4 Real Analysis

Let $\{a_{1k}\}$, $\{a_{2k}\}$, ..., and $\{a_{nk}\}$ be sequences that converge to a_1, a_2, ..., and a_n respectively. Then the sequence of vectors, $\{[a_{1k}, a_{2k}, \ldots, a_{nk}]'\}_{k=1}^{\infty}$ converges to the vector $[a_1, a_2, ..., a_n]'$. This is the **convergence** of vectors.

A **closed set** is a set of vectors $X \subset \mathbb{R}^n$ where the limit of every convergent sequence $\{x_k\} \subset X$ also lies in X. If for all $x \in X \subset \mathbb{R}^n$, there exists an open box $S \subset X$ such that $x \in S$, then X is said to be an **open set**.

A **bounded set** is a set X for which there is an open box $S = S_1 \times S_2 \times \cdots \times S_n$ such that $X \subset S$ and each S_i, $i = 1, ..., n$, is the open interval $(-z, z)$, where $z > 0$. A subsequence $\{x_{m(k)}\}$ of a sequence $\{x_k\}$ is a sequence of some (or all) of the elements of $\{x_k\}$ appearing in the order in which they appear in $\{x_k\}$.

A **compact set** is a set is a set $X \subset \mathbb{R}^n$ such that every sequence in X has a convergent subsequence whose limit is in X.

A **convex set** is a set $X \subset \mathbb{R}^n$ where if $x \in X$ and $y \in X$ then $\alpha x + (1-\alpha)y \in X$ for all $\alpha \in (0, 1)$.

The **supremum** of a set $X \subset \mathbb{R}$ is the lowest number sup X such that every number greater than sup X is greater than every number in X. This is also called the "lowest upper bound" of X. The **infimum** is the "greatest lower bound". A fact that we will not prove is that if X is a bounded set (i.e. there is a number $z > 0$ such that $X \subseteq [-z, z]$) then both $\inf X$ and $\sup X$ are elements of $\mathbb{R}$.

Exercise 29. Show that if $X \subset Y \subset [a, b]$ for some interval $[a, b]$ then $\inf Y \leq \inf X$ and $\sup Y \geq \sup X$.

Exercise 30. Show that if A and B are both convex sets, their intersection is convex but not necessarily their union.

Exercise 31. A real-valued function f defined on some interval $[a, b] \subseteq \mathbb{R}$ is said to be **convex** if for all $x \neq y$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$ for all $\alpha \in (0, 1)$ and **strictly convex** if the inequality holds strictly. (And similarly, f is said to be **concave** if $-f$ is convex, and **strictly concave** if $-f$ is strictly convex.) Show that if f is convex or concave it is continuous on (a, b). Show that if it is strictly convex or strictly concave, then it is differentiable on (a, b). Show that if f is convex, then for all $x_0 \in [a, b]$ there exists a line $y = f(x_0) + m(x - x_0)$ through $(x_0, f(x_0))$ such that $f(x) \geq f(x_0) + m(x - x_0)$ for all $x \in [a, b]$. Show that if f is concave, then for all $x_0 \in [a, b]$ there exists a similar line through $(x_0, f(x_0))$ such that f is weakly lower than the line.

4.1 Existence of Extreme Values

Our objective in this part is to establish the result that a continuous real-valued function defined on a compact set achieves both a maximum and minimum on that set. This is a central result that underpins optimization theory. We begin with:

Theorem 10. (Heine-Borel Theorem) *A set $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

I have put the proof for you to read in the appendix to this section. The Heine-Borel theorem implies the following result, sometimes called the "extreme value theorem."

Theorem 11. (Weierstrass theorem) *Let $X \subset \mathbb{R}^n$ be a compact set and f be a continuous real-valued function on X. Then f attains a minimum and maximum on X.*

Proof. The first step is to show that the codomain $f(X)$ of f is compact. Let $\{y_k\} \subset f(X)$ be a sequence. For each k pick $x_k \in X$ such that $f(x_k) = y_k$ (which you can do by construction). This gives us a sequence $\{x_k\} \subset X$. Since S is compact you can pick an infinite subsequence $\left\{x_{m(k)}\right\} \subset \{x_k\}$ that converges to some $x \in X$. Let $y = f(x)$ and $y_{m(k)} = f(x_{m(k)})$. Since $\left\{x_{m(k)}\right\}$ converges to x and f is continuous, the infinite sequence $\left\{f(x_{m(k)})\right\}$ converges to $f(x)$. But $f(x) \in f(X)$ so $f(X)$ is compact.

