a)}{1!}(b - a) + ... + \frac{f^{(n-1)}(a)}{(n-1)!}(b - a)^{n-1} + R_n \tag{6.13}$$

where

$$R_n = \int_a^b \frac{(b - t)^{n-1}}{(n - 1)!} f^{(n)}(t)dt. \tag{6.14}$$

Here is the proof. The fundamental theorem of calculus tells us that $f(b) = f(a) + \int_a^b f'(t)dt$. Notice that this is equation (6.13) for $n = 1$. Now assume that (6.13) is true. Perform the integration in (6.14) by parts where the function whose derivative is not taken is $f^{(n)}(t)$, and the derived function is $\frac{(b-t)^{n-1}}{(n-1)!}$. If you do this you get

$$R_n = \frac{(b-a)^n}{n!} f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)dt. \tag{6.15}$$

Plug this back into (6.13) and, by induction, you have completed the proof.

Exercise 48. Assume in the above theorem that $S$ contains 0 and remember that because $f^{(n)}(x)$ is continuous and $[a, b]$ is compact, the function is bounded. Argue that under these conditions, the **Taylor polynomial**

$$P_{n-1}(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + ... + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} \tag{6.16}$$

is a good approximation of $f$ when $n$ is large.

Exercise 49. Recall from the Weierstrass Theorem that because it is continuous, $f$ attains a maximum and a minimum on the interval $[a, b]$. Use the intermediate value theorem to show that there is a number $c \in [a, b]$ such that

$$R_n = f^{(n)}(c)\frac{(b-a)^n}{n!}. \tag{6.17}$$

What does this say for (6.13)?

**Taylor's theorem in $\mathbb{R}^n$.** Let $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$ is an open set. If $f$ is $C^1$ then for any $x \in S$ and $y \in S$, we can write

$$f(y) = f(x) + \nabla f(x)(y - x) + R_1(x, y), \tag{6.18}$$

where $R_1$ is a function with the property

$$\lim_{y \to x} \left( \frac{R_1(x, y)}{||x - y||} \right) = 0. \tag{6.19}$$

If $f$ is $C^2$ then for any $a \in S$ and $b \in S$, we can write

$$f(y) = f(x) + \nabla f(x)(y - x) + \frac{1}{2}(y - x)'Hf(x)(y - x) + R_2(x, y), \tag{6.20}$$

where $Hf(x)$ is the Hessian matrix and $R_2$ is a function with the property

$$\lim_{y \to x} \left( \frac{R_2(x, y)}{||x - y||^2} \right) = 0. \tag{6.21}$$

Here $\lim_{y \to x}$ is abusive limit for the convergence of vectors that was defined yesterday. I will prove only the first part of the statement, leaving the second part as an exercise. Here goes. Since $R_1(x, y)$ was not defined (it is a residual term) all we have to show is that (64) is true. Now fix $x$ and define $h(y) = f(y) - f(x) + \nabla f(x)(y - x)$, and notice that $h(x) = 0$. Then showing (64) is equivalent to showing that for any $\epsilon > 0$ there is a $\delta$ such that $y \in S$ and

$$||y - x|| < \delta \text{ imply } |h(y)| < \epsilon ||x - y||,$$

and this is what we'll show.

By the continuity of $\nabla h$ (why is $\nabla h$ continuous?) there is a $\delta > 0$ such that $y \in S$ and

$$||y - x|| \leq \delta \text{ imply } ||\nabla h(y)|| < \epsilon.$$

Now for any $y \in S$ such that $||y - x|| < \delta$, define $g : [0, 1] \to \mathbb{R}$ by

$$g(t) = h[(1 - t)x + ty].$$

(For math geeks, why is $g$ well defined?) Then $g(0) = h(x) = 0$. Now $g$ is differentiable because $h$ is, and

$$g'(t) = \nabla h[(1 - t)x + ty](y - x). \tag{6.22}$$

Now notice that $||(1-t)x+ty-x|| = t||y-x|| < \delta$ for all $t \in [0, 1]$. So by continuity, $||\nabla h[(1 - t)x + ty]|| \leq \epsilon$ for all $t \in [0, 1]$, which means that $|g'(t)| < \epsilon ||y - x||$ for all $t \in [0, 1]$.

By the mean value theorem in $\mathbb{R}$, there is a $c$ such that

$$g(1) = g(0) + g'(c).$$

Therefore, $|h(y)| = |g(1)| = |g'(c)| < \epsilon ||y - x||$, and we are done.

Exercise 50. Write down three terms of the Taylor series and evaluate the it at $(x, y) = (1, 1)$ for $f(x, y) = \ln(xy)$.

# Chapter 7

# Optimization

## 7.1 Optimization in $\mathbb{R}$

Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ be differentiable function on the interior of $S$. The interior of a set $A \subset \mathbb{R}^n$ is the set

$$A^* = \{x \in A | \exists \text{ an open box } B \text{ such that } x \in B \text{ and } B \subset A\}.$$

If $f$ achieves a maximum or minimum in the interior of $S$ then $f'(x) = 0$ at the point $x$ at which it achieves a maximum or minimum. If $x \in S^*$ is a maximum then $f''(x) \leq 0$. If it is a minimum then $f''(x) \geq 0$. Note that $f'(x) = 0$ and $f''(x) \leq 0$ are only necessary conditions for $x \in S^*$ being a maximum. The next exercise demonstrates how they are insufficient.