The second step is to show that because $f(X)$ is compact $\sup f(X) \in f(X)$ and $\inf f(X) \in f(X)$, and these are the maximum and minimum we need. First of all, boundedness (from Heine-Borel) tells us that $\sup f(X) < \infty$ and $\inf f(X) > -\infty$. Now, let N_k be the interval $(\sup f(X) - 1/k, \sup f(X)]$ where $k = 1, 2, \ldots$. Let $f(X)_k = f(X) \backslash N_k$. Then $f(X)_k$ is not empty for each k, otherwise we would have an upper bound strictly smaller than $\sup f(X)$. Now for each $f(X)_k$ pick any $y_k \in f(X)_k$. The sequence $\{y_k\}$ must converge to $\sup f(X)$. Since $f(X)$ is closed (again, Heine-Borel) $\sup f(X) \in f(X)$. The argument for inf is almost identical. $\square$

4.2 Intermediate & Mean Value Results

We now turn to a set of results known as intermediate and mean-value results, along with an application of these results to the method known as "L'Hopital's rule."

Theorem 12. (intermediate value theorem) *Let f be a continuous real-valued function defined on an interval $[a, b]$. For $f(x) > f(y)$ and any $c \in (f(y), f(x))$, there exists a number $z \in (\min\{x, y\}, \max\{x, y\})$ such that $f(z) = c$.*

Proof. Let $g(x) = f(x) - c$. Construct a sequence of intervals $\{S_i\}_{i=0}^{\infty}$ as follows. Begin with $S_0 = [a, b]$. If $g(x) = 0$ at the midpoint of this interval, then we're done. Otherwise, g changes sign between the endpoints on either the right half or the left. Pick the half that it changes sign on and call the interval S_1. If it is 0 at the midpoint, then again we're done. If not, again pick the half on which it changes sign and call it S_2, and so on. Either we reach a point where $g(x)$ takes a value of 0 at the midpoint of an interval, or we obtain an infinite sequence of intervals. In the latter case, the sequence of left endpoints and the sequence of right endpoints both converge to the same limit z between x and y. By continuity and the change of sign condition, $g(z) = 0$. □

Theorem 13. (generalized intermediate value theorem) *Let $X \subset \mathbb{R}^n$ be a convex set, $f : X \to \mathbb{R}$ a continuous function, and x and y points such that $f(x) < f(y)$. Then for any c such that $f(x) < c < f(y)$ there is an $\alpha \in (0, 1)$ such that $f((1-\alpha)x+\alpha y) = c$.*

Proof. Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(\beta) = f((1 - \beta)x + \beta y)$ for $\beta \in [0, 1]$. Since f is continuous, g is continuous and $g(0) = f(x)$, $g(1) = f(y)$, and $g(0) < c < g(1)$. By the intermediate value theorem there is $\alpha \in (0, 1)$ such that $c = g(\alpha) = f((1-\alpha)x+\alpha y)$. □

Theorem 14. (Rolle's theorem) *If f is continuous on the interval $[a, b]$, differentiable everywhere on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. If f is constant then $f'(x) = 0$ for all x, and we'd be done. So for challenge's sake, let f not be constant. By the Weierstrass Theorem, f attains maximum and minimum values on $[a, b]$. Since f is not constant, either the maximum is greater than $f(a)$ or the minimum is less than $f(a)$ (or both). If the maximum value is greater than $f(a)$ then any point x at which it is attained lies in (a, b) (it can't be b because $f(a) = f(b)$ by assumption). The numerator of

$$\frac{f(y) - f(x)}{y - x} \tag{24}$$

where $x \neq y \in [a, b]$ is always non-positive and the denominator can have either sign. Now take the abusive limit as $x \to y$ on the centered expression above. Due to different signs on different sides, the limit cannot be positive or negative. But it exists by the assumption that f is differentiable on (a, b). So it must be 0. The argument is similar if f's minimum is less than $f(a)$. □

Theorem 15. (mean value theorem) *If f is continuous on the interval $[a, b]$ and differentiable everywhere on (a, b), then there exists $c \in (a, b)$ such that*

$$(b - a)f'(c) = f(b) - f(a).$$

Proof. Note that we have the same assumptions as in Rolle's theorem, except the assumption that $f(a) = f(b)$. We need to show that there is a point $c \in (a, b)$ with