Exercise 51. Let $f : [-2, 5] \to \mathbb{R}$ be defined by $f(x) = x^3 - 4.5x + 6x + 1$. By the Weierstrass theorem we know that $f$ achieves a maximum and minimum on its domain. We can find the values of $x$ where $f'(x) = 0$. Which of these are maximum values of $f$ and minimum values of $f$? Calculate $f(x)$ at the points where $f'(x) = 0$. Then calculate $f(-1)$ and $f(4)$.

What are the issues with sufficiency? The problem is that of global versus local minima and maxima. One popular way of avoiding this problem is to assume that $S$ is a convex set and $f$ is either a concave or convex function. In that case, if $f'(x) = 0$

at any particular $x$ then that $x$ is a global maxima if $f$ is concave and a global minima if $f$ is convex. If $f$ is strictly concave then the global maxima or minima would be unique. If there is no $x \in S^*$ such that $f'(x) = 0$ then the maxima and minima are on the boundary of $S$, i.e. the set $S - S^* = \{x \in S | x \notin S^*\}$.

These insights from the one-dimensional case help motivate results for the multivariate case.

## 7.2 Optimization in $\mathbb{R}^n$

Consider the function $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}^n$ is a closed and convex set. As before, we say that $f$ is concave if $\forall x \in S$ and $\forall y \in S$ such that $y \neq x$ we have,

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y) \qquad \forall \alpha \in [0,1]; \tag{7.1}$$

and $f$ is strictly concave if in the inequality above, you replace $\geq$ with just $>$ and the interval $[0,1]$ with the interval $(0,1)$. It is convex if the function $-f$ is concave and strictly convex if $-f$ is strictly concave.

Exercise 52. Continue to let $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}^n$ is a closed and convex set. Prove that $f$ is concave if and only if the function $g(t) = f(x + tz)$ is concave on the set $T = \{t \in \mathbb{R} | x + tz \in S\}$, where $x \in S$, $z \in S$ and $z \neq 0_n$.

Before proceeding any further, I will define an important concept. An $n \times n$ matrix $A$ is said to be **negative semi-definite** if for all $x \in \mathbb{R}^n$,

$$x'Ax \leq 0.$$

If the inequality is strict for all $x \neq 0_n$ then $A$ is **negative definite**. $A$ is **positive definite** if $-A$ is negative definite and **positive semi-definite** if $-A$ is negative semi-definite. Now we have a series of fun and amazing results beginning with the multidimensional concavity lemma.

**Multidimesnional Concavity Lemma.** Consider the function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$ is convex and has a nonempty interior. Assume that $f$ has a Hessian where all the entries are continuous. Then the following three statements are equivalent: (i) $f$ is concave, (ii) $Hf(x)$ is negative semi-definite for all $x \in S$, and (iii) $f(x) \leq f(y) + \nabla f(y)(x - y)$ for all $x, y \in S$. In addition, if $Hf(x)$ is negative definite for all $x \in S$ then $f$ is strictly concave.

I demonstrate only that (i) and (ii) are equivalent, as well as the additional claim about strict concavity. I first show that (i) implies (ii), and hope that you are convinced that it is enough to establish the claim on the interior of $S$ since continuity takes care of the boundary points. Now let $g$ be the function and $T$ be the set both defined for Exercise 52. Notice that

$$g'(t) = \nabla f(x + tz) \cdot z = \sum_{i=1}^{n} \frac{\partial f(x + tz)}{\partial x_i} z_i \tag{7.2}$$

because $g(t) = f(x + tz)$ implies that

$$\lim_{h \to 0} \frac{g(t + h) - g(t)}{h} = \lim_{h \to 0} \frac{f(x + tz + hz) - f(x + tz)}{h} \tag{7.3}$$

and the right hand side is the definition of directional derivative of $f$ at $x + tz$ in the direction of $z$. (Remember that that is $\nabla f(x) \cdot z$.) Now take the second derivative of $g$ knowing that that is the directional derivative of each of the partial derivatives of $f$ added:

$$g''(t) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial f(x + tz)}{\partial x_j x_i} z_i z_j. \tag{7.4}$$

But this is just the equation $g''(t) = z'Hf(x + tz)z$. Now $f$ is concave, which by Exercise 52 means that $g$ is concave, and because $z$ is arbitrary (as long as we are in the set $T$), that means that $z'Hf(x + tz)z \leq 0$, i.e. $Hf$ is negative semi-definite since the argument holds for any $x$ and any $z$ such that $x + tz$ is in the domain of $f$. Proving (ii) implies (i) is simply writing the argument backwards, and the claim about strict concavity is merely replacing the $\leq$ sign by the $<$ sign in that proof.

Exercise 53. Complete the proof of the multidimensional concavity lemma.

**First order conditions theorem.** If the differentiable function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$, reaches a *local* interior extremum (maximum or minimum) at $x^*$ then $x^*$ satisfies $\nabla f(x^*) = 0_n$.

The proof goes as follows. Again write $f(x^* + tz)$ in terms of the familiar $g$ of Exercise 52 (yes, this time $x^*$ instead of $x$). Note that $g(0) = f(x^*)$. Because $g$ coincides with some value of $f$ for every $t$, then $g(t)$ must reach a local extremum at $t = 0$ if $f$ reaches an extremum at $x$. This means that $g'(0) = 0$. Now recall from the proof of the multidimensional concavity lemma why it should be that (7.2) should hold with $x$ replaced by $x^*$. Combined with $g'(0) = 0$, this implies that $\nabla f(x^*)z = 0$. Since this holds for any $z$, it must hold for all $e_j$, $j = 1, .., n$. Therefore, each of the partials are 0, and we have $\nabla f(x^*) = 0_n$.

**Second order conditions theorem.** If the $C^2$ function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$, reaches a *local* interior maximum (resp. minimum) at $x^*$, then $Hf(x^*)$ is negative (positive) semidefinite.

Here is the proof. Recall that (7.4) must be true. Then if $f$ is maximized at $x = x^*$ it must be that $g$ is maximized at $t = 0$, and thus $g''(0) \leq 0$. That means that the right hand side of (7.4) is nonpositive, or that the Hessian of $f$ at $x^*$ is negative semidefinite. (This is again because $z$ is arbitrary.) A similar argument holds for the "minimized" case.

**Global maximum theorem.** If the $C^2$ function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$, has a negative semidefinite Hessian at every $x$ in the interior of $S$, then $\nabla f(x^*) = 0$ implies that $f$ achieves a *global* maximum at $x^*$.

Proof. Because the Hessian is everywhere negative semidefinite, $f$ is concave by the multidimensional concavity lemma. The same lemma tells us that $f(x) \leq f(x^*) + \nabla f(x^*)(x - x^*)$ for all $x \in S$. Now if $\nabla f(x^*) = 0_n$ then $f(x) \leq f(x^*)$ for all $x \in S$. So $f$ reaches a global maximum.

A similar theorem holds for global minima. However, in most applications if what you are using the first and second order conditions to find interior minima of $f$, you can do this by finding interior maxima of $-f$.

Exercise 54. Re-read the proof of the global maximum theorem. Then prove that if the Hessian of $f$ is negative definite at every $x$ in its domain, and $x^*$ maximizes $f$, then $x^*$ is the *unique* global maximizer of $f$.

**Definiteness of the Hessian.** Let $D_1(x)$ be the determinant of the top left $1 \times 1$ submatrix of the $n \times n$ matrix $Hf(x)$, $D_2(x)$ be the determinant of the top left $2 \times 2$ submatrix of $Hf(x)$,..., and $D_k(x)$ be the determinant of the top left $k \times k$ submatrix of $Hf(x)$. If $(-1)^i D_i(x) > 0$ for all $i = 1, ..., n$ then $Hf(x)$ is negative definite, and if this is true for all $x$ in the domain of $f$ then $f$ is strictly concave. If $D_i(x) > 0$ for all $i = 1, ..., n$ then $Hf(x)$ is positive definite, and if this is true for all $x$ in the domain of $f$ then $f$ is strictly convex.

Exercise 55. Provide an argument for the definiteness of the claim above.

## 7.3 Lagrange's Theorem

Let $S \subset \mathbb{R}^n$. Let $f : S \to \mathbb{R}$ and $g_i : S \to \mathbb{R}$, $i = 1, ..., k$, be $C^1$ functions. Let $x^*$ be a point in the interior of $S$ and suppose that $x^*$ is an optimum (local maximum or local minimum) of $f$ subject to the constraints, $g_i(x) = 0$, $i = 1, ..., k$. If the gradient vectors $\nabla g_i(x^*)$, $i = 1, ..., k$, are linearly independent then there exists a vector $\Lambda = (\lambda_1, ..., \lambda_k)' \in \mathbb{R}^k$ such that

$$\nabla f(x^*) + \sum_{i=1}^{k} \lambda_i \nabla g_i(x^*) = 0 \tag{7.5}$$

Intuition. Recall that $S^*$ denotes the interior of $S$ and write the problem as

$$\max_{x \in S^*} f(x) \text{ subject to } g_i(x) = 0,\ i = 1, ..., k.$$

Now define the **Lagrangean** for this problem,

$$\mathcal{L}(x, \Lambda) = f(x) + \textstyle\sum_{i=1}^{k} \lambda_i g_i(x),$$

and apply the first order conditions theorem to $\mathcal{L}$ while deleting the last equation:

$$\nabla_n \mathcal{L}(x, \Lambda) = \nabla f(x) + \sum_{i=1}^{k} \lambda_i \nabla g_i(x) = 0, \tag{7.6}$$

where $\nabla_n \mathcal{L}$ simply means the vector with only the first $n$ entries of usual gradient of $\mathcal{L}$, since differentiating with respect to $\Lambda$ gives us back the constraints. Thus the information that we discard was something we already knew.

Now we have to convince ourselves of two things. First, we must be convinced that the solution to the maximization problem, $x$ is a vector of the first $n$ entries of a critical point of $\mathcal{L}(x, \Lambda)$. To convince yourself, understand that the directional derivative $\nabla f(x) \cdot h = 0$ at the maximum for all small length vectors $h$ that take $x$ to points in which the constraints $g_i$ continue to be satisfied. If the $g_i$ continue to be satisfied after moving a small amount in the direction $h$ then, then no change occurs in any $g_i$, i.e. $\nabla g_i \cdot h = 0$ for all $i = 1, ..., k$. But that means that $\nabla_n \mathcal{L}(x, \Lambda) \cdot h = \nabla f(x) \cdot h$ for any movement $h$ that keeps the constraints satisfied. This is good. It says that you cannot increase or decrease the objective function $f$ by making small movements in any "permissible" direction. So we must be at a critical point of $\mathcal{L}$ (except that we don't know the $\lambda_i$). Suppose we *were* at a maximum or minimum at $x$. Then the constraints would be satisfied and so would (7.5).

The second thing that we must convince ourselves is that the vector $\Lambda$ exists. That's a little harder, and you will have to wait until we see the proof of the theorem.