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{25}$$

But this is easy. Let g be a function on $[a, b]$ defined by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) \tag{26}$$

and notice that $g(a) = g(b) = f(a)$. By Rolle's theorem, there is a point $c \in (a, b)$ such that

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}, \tag{27}$$

and we are done. □

Exercise 32. Suppose that $f : (a, b) \to \mathbb{R}$ is a differentiable function, and so is $f' : (a, b) \to \mathbb{R}$. ($f$ is "twice differentiable" if its derivative is a differentiable function.) Suppose also that $f''(x) < 0$ for all $x \in (a, b)$. Show that f must be strictly concave. If instead f is convex and twice differentiable, show that $f''(x) \geq 0$ for all $x \in (a, b)$.

Exercise 33. Let $X \subset \mathbb{R}^n$ be a convex and open set and let $f : X \to \mathbb{R}$ be a differentiable function. Then the **generalized mean value theorem** states that for any $x \in X$ and $y \in X$ there is an $\alpha \in (0, 1)$ such that

$$f(x) - f(y) = \nabla f((1 - \alpha)x + \alpha y)(x - y)$$

Prove this by defining g exactly the same as in the proof of the generalized intermediate value theorem. *Hint*: Notice that $g'(\alpha) = \nabla f((1 - \alpha)x + \alpha y) \cdot (b - a)$.

Theorem 16. (another mean value theorem) *Suppose that there are continuous functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$. Suppose that these functions are differentiable at every point in (a, b) and that $g'(x) \neq 0$ for all $x \in (a, b)$. Then, there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \tag{28}$$

Proof. With the assumptions of the theorem, the mean value theorem implies that $g(b) - g(a) \neq 0$. Now define $h : [a, b] \to \mathbb{R}$ by

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]. \tag{29}$$

I'll give you \$100 if $h(a)$ is not equal to $h(b)$. Now again, by the mean value theorem, there is $c \in (a, b)$ such that

$$h'(c) = f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] = 0. \tag{30}$$

Since we agreed that $g(b) - g(a) \neq 0$, you can divide both sides of this final equality by $[g(b) - g(a)]g'(c)$ to prove the stated result. □

Theorem 17. (L'Hopital's Rule) *Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be differentiable everywhere on (a, b) and that $g(x) \neq 0$ and $g'(x) \neq 0$ for $x \in (a, b)$. Then it is not so unfortunate that $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$, for it is the case that:*

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)} \tag{31}$$

so long as the limit on the right hand side exists.

Proof. I will try to derive (31) from the assumptions. To make sure f and g are continuous at a, we need $f(a) = g(a) = 0$. This does not have to be the case, but we can just redefine f and g to be so if it isn't. Call the limit on the right side of (31), L. By the properties of limits, for any $\epsilon > 0$, we can find an interval $T = (a, a + \delta)$ such that

$$\left| \frac{f'(c)}{g'(c)} - L \right| < \epsilon \tag{32}$$

for $c \in T$. Invoking the other mean value theorem, we can then argue

$$\left| \frac{f(x) - f(a)}{g(x) - g(a)} - L \right| < \epsilon \tag{33}$$

for all $x \in T$. Then using that $f(a) = g(a) = 0$, we've derived (31). □

Even though we didn't prove it, L'Hopital's rule would still be true if $a = -\infty$ or $a = \infty$, and/or if $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$ instead of 0. Furthermore, a does not have to be approached from the right (which is not possible in the case of $a = \infty$ anyway).

4.3 The Implicit and Inverse Function Theorems

The implicit and inverse function theorems are important results. For example, the implicit function theorem will be invoked in the proof of Lagrange's theorem, which we use in optimizing a function subject to equality constraints.

Theorem 18. (implicit function theorem) *Given $n \geq 1$, let a typical point of the set $\mathbb{R}^{n+1}$ be denoted by (x, y), where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Let $S \subset \mathbb{R}^{n+1}$ be an open box, and let $F : S \to \mathbb{R}$ be a differentiable function with continuous partial derivatives. Let (x^*, y^*) be a point in S such that*

$$\frac{\partial F(x^*, y^*)}{\partial y} \neq 0 \tag{34}$$

and let $F(x^*, y^*) = 0$. Then there is an open box $B \subset \mathbb{R}^n$ such that $x^* \in B$, and a differentiable function $g : B \to \mathbb{R}$ whose partial derivatives are continuous, such that $g(x^*) = y^*$, and $F(x, g(x)) = 0$ for all $x \in B$. The derivative of g at any $x \in B$ is:

$$\frac{\partial g}{\partial x_j} = -\frac{\partial F/\partial x_j}{\partial F/\partial y}. \tag{35}$$

Proof. We first construct the function g so that $F(x, g(x)) = 0$, and then we show that it has the remaining properties stated in the theorem. It is the function g that we call the "implicit function" in the name of the theorem.