**Caution with Lagrange.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = -y$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be the function $g(x, y) = y^3 - x^2$. Notice that the maximum of $f$ subject to $g(x, y) = 0$ is at $(0, 0)$, since if $y$ is negative $x$ would have to be the square root of a negative number for the constraint to be met. If you set up the Lagrangean, $\mathcal{L}$ of this problem (defined in the previous lecture note) and take the first order conditions then you get

$$\begin{aligned} -2\lambda x &= 0 \\ -1 + 3\lambda y^2 &= 0 \\ -x^2 + y^3 &= 0 \end{aligned}$$

Now for the first equation to be true, either $\lambda = 0$ or $x = 0$. If $\lambda = 0$ then the second equation is a contradiction. If $x = 0$ then the third implies that $y = 0$. Plug this back into the second and obtain a contradiction again.

Exercise 56. What went wrong in the example above?

Now suppose that $f$ is defined by

$$f(x,y) = \frac{1}{3}x^3 + \frac{3}{2}y^2 + 2x \tag{7.7}$$

instead, and $g$ was the benign $g(x,y) = x - y$. The constraint qualification is met since $\nabla g(x,y) = (1,-1)$ for all $x$ and $y$. Now set up the Lagrangean and take the first order conditions:

$$\begin{aligned} x^2 + 2 - \lambda &= 0 \\ -3y - \lambda &= 0 \\ x - y &= 0 \end{aligned}$$

There are two solutions: $(x,y) = (2,2)$ or $(1,1)$, and seeing that $f(2,2) = 2/3$ while $f(1,1) = 5/6$ you could guess that $(2,2)$ is a minimum and $(1,1)$ is a maximum. But in fact, $f(0,0) = 0$ and $f(3,3) = 1.5$.

Exercise 57. What went wrong here?

## 7.4 The Karush-Kuhn-Tucker Theorem

In most economic applications constraints to a maximization problem take the form of inequalities rather than equalities. Begin with the simple problem

$$\max_x f(x) \text{ subject to } x \geq 0$$

and notice that if $x^*$ is the solution then it satisfies one of the following three cases:

1. $x^* = 0$ and $f'(x^*) < 0$,
2. $x^* = 0$ and $f'(x^*) = 0$,
3. $x^* > 0$ and $f'(x^*) = 0$.

These imply that

1. $f'(x^*) \leq 0$,
2. $x^*[f'(x^*)] = 0$,
3. $x^* \geq 0$.

In the world of $\mathbb{R}^n$, these correspond to

1. $\frac{\partial f(x^*)}{\partial x_i} \leq 0$,
2. $x_i^* \left[\frac{\partial f(x^*)}{\partial x_i}\right] = 0$, and
3. $x_i^* \geq 0$,

which must hold for all $i = 1, ..., n$ if $x^*$ maximizes $f(x)$ subject to $x_i \geq 0$ for all $i$. Now convince yourself that the problem

$$\max_{x_1,x_2} f(x_1, x_2) \text{ subject to } g(x_1, x_2) \geq 0,$$

is equivalent to

$$\max_{x_1,x_2,z} f(x_1, x_2) \text{ subject to } g(x_1, x_2) - z = 0 \text{ and } z \geq 0.$$

(Where have we seen the use of a *slack* variable like $z$ before?) Take the first order conditions of the Lagrangian, $\mathcal{L}$ of this latter beast while ignoring the inequality constraint, and arrive at

$$\begin{aligned} \frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} &= 0 \\ \frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} &= 0 \\ \frac{\partial f}{\partial \lambda} = g(x_1, x_2) - z &= 0 \end{aligned}$$

These give the critical points of $\mathcal{L}$. The only additional necessary conditions come from the inequality constraint $z \geq 0$. The insight from the previous discussion tells us the equivalent of properties $(1) - (3)$ for $z$, i.e that:

$$\begin{aligned} -\lambda &\leq 0 \\ z(-\lambda) &= 0 \\ z &\geq 0 \end{aligned}$$

Summarizing, we have:

$$\begin{aligned}
\frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} &= 0 \\
\frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} &= 0 \\
\lambda g(x_1, x_2) &= 0 \\
\lambda &\geq 0 \\
g(x_1, x_2) &\geq 0
\end{aligned}$$

These are called the **Karush-Kuhn-Tucker** conditions. More generally, we have:

**The Karush-Kuhn-Tucker Theorem.** Let $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}$ and $g_i : S \to \mathbb{R}$, $i = 1, ..., k$, be $C^1$ functions. Let $x^*$ be a point in the interior of $S$ and suppose that $x^*$ is an optimum (local maximum or local minimum) of $f$ subject to the constraints, $g_i(x) \geq 0$, $i = 1, ..., k$. If the gradient vectors $\nabla g_i(x^*)$, $i = 1, ..., k$, are linearly independent then there exists a vector $\Lambda = (\lambda_1, ..., \lambda_k)' \in \mathbb{R}^k$ such that

$$\nabla f(x^*) + \textstyle\sum_{i=1}^{k} \lambda_i \nabla g_i(x^*) = 0 \text{ and } \lambda_i g_i(x^*) = 0 \text{ for all } i = 1, ..., k.$$

In addition, the $\lambda_i$ are all nonnegative if $x^*$ is a maximum and nonpositive if it is a minimum.

Exercise 58. Re-read the cautionary examples for Lagrange and consider the problem of finding $(x, y)$ to maximize $f(x, y) = -(x^2 + y^2)$ subject to $h(x, y) = (x-1)^3 - y^2 \geq 0$. Find a solution to this problem just by looking at the functions. (You don't have to set up any Lagrangean or Karush-Kuhn-Tucker conditions). Now show that this problem cannot be analyzed using the Karush-Kuhn-Tucker theorem. Which one of the assumptions is violated?