Due to (34) we can assume without loss of generality that $\partial F(x^*, y^*)/\partial y > 0$. By continuity of $\partial F/\partial y$, there is a small open box $A \subset \mathbb{R}^{n+1}$ containing (x^*, y^*) such that

$$\frac{\partial F(x, y)}{\partial y} > 0, \qquad \forall (x, y) \in A \tag{36}$$

Thus $F(x^*, \cdot)$ is increasing in y in a neighborhood of y^*, which means we can find y_1, y_2 satisfying $F(x^*, y_1) < 0 < F(x^*, y_2)$ and $y_1 < y^* < y_2$. Again by the continuity of F we can find an open box $B \in \mathbb{R}^n$ containing x^* such that $B \times [y_1, y_2] \subset A$ and $F(x, y_1) < 0 < F(x, y_2)$ for all $x \in B$. By the intermediate value theorem, for each $x \in B$, there is $y \in (y_1, y_2)$ such that $F(x, y) = 0$. Uniqueness of this y is guaranteed by (36). This uniqueness allows us to define the continuous function $g(x) = y$ having the properties described in the theorem, except that we have yet to show (35) and the fact that that the partial derivatives are continuous.

To that end, fix $x \in B$, and let $y = g(x)$. Then by definition of the derivative (please re-read the definition of the derivative in multiple dimensions), we have

$$F(x + se_j, y + t) - F(x, y) = s\frac{\partial F(x, y)}{\partial x_j} + t\frac{\partial F(x, y)}{\partial y} + \varepsilon\sqrt{s^2 + t^2}$$

where ε is a "correction term" that goes to 0 as $\sqrt{s^2 + t^2}$ goes to 0. Now pick s small enough so that $x + se_j \in B$ and set $t = g(x + se_j) - g(x)$. For this value of t, the left hand side equals of the centered equality above equals 0 given the way g was constructed. Therefore, rearranging the equality we get

$$t\frac{\partial F(x,y)}{\partial y} = -s\frac{\partial F(x,y)}{\partial x_j} - \varepsilon\sqrt{s^2+t^2}, \tag{37}$$

which, after substituting t on the left and dividing both sides by $\partial F(x,y)/\partial y \neq 0$, in turn rearranges to

$$\frac{g(x+se_j)-g(x)}{s} = -\frac{\partial F(x,y)/\partial x_j}{\partial F(x,y)/\partial y} - \frac{\varepsilon}{\partial F(x,y)/\partial y}\frac{\sqrt{s^2+t^2}}{s} \tag{38}$$

Keeping in mind that our choice of t goes to 0 as s goes to 0, take the limit as $s \to 0$ on both sides. The question is: Can we eliminate the right hand term by doing this? The answer is yes if $\sqrt{s^2+t^2}/s$ is bounded for small values of s, in which case its behavior cannot overwhelm the convergence of ε to 0. But in fact this term *is* bounded. I show this in the second appendix to this chapter. Please read it, and come back here.

Therefore, taking the limit as $s \to 0$ on both sides of (38) gives us the j'th partial derivative of g on the left side, and and the right side of (35) on the right, as required. We know that these partials are continuous since $\partial F(x,y)/\partial y$ is non-vanishing, and the partials of F are continuous. Thus g is differentiable by the third true fact. □

Exercise 34. Use the implicit function theorem to find dy/dx along the circle, $x^2+y^2 = 1$. Where does dy/dx not exist?