## 7.5 Proof of Lagrange's Theorem

We prove the statement only for $k = 1$. Let $g(x) = g_1(x)$ so that we can drop the subscript from now on. Let the local optimum be $x^*$. The rank condition on $g$ tells us that $\nabla g(x^*) \neq 0_n$. Without loss of generality, assume that the first component of this vector is nonzero. Denote the first coordinate of a vector

$$x \in \mathcal{D} \equiv S \cap \{x \in \mathbb{R}^n | g(x) = 0\}$$

by $w$ and the last $n-1$ of them by $z$. Write $x = (w, z)$. Let $x^* = (w^*, z^*)$ denote the local optimum. Let $\nabla f_w(w, z)$ denote the derivative of $f$ with respect to $w$ alone and $\nabla f_z(w, z)$ denote the derivative with respect to $z$ alone. The derivative of $g$ is partitioned analogously into $\nabla g_w(w, z)$, which is a number, and $\nabla g_z(w, z)$ which is a vector of size $n - k$.

To prove the theorem we must show that there exists $\lambda \in \mathbb{R}$ such that

1. $\nabla f_w(w^*, z^*) + \lambda \nabla g_w(w^*, z^*) = 0$
2. $\nabla f_z(w^*, z^*) + \lambda \nabla g_z(w^*, z^*) = 0$

To prove this we need to appeal to the Implicit Function Theorem. This says that there is an open box $B \in \mathbb{R}^{n-1}$ containing $z^*$ and a $C^1$ function $h : B \to \mathbb{R}$ such that $h(z^*) = w^*$ and $g(h(z), z) = 0$ for all $z \in B$. Also,

$$\nabla h(z) = -\frac{\nabla g_z(h(z), z)}{\nabla g_w(h(z), z)} \tag{7.8}$$

which you will recognize as none other than formula (4.12).

Define $\lambda$ now as

$$\lambda = -\frac{\nabla f_w(w^*, z^*)}{\nabla g_w(w^*, z^*)}$$

which rearranges to

$$\nabla f_w(w^*, z^*) + \lambda \nabla g_w(w^*, z^*) = 0.$$

That's the first thing we had to show, which is a bit simpler than the second. Define the function $\phi : B \to \mathbb{R}$ by $\phi(z) = f(h(z), z)$. Since $f$ has a local optimum at $(w^*, z^*) = (h(z^*), z^*)$, then $\phi$ has a local optimum at $z^*$. Since $B$ is open, $z^*$ is an unconstrained local optimum of $\phi$ and the first-order conditions for an unconstrained optimum imply $\nabla\phi(z^*) = 0_{n-1}$, i.e. by the chain rule:

$$\nabla f_w(w^*, z^*)\nabla h(z^*) + \nabla f_z(w^*, z^*) = 0. \tag{7.9}$$

Substitute (7.8) in this to get

$$\nabla f_z(w^*, z^*) + \lambda \nabla g_z(w^*, z^*) = 0.$$

and that's it.

Exercise 59. We did not prove the Chain rule appearing in (7.9) in the multidimensional case. In the case of two variables, let $x(t), y(t)$ be two differentiable functions of $t$ and let $f(x, y)$ be a differentiable function. For the purposes of this demonstration, define $\partial x = x(t+h) - x(t)$ and $\partial y = y(t+h) - y(t)$. Then

$$\begin{aligned} f'(x(t), y(t)) &= \lim_{h\to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t))}{h} \\ &= \lim_{h\to 0} \frac{f(x+\partial x, y+\partial y) - f(x, y+\partial y) + f(x, y+\partial y) - f(x, y)}{h} \\ &= \lim_{h\to 0} \frac{f(x+\partial x, y+\partial y) - f(x, y+\partial y)}{h} + \lim_{h\to 0} \frac{f(x, y+\partial y) - f(x, y)}{h}. \end{aligned}$$

On the right is the definition of the partial of $f$ with respect to $y$, which by the single variable chain rule is

$$\frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Apply the mean value theorem to the limit on the left by picking an $x' \in [x, x+\partial x]$ such that the limit is equal to

$$\lim_{h\to 0} \frac{\partial x}{h}\frac{\partial f(x')}{\partial x} = \frac{\partial f}{\partial x}\frac{dx}{dt}.$$

That gives us

$$f'(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Use an extended argument based on this demonstration to argue that (7.9) is true.

## 7.6 Second Order Conditions

Let us make a few modifications to Lagrange's Theorem. Instead of assuming that $f$ and the $g_i$ are $C^1$, let us assume that they are all $C^2$ since we will want to take their second derivatives. Let us also assume that $S$ is an open set so that it is equal to its interior. It will be useful now to state the problem as

$$\max_{x \in \mathcal{D}} f(x),$$

(or the corresponding minimization; recall how $\mathcal{D}$ is defined in the previous section). The Lagrangian for the problem is:

$$\mathcal{L}(x, \Lambda) = f(x) + \sum_{i=1}^{k} \lambda_i g_i(x).$$

Taking the second derivative we get

$$H\mathcal{L}(x, \Lambda) = Hf(x) + \sum_{i=1}^{k} \lambda_i H g_i(x).$$

Let $(x^*, \Lambda^*)$ denote a critical point of $\mathcal{L}(x, \Lambda)$, and for simplicity of notation let $A = H\mathcal{L}(x^*, \Lambda^*)$ and $B = [\nabla g_1(x^*), ..., \nabla g_k(x^*)]'$. In other words $B$ is the $k \times n$ matrix created by putting all of the vectors $\nabla g_i(x^*)$ together into columns (in order), and then taking the transpose. Here is **Lagrange's second order theorem**:

Let $(x^*, \Lambda^*)$ be a critical point of $\mathcal{L}(x, \Lambda)$ and suppose that $B$ has rank $k$ (that's the same as saying the $\nabla g_i(x^*)$ are linearly independent). Let

$$\mathcal{Z}(x^*) = \{z \in \mathbb{R}^n | Bz = 0_k\}.$$

Then

1. If $f$ has a local maximum at $x^* \in \mathcal{D}$, then $z'Az \leq 0$ for all $z \in \mathcal{Z}(x^*)$.

2. If $f$ has a local minimum at $x^* \in \mathcal{D}$, then $z'Az \geq 0$ for all $z \in \mathcal{Z}(x^*)$.

3. If $z'Az < 0$ for all $z \in \mathcal{Z}(x^*) - \{0_n\}$, then $x^* \in \mathcal{D}$ is a local maximum of $f$.

4. If $z'Az > 0$ for all $z \in \mathcal{Z}(x^*) - \{0_n\}$, then $x^* \in \mathcal{D}$ is a local minimum of $f$.

That's the theorem, and it should look fairly intuitive to you. It is not much different from the second order conditions theorem for the unconstrained case, except now we can't talk of $A$ being negative or positive semidefinite since the permissible $z$s are restricted to directions where the constraints are still satisfied. That is, we must have $\nabla g_i(x^*) \cdot z = 0$, which is to say that the value of the constraint doesn't change along $z$. (And if that starting value was 0, then it remains 0...)

How do you test any of the conditions on the matrix $A$ in any of the four proposition above? For $l \leq n$, let $A_{ll}$ denote the top left $l \times l$ submatrix of $A$ and $B_{kl}$ the top left $k \times l$ submatrix of $B$. Now let $\pi$ denote a permutation, i.e. a jumbling up of the order of numbers $1, ..., n$ and let $A^\pi$ denote the matrix resulting from jumbling up the order of rows of $A$, and the order of columns in exactly the same way as the rows were jumbled. Now convince yourself that $A^\pi$ is also a symmetric matrix like the Hessian, $A$. Let $B^\pi$ denote the $k \times n$ matrix obtained by jumbling only the columns of $B$ in the same way that the rows and columns of $A$ were jumbled. As before let $A^\pi_{ll}$ denote the top left $l \times l$ submatrix of $A^\pi$, and $B^\pi_{kl}$ the top left $k \times l$ submatrix of $B^\pi$. Now define $C_l$ to be the $k + l \times k + l$ matrix obtained:

$$C_l = \begin{bmatrix} 0_{nn} & B_{kl} \\ (B_{kl})' & A_{ll} \end{bmatrix}$$

where $0_{nn}$ denotes the $n \times n$ matrix of 0s in every entry. And, define $C^\pi_l$ analogously where $A_{ll}$ is replaced by $A^\pi_{ll}$ and $B_{kl}$ by $B^\pi_{kl}$ (and its transpose...). Assume that $\det B_{kk} \neq 0$, which we can do without loss of generality since the rank of $B$ is $k$.

**The Lagrange characterization theorem.** The following are true.

1. $x'Ax \geq 0$ for every $x$ that $Bx = 0$ if and only if for all permutations $\pi$ of the first $n$ integers, and for all $r \in \{k+1, ..., n\}$, you got $(-1)^k \det C^\pi_r \geq 0$.

2. $x'Ax \leq 0$ for every $x$ that $Bx = 0$ if and only if for all permutations $\pi$ of the first $n$ integers, and for all $r \in \{k+1, ..., n\}$, you got $(-1)^r \det C^\pi_r \geq 0$.

3. $x'Ax > 0$ for all $x \neq 0$ such that $Bx = 0$ if and only if for all $r \in \{k+1, ..., n\}$, we have $(-1)^k \det C_r > 0$.

4. $x'Ax < 0$ for all $x \neq 0$ such that $Bx = 0$ if and only if for all $r \in \{k+1, ..., n\}$, we have $(-1)^r \det C_r > 0$.

## 7.7 The Kuhn-Tucker Theorem

Exercise 58 in Section 7.4 demonstrates that the same caution should be taken when applying the Karush-Kuhn-Tucker theorem as should be taken when applying Lagrange's theorem. However, the following useful theorem should ease some of your fears.

**The Kuhn-Tucker Theorem.** Let $S \subset \mathbb{R}^n$ be an open set and $f : S \to \mathbb{R}$ be a concave $C^1$ function and $g_i : S \to \mathbb{R}$, $i = 1, ..., k$, be $C^1$ functions such that

$$X = \{x \in S | g_i(x) \geq 0 \forall i = 1, ..., n\}$$

is convex. If $x^* \in X$ and there are numbers $\lambda_1, ..., \lambda_k \geq 0$ such that

$$\nabla f(x^*) + \textstyle\sum_{i=1}^k \lambda_i \nabla g_i(x^*) = 0 \text{ and } \lambda_i g_i(x) = 0 \text{ for all } i = 1, ..., k.$$

then $x^*$ solves the program

$$\max\nolimits_{x \in X} f(x).$$

It may be worthwhile to go ahead and prove this, since it is not too hard. Suppose the theorem wasn't true, i.e. there was an $x \in X$ such that $f(x) > f(x^*)$. Let $v = x^* - x$ and begin by writing the definition of the directional derivative of $f$ at $x^*$ in the direction $-v$:

$$\begin{aligned} -\nabla f(x^*) \cdot v &= \lim_{t \to 0} \frac{f(x^* - tv) - f(x^*)}{t} \\ &= \lim_{t \to 0} \frac{f((1-t)x^* + tx) - f(x^*)}{t} \\ &\geq \lim_{t \to 0} \frac{(1-t)f(x^*) + tf(x) - f(x^*)}{t} \qquad (7.10) \\ &= f(x) - f(x^*) > 0 \end{aligned}$$

where (7.10) is due to the concavity of $f$ and the fact that small $t > 0$ probably means $t < 1$ at least. The last line is due to the assumption.