Theorem 19. (inverse function theorem) *Let f be differentiable at every point on (a,b), and let $f'(x) \neq 0$ for all $x \in (a,b)$. Assume that f is invertible and let its inverse be $g = f^{-1}$. Then g is differentiable at $f(x)$ and*

$$g'(f(x)) = \frac{1}{f'(x)}. \tag{39}$$

Proof. To see why g is differentiable at $f(x)$, let $f(x) = y$ and $f(x') = y' \neq y$ for some x and x' both in S. You can make this assumption because there is an open interval that contains x, and $y' \neq y$ for any $x' \neq x$ that we choose in this interval. This is because $f'(x) \neq 0$. Now since g and f are inverses of each other,

$$\frac{g(y')-g(y)}{y'-y} = \frac{x'-x}{f(x')-f(x)}. \tag{40}$$

Now recall that the inverse of a continuous function is continuous, take the abusive limits on both sides (here, $\lim_{x'\to x}$) and note that the left side is the limit as $f(x') \to f(x)$ (by continuity) which gives us the definition of $g'(y)$ while the right side is $1/f'(x)$. □

Alternatively, if we knew already that g is differentiable at $f(x)$, then this follows from the chain rule. Since $g \circ f(x) = x$, we have $[g \circ f]'(x) = 1$. Then plugging this into (17), we have $1 = g'(f(x))f'(x)$, which rearranges to give (39). But of course we had to first establish that g is differentiable at $f(x)$.

Exercise 35. Show that a continuous strictly increasing function f defined on an interval $[a, b]$ has a continuous strictly increasing inverse.

Exercise 36. Use the inverse function theorem to find the derivative of $\log x$.

Exercise 37. Find the derivative of $f(x) = \frac{\log(3x^2+2)}{e^{6x}+1}$.

Exercise 38. Is the inverse of $f(x) = x^3$ differentiable everywhere?

Exercise 39. Use L'Hopital's rule to calculate $\lim_{x\to 0^+} x^x$. (*Hint*: Do some easier L'Hopital's rule problems from a textbook of your liking first, remember the properties of log and e, and then think about continuity.)

Exercise 40. Prove that for a sequence of numbers $\{x_n\}_{n=1}^{\infty}$ that converges to x,

$$\lim_{n\to\infty} (1 + x_n/n)^n = e^x.$$

Hint: One way to do this is to use the binomial theorem and L'Hopital's rule; but there are other ways.

4.4 Application: The Swan-Solow Model

To analyze production in an economy, assume that there are only two inputs: capital, $K(t)$ and labor, $L(t)$ where t denotes time. The amount of output produced is a function of these inputs and at any time t, it is given by

$$Y(t) = F(K(t), L(t)), \tag{41}$$

F being the production function. Some of the output is consumed, $C(t)$ and the remainder, $I(t)$ is invested to generate capital for future production. We assume that the economy is "closed" (there is no interaction with other foreign economies) and output and capital input are the same single type of goods. Let s be the fraction of output saved at any time—the saving rate—which we assume is constant. All savings are invested. Capital is not permanent. It depreciates at the rate of $\delta > 0$ so that the rate of increase of capital with time is given by

$$\frac{dK(t)}{dt} = I(y) - \delta K(t) = sF(K(t), L(t)) - \delta K(t), \tag{42}$$

which is called the *flow equation*. Population grows over time; that's why L depends on t. Assume that the labor force grows at a constant rate

$$\frac{1}{L(t)}\frac{dL(t)}{dt} = n \geq 0$$

and each worker has equal productivity for a given amount of capital. Assume that at the beginning of time, $t = 0$, there is only 1 worker in the economy.

Exercise 41. Find an expression for the number of workers at any time, t. (*Hint*: Try to find it first for $n = 1$.)

Momentarily forget about the dependence on time of each of the variables, K, L, Y etc. Assume that the production function, F, is **neoclassical**, which means:

1. for all $K > 0$ and $L > 0$, $\frac{\partial F}{\partial K} > 0$, $\frac{\partial^2 F}{\partial K^2} < 0$, $\frac{\partial F}{\partial L} > 0$, and $\frac{\partial^2 F}{\partial L^2} < 0$,
2. $F(\lambda K, \lambda L) = \lambda F(K, L)$ for all $\lambda > 0$,
3. $\lim_{K\to 0} \frac{\partial F}{\partial K} = \lim_{L\to 0} \frac{\partial F}{\partial L} = \infty$ and $\lim_{K\to\infty} \frac{\partial F}{\partial K} = \lim_{L\to\infty} \frac{\partial F}{\partial L} = 0$.

Notice that the second condition lets us write

$$Y = F(K, L) = L \cdot F(K/L, 1) = Lf(k), \text{ or } y = f(k)$$

where $y = Y/L$ is the output per capita, and $k = K/L$ is the capital-labor ratio.

Exercise 42. Find expressions for $\partial Y/\partial K$ and $\partial Y/\partial L$ in terms of f, f' and k.

Exercise 43. Argue that $f(0) = 0$.

Exercise 44. Use the information provided so far to show that

$$\frac{dk}{dt} = sf(k) - (n + \delta)k. \tag{43}$$

A **steady state** is defined as a point where the growth rate of per capita captial $\frac{dk}{dt}/k$ is constant. We can show that at any steady state $\frac{dk}{dt} = 0$, i.e. per capita capital does not grow. Divide both sides of (43) by k to get

$$\frac{dk/dt}{k} = sf(k)/k - (n + \delta). \tag{44}$$

At a steady state, the left hand side is constant. That means that $f(k)/k$ should be constant on the right hand side. This implies

$$\frac{d(f(k)/k)}{dt} = \frac{dk/dt}{k}\left(f'(k) - \frac{f(k)}{k}\right) = 0. \tag{45}$$

The term in the parenthesis between the two equal signs is negative, so that if k is finite then $dk/dt = 0$.

Exercise 45. Why is the term in parenthesis on the right hand side of (45) negative?

Plugging this result into (44), it must be the case that $sf(k^*) = (n+\delta)k^*$, where k^* is the steady state level of capital. (Since $dk/dt = 0$, capital doesn't grow in the steady state so it is a constant amount, k^*.) Since k is constant in the steady state, so is y, and since the saving rate is constant, the level of consumption is also constant.

Exercise 46. At what rate do K and Y grow in the steady state?

The growth rate of k, γ_k is given by (44). The first term of the expression has derivative

$$\frac{d}{dk}\left[\frac{sf(k)}{k}\right] = \frac{s[kf'(k) - f(k)]}{k^2}, \tag{46}$$

which is negative because the term in the square brackets is negative for the same reason as you gave in Exercise 45. Therefore, $f(k)/k$ has a downward sloping graph, which cuts $n+\delta$ at the steady state level of capital. We are sure it cuts $n+\delta$, i.e. the steady state exists, for the following reason:

$$\lim_{k\to 0} \frac{sf(k)}{k} = \lim_{k\to 0} sf'(k) = \infty \tag{47}$$

from neoclassical condition 3 and L'Hopital's rule, and similarly $\lim_{k\to\infty}[sf(k)/k] = 0$. Then we can apply the intermediate value theorem. In fact, because of this and the downward sloping property of $sf(k)/k$, there is one and only one steady state level of capital, k^*. If $k < k^*$ then the growth rate of capital is positive and it grows (at progressively slower rates) as it approaches the steady state. If $k > k^*$ the growth of capital is negative and it shrinks (again at progressively slower rates) as it reaches the steady state. Therefore, the steady state is globally stable.

In the long run, economies are supposed to be at their steady states, but unfortunately, there is no growth in y at this point. Therefore, in the model, *capital accumulation cannot be the reason for long run economic growth.* Yet we know that economies have been growing at a significantly positive rate on average for a very long time.

Appendix A: Proof of the Heine-Borel Theorem

We begin the proof by first establishing two lemmata.

Lemma 1. *Let* $\{x_i\}$ *be an increasing sequence contained in a bounded set* $X \subset \mathbb{R}$*. Then the sequence converges to a limit that equals* $\sup\{x_i\}$*.*

Proof. For $\epsilon > 0$ we know that $\sup\{x_i\} - \epsilon$ is not an upper bound for the sequence, so there exists $x_N \in \{x_i\}$ such that for all $n > N$ we have $\sup\{x_i\} - \epsilon < x_n \leq \sup\{x_i\}$ (which follows because the sequence is increasing). This rearranges to $0 < \sup\{x_i\} - x_n < \epsilon$. Since ϵ was arbitrary, this shows that $\sup\{x_i\}$ is a limit of the sequence $\{x_i\}$. □

Lemma 2. *Let* $Z := [-z, z]$ *for some* $z > 0$*. Then for all* n*, the set* Z^n *is compact.*

Proof. We show that Z is compact; the fact that Z^n is compact follows immediately from this and the definition of convergence of vectors.