Now, due to the convexity of $X$, we have $x^* - tv = (1 - t)x^* + tx \in X$. So for each $g_i$ such that $g_i(x^*) = 0$ we have

$$-\nabla g_i(x^*) \cdot v = \lim_{t \to 0} \frac{g_i(x^* - tv) - g_i(x^*)}{t} \geq 0. \qquad (7.11)$$

For all the rest, we have $\lambda_i = 0$ as per one of the Karush-Kuhn-Tucker conditions. Gathering all observations, we have the following contradiction:

$$0 > \nabla f(x^*) \cdot v = -\left(\sum_{i=1}^{k} \lambda_i \nabla g_i(x^*)\right) \cdot v = -\left(\sum_{i=1}^{k} \lambda_i \nabla g_i(x^*) \cdot v\right) \geq 0. \qquad (7.12)$$

And that is the proof.

Exercise 60. The following is the canonical producer's cost minimization problem where $w_1$ is the cost of the first input, $x_1$ is its quantity, $w_2$ is the cost of the second input, $x_2$ its quantity, and $y$ the desired minimum level of output:

$$\min_{x_1,x_2} w_1x_1 + w_2x_2 \text{ subject to } (x_1)^2 + (x_2)^2 \geq y.$$

## 7.8 Envelope Theorem

Consider the problem

$$\text{(P1) } \max_x f(x,a) \text{ subject to } g(x,a) = 0 \text{ and } x \geq 0,$$

where $x$ is the usual vector of variables (size $n$), and $a$ is a vector of parameters (size $m$). Suppose that for each vector $a$, the solution to this problem is unique, and denote it $x(a)$. Now define the **maximum value function**,

$$M(a) = f(x(a), a);$$

in other words, $M$ is a function of $a$ subject to $x$ having been chosen to solve (P1). Now suppose we would like to analyze how $M$ varies as $a$ varies. We can appeal to the envelope theorem.

**Envelope theorem.** Consider (P1) and assume that $f$ and $g$ are $C^1$ in $a$. For each $a$ let $x(a)_j \geq 0$ for all $j = 1, ..., n$ and assume that the $x(a)_j$ are also $C^1$ in $a$. Let $\mathcal{L}(x, a, \lambda)$ be the Lagrangean for (P1) and let $(x(a), \lambda(a))$ solve the Karush-Kuhn-Tucker conditions for the problem. Let $M(a)$ be the maximum value function for $f$. Then

$$\frac{\partial M(a)}{\partial a_j} = \frac{\partial \mathcal{L}}{\partial a_j} \text{ evaluated at } (x(a), \lambda(a)), \text{ for all } j = 1, ..., m.$$

The proof is as follows. First write the Lagrangian

$$\mathcal{L} = f(x,a) + \lambda g(x,a).$$

If $x(a)$ solves (P1) then Karush-Kuhn-Tucker says

$$\nabla_n f(x(a),a) + \lambda(a)\nabla_n g(x(a),a) = 0 \tag{7.13}$$
$$g(x(a),a) = 0 \tag{7.14}$$

which define solutions $(x(a), \lambda(a))$. Then note

$$\frac{\partial \mathcal{L}(x(a),\lambda(a))}{\partial a_j} = \frac{\partial f(x(a),a)}{\partial a_j} + \lambda(a)\frac{\partial g(x(a),a)}{\partial a_j}. \tag{7.15}$$

Also note,

$$\begin{aligned}
\frac{\partial M(a)}{\partial a_j} &= \sum_{i=1}^{n}\left[\frac{\partial f(x(a),a)}{\partial x_i}\right]\frac{\partial x_i(a)}{\partial a_j} + \frac{\partial f(x(a),a)}{\partial a_j} \\
&= \lambda(a)\sum_{i=1}^{n}\left[-\frac{\partial g(x(a),a)}{\partial x_i}\frac{\partial x_i(a)}{\partial a_j}\right] + \frac{\partial f(x(a),a)}{\partial a_j} \\
&= \lambda(a)\frac{\partial g(x(a),a)}{\partial a_j} + \frac{\partial f(x(a),a)}{\partial a_j} \\
&= \frac{\partial \mathcal{L}(x(a),\lambda(a))}{\partial a_j}
\end{aligned}$$

where the first equality comes from the chain rule you argued in Exercise 59; the second from substituting (7.13); the third from substituting the derivative of the left hand side of (7.14) and applying the chain rule; and the fourth from substituting (7.15). And that's what we had to show.

Exercise 61. Verify that the envelope theorem is true for the following problem.

$$\max_{x_1,x_2} x_1 x_2 \text{ subject to } a - 2x_1 - 4x_2 = 0 \text{ and } x_i \geq 0 \text{ for } i = 1,2.$$

You can dispense with the nonnegativity constraints since the solution will satisfy the Karush-Kuhn-Tucker conditions.