Consider an infinite sequence $\{x_i\} \subset Z$. For each j, let $z_j = \inf\{x_j, x_{j+1}, ...\}$. Then $\{z_j\}$ is an increasing sequence that is bounded by $Z \subset \mathbb{R}$. (This follows from Exercise 29.) According to the previous lemma, it therefore converges to a limit, $z := \sup\{z_j\}$. We now show that z is the limit of a subsequence of $\{x_i\}$. Given $\epsilon > 0$, and any number N, there is $n > N$ such that $|z_n - z| < \epsilon/2$ since z is the limit of sequence $\{z_n\}$. Since $z_n = \inf\{x_n, x_{n+1}, ...\}$ there is $m \geq n$ such that $|x_m - z_n| < \epsilon/2$. Combining these using the triangle inequality,

$$|x_m - z| < |x_m - z_n| + |z_n - z| < \epsilon.$$

Thus we can construct a subsequence of $\{x_i\}$ that converges to z. Note that $z \in Z$ since ϵ was arbitrary; otherwise, some elements of the subsequence would lie outside Z. □

Now we can prove the Heine-Borel theorem.

First let us show that a compact set, X, is closed and bounded. To show that it is closed, take any convergent sequence $\{x_k\} \subset X$. Since X is compact, this sequence has a convergent subsequence $\left\{x_{m(k)}\right\}$ whose limit is in X. By the uniqueness of the limit, this is also the limit of $\{x_k\}$. Hence X is closed.

If X is not bounded, then for each n, there is $x_n \in X$ such that $||x_n|| > n$. Then the sequence $\{x_n\} \subset X$ does not have a convergent subsequence. To see this, suppose that it did and let x be the limit of such a subsequence. For all $m > 2||x||$ we have

$$||x_m - x|| \geq ||x_m|| - ||x|| \geq m - ||x|| > m$$

where the first inequality follows from the triangle inequality for vectors (see Exercise 9). This however, violates the condition for convergence (that for all ϵ there is a number N such that $||x_n - x|| < \epsilon$ for all $n \geq N$; simply take ϵ to be smaller than m). This proves that X must be bounded.

We must now show the reverse: that if X is closed and bounded set then it must be compact. By boundedness, there is a number $z > 0$ such that $|x_i| \leq z$ for all $x \in X$ and all i, where x_i is the ith component of the vector x. Lemma 2 above shows that $Z \equiv [-z, z] \times \cdots \times [-z, z]$ is compact. Obviously, $X \subset Z$. If we can show that a closed subset of a compact set is also compact, then we are done. To do this last step, take any sequence in X. Since $X \subset Z$, this is also a sequence in Z, which is a compact set. So it must have a convergent subsequence with limit in Z. But since X is closed, and this subsequence lies in X, the limit must also lie in X. Therefore, X is compact.

Appendix B: Finishing the Implicit Function Theorem Proof

We prove the claim that $\sqrt{s^2 + t^2}/s$ in the proof of the theorem is bounded for small values of s.

Note that for s small enough we have

$$|\varepsilon| < \min\left\{\frac{1}{2}\left|\frac{\partial F(x,y)}{\partial y}\right|, \left|\frac{\partial F(x,y)}{\partial x_j}\right|\right\} \tag{48}$$

since ε goes to 0 and s goes to 0. Then, we have

$$\begin{aligned}
\left|\frac{\partial F(x,y)}{\partial y}\right| |t| &\leq |s| \left|\frac{\partial F(x,y)}{\partial x_j}\right| + |\varepsilon|\sqrt{s^2 + t^2} \\
&\leq |s| \left|\frac{\partial F(x,y)}{\partial x_j}\right| + |s||\varepsilon| + |t||\varepsilon| \\
&\leq |s| \left|\frac{\partial F(x,y)}{\partial x_j}\right| + |s| \left|\frac{\partial F(x,y)}{\partial x_j}\right| + |t|\frac{1}{2}\left|\frac{\partial F(x,y)}{\partial y}\right|
\end{aligned}$$

where the first inequality follows from (37), the second from the triangle inequality, and the third from (48). Solving the final inequality for $|t|$ gives

$$|t| \leq 4|s| \left|\frac{\partial F(x,y)/\partial y}{\partial F(x,y)/\partial x_j}\right|$$

Therefore, we have

$$\left|\frac{\sqrt{s^2 + t^2}}{s}\right| \leq \frac{|s| + |t|}{|s|} = 1 + 4\left|\frac{\partial F(x,y)/\partial y}{\partial F(x,y)/\partial x_j}\right|$$

which shows that $\sqrt{s^2 + t^2}/s$ is indeed bounded for small values of s.