## 7.9 Application: The Agricultural Household

Consider a household that has preferences over its aggregate consumption, $c$ and leisure, $l$. Let $\mathbb{R}^2$ be the domain of preferences, and suppose that preferences satisfy Assumptions (1)-(4) in Section 2 of the handout on Decision Theory. Therefore, there is a continuous utility function $U(c, l)$ that represents the household's preferences. We will return to the issue of how consumption and leisure are divided among its members and the broader issue of whether such group preferences makes sense. The household can produce a good on its farm according to the production function $F(L, A)$, where $L$ is total farm labor and $A$ is total cultivated area. Let the price of this good be normalized to 1. Let $E^L$ be the household's total endowment of time and $E^A$ the size of its plot; and let $r$ be the price of renting a unit of land and $w$ be the wage of labor. The household's problem is then to solve

$$\begin{aligned}
\max_{c,l,A^h,L^h,A^m,L^m,A^f,L^f} U(c,l) \qquad & \text{subject to} \\
c + wL^h + rA^h \leq F(L,A) + wL^m + rA^m & \\
L = L^f + L^h & \\
A = A^f + A^h & \\
E^A = A^f + A^m & \\
E^L = L^f + L^m + l & \\
c, l, A^h, L^h, A^m, L^m, A^f, L^f \geq 0 &
\end{aligned}$$

where $A^h$ is hired land, $L^h$ is hired labor, $A^m$ is land rented out, $L^m$ is labor supplied to other employers, $A^f$ is the part of the household's plot that is cultivated by the household, and $L^f$ is the part of the household's labor that is used on the farm.

Exercise 62. Show that the household's problem is equivalent to

$$\max_{c,l,L,A} U(c,l) \qquad \text{subject to}$$

$$c + wl \leq \Pi + wE^L + rE^A \tag{7.16}$$

$$\Pi = F(L,A) - wL - rA \tag{7.17}$$

$$c, l, L, A \geq 0.$$

For the reason you argued in Task 3 of the Handout on Decision theory, the maximized value of $U$ is increasing in $\Pi$ as defined by (7.17), and (7.16) holds with equality at the solution.

Now since $U$ does not depend on $L$ or $A$ and $\Pi$ does not depend on $c$ or $l$, the solution to the household's problem can be found by solving

$$\max_{L,A} F(L,A) - wL - rA \text{ subject to } L, A \geq 0,$$

then letting $\Pi^*$ be the maximized value of this function, and then solving

$$\max_{c,l} U(c,l) \text{ subject to } c + wl = \Pi^* + wE^L + rE^A.$$

If every household problem can be broken down into two separate maximization problems like this (one profit maximization problem, another utility maximization problem that follows) then the rural agricultural economy is said to satisfy **separability**. It should be noted that separability has not been observed anywhere in Africa, Latin America, South and Southeast Asia, except for Indonesia. As a consequence, the profit maximization problem and utility maximization problem are intertwined and typically profit cannot be maximized.

Exercise 63. Go back to the original household problem and show that if there is no land market (i.e. if $A$ is replaced by $E^A$ and $r = 0$) separability would still hold. Then show that if the land market is in place, but there is no labor market (i.e. if $L$ is replaced by $E^L$ and $w = 0$) again separability would still hold.

Why would separability not hold? In Exercise 63 you showed that separability is robust to one missing market, either land or labor. But it turns out to be the case that it is not robust to imperfections in both land and labor markets. Of course, for anyone who has traveled in rural parts of Africa or South Asia, it is clear that it is more reasonable to assume that land markets do not exist, while there may be imperfections in the labor market as follows.

If there is no land market and if there is some involuntary unemployment, then one version of the household's problem is

$$\max_{c,l,L^h,L^f} U(c,l) \text{ subject to}$$

$$c = F(L^h + L^f, E^A) - wL^h + wL^m \tag{7.18}$$
$$l + L^f + L^m = E^L \tag{7.19}$$
$$L^m \leq M \tag{7.20}$$

where $M$ is the maximum labor the household can provide to the market due to involuntary rationing, and we have continued to assume that preferences are such that at the solution (7.18) is an equality rather than inequality. You have shown that if (7.20) is not binding then separability holds. If $L = L^f + L^h$, and you plug (7.19) into (7.18), the only relevant constraint is

$$c + wl = F(L, E^A) - wL + wE^L$$

and the household chooses $c$, $l$ and $L$ to maximize utility. Now suppose that $F$ satisfies the first two of the neoclassical assumptions, and in particular let

$$F(L, E^A) = Z(E^A)^\alpha L^{1-\alpha}, \tag{7.21}$$

where $0 < \alpha < 1$ and $Z$ is a technological constant.

Exercise 64. Let $L^*$ be the optimal labor on $E^A$ of land with the above assumptions when (7.20) does not bind. Show that the labor intensity of cultivation,

$$\frac{L^*}{E^A} = \left(\frac{Z(1-\alpha)}{w}\right)^{\frac{1}{\alpha}}.$$

Notice that it does not depend on plot size. Provide an intuitive explanation for this.

Now suppose (7.20) binds, and in particular assume that $M$ is so small that households wish to supply as much labor as they can so that $L^m = M$ and $L^h = 0$. (Why would $L^h = 0$?) The problem is now rather simple. It is

$$\max_{c,l} U(c, l) \text{ subject to } c = F(E^L - M - l, E^A) + wM.$$

Exercise 65. For concreteness, assume that for some $\gamma > 0$

$$U(c, l) = \ln c + \gamma \ln l \tag{7.22}$$

and find the first order conditions of the maximization problem above with $F$ given as in (7.21). Combine the first order conditions into one equation and use the implicit function theorem to show that that the labor intensity of cultivation is decreasing in plot size.

Smaller plots are cultivated more intensely than larger plots, which is observed in most parts of Africa and South Asia. Since the marginal productivity of labor is decreasing, this is an inefficient outcome.

From a policy standpoint, is non-separability a bad thing, or is it ambiguous? If it is a bad thing, and it is the case that labor market imperfections are causing non-separability, how would you go about remedying this problem? If you do not think that it is the labor market and lack of land market that are causing non-separability, what do you think it is? Can you write a model to explain it in a different way?