1 Preliminaries

1.1 Mathematical Statements

The table at the bottom of the page lists some common mathematical symbols and their abbreviations. Mathematical statements in this course will seldom involve abbreviations or symbols other than the ones listed in the table (except ones that you surely already know such as $=$, $\leq$, etc.). When new symbols arise, I will explain them. As an example, a typical statement is

$$\forall x \in X \text{ and } \forall y \in Y, \exists z \in Z \text{ s.t. } x + y = z,$$

which you will read "For every x in the set X and every y in the set Y, there is an element z in the set Z such that x plus y equals z." Thus a mathematical statement is nothing more than a statement in the English language (or any other language for that matter), where the vocabulary is limited to words like "for all," "there is," and "such that." The objective is to determine which statements are true, and which are not.

It will be important to note the difference between "if" and "only if." S_1 **if** S_2 means that you can derive S_1 from S_2, but it may not be the case that you can derive S_2 from S_1. On the other hand, S_1 **only if** S_2 means that you can derive S_2 from S_1 but that you may not be able to derive S_1 from S_2. S_1 if and only if S_2 means that S_1 can be derived from S_2 and S_2 can be derived from S_1. When this happens, S_1 and S_2 are equivalent: one statement does not say any more or any less than the other statement.

Every mathematical statement has a negation. The **negation** of statement S_1 is written $\neg S_1$ (read "not S_1"). For example, you can negate the statement

"Every left-handed man in Palo Alto has a beard,"

by presenting a left-handed man in Palo Alto who does not have a beard. You cannot negate this statement by presenting a right-handed man in Palo Alto who has a beard; or by presenting left-handed woman in Palo Alto without a beard. You may find it

Symbol	How to read it
$\in$	"in the set," or "is in the set" depending on context
$\exists$	"there is a(n)"
$\forall$	"for all" or "for every"
s.t.	"such that"
w.l.o.g.	"without loss of generality"

hard to believe that people make these errors, but they do. The negation of the above statement is the statement

"There is a left-handed man in Palo Alto who does not have a beard."

The statement "S_1 is true if S_2 is true" is equivalent to the statement "$\neg S_2$ is true if $\neg S_1$ is true," or "S_2 is not true if S_1 is not true." The latter two statements are the **contrapositive** of the first statement. A statement is always equivalent to its contrapositive.

The **converse** of the statement "S_1 is true if S_2 is true" is the statement "S_2 is true if S_1 is true." A statement is not equivalent to its converse. To see this, let S_1 be the statement "U2 rocks," and S_2 be the statement "All Irish bands rock."

Proofs We will occasionally do proofs. Often we will prove a statement directly from a set of other statements. This is called "**direct** proof." But this may not always be convenient. If we assume that S_1 is true and would like to prove that S_2 is true, then one way of doing this is to begin the proof by assuming that S_2 is not true. Then we show that $\neg S_2$ implies $\neg S_1$. But this cannot be, since we assumed S_1 is true. Therefore, $\neg S_2$ must be false, i.e. S_2 must be true, and that completes the proof. You may have already noticed that this is simply proving the statement "S_1 implies S_2" by proving the contrapositive. This kind of proof is called a proof by **contradiction**. It is closely related to the method of proof called **reductio ad absurdum**, which allows us to conclude that the statement S_1 is false if S_1 implies a statement S_2 and its negation $\neg S_2$. (Both S_2 and $\neg S_2$ cannot simultaneously be true, so S_1 must be false.)

The third common method of proof is proof by **induction**. Suppose you wanted to prove that a sequence of statements S_1, S_2, S_3 ... are all true if S_0 is true. If the sequence never terminates, you have no hope of doing this in your lifetime if you try to prove each statement one at a time. But there is a shortcut. First you show that S_1 is true if S_0 is true. Then you show that for any positive integer k, S_k implies S_{k+1} when S_0 is true. That completes the proof. The reason this works is you can substitute 1 for k, and since you showed S_1 is true, S_2 must be true. Then substitute 2 for k, and you get the result that S_3 is true, etc. Under **strong induction** you show that S_1 is true. Then you show that for every positive integer k, $S_1, \ldots, S_k$ implies S_{k+1}.

There are also other methods of proof. Do a google search.

Numbers, etc. When I say "number" I always mean a real number, except in some cases where I mean a natural number/positive integer. The set of real numbers will be denoted $\mathbb{R}$, which consists of all the numbers you know, except the imaginary numbers (e.g. 32, -0.73, 0, 19/7, 4π, and e^{23} are all in $\mathbb{R}$, but $5i$ is not). Beyond this, I do not want to go into much detail about what a real number is. When the symbols $\geq$, $<$, $\leq$, $>$ are used, two numbers are being compared. When I say that a is **weakly** larger than b, then $a \geq b$; when I say that a is **strictly** larger than b, then $a > b$. The concepts of "weakness" and "strictness" will extend to other settings where equality is and is not allowed (as when a set may be either a strict or weak subset of another set).

If r is a number and we write $|r|$ (read "absolute value of r") then that means r if r is nonnegative and $-r$ if it is negative. The two sides of the equal sign, $=$, may have numbers or other kinds of objects such as sets. The context will make that clear. The notation ":=" means the left hand side is being defined as the right hand side; vice versa for "=:". Sometimes I will use "$\equiv$" instead of "$=$" to denote that to two sides of the equality are equivalent, or identically equal to each other. One common shorthand that I will use is "$\forall i = 1, 2, 3 \ldots$" which you read as "for every positive integer i."

Exercise 1. A number is **rational** if and only if it can be expressed as the ratio of two integers. Prove, by contradiction, that $\sqrt{2}$ is irrational. *Hint:* If $\sqrt{2}$ is rational, then it can be expressed as the ratio of two integers that are not both even. Express it as such, take squares, etc. and remember that only even numbers have even squares.

We will denote "**infinity**" by ∞. It is not a number, but we will sometimes treat it informally as one—for example, when we say a set as an "infinite number" of elements. It is a "quantity" bigger than every number, just as $-\infty$ represents a "quantity" smaller than every number. The set of natural numbers is denoted $\mathbb{N} := \{1, 2, 3, ...\}$. The set of integers is $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$. Positive numbers are larger than 0 and negative numbers are smaller. Subscripts such as + or - on a set of numbers indicate nonnegative and nonpositive subsets; when written twice such as $++$ or $--$, they indicate positive and negative subsets; so, e.g., $\mathbb{N} = \mathbb{Z}_{++}$.

For any number $n \in \mathbb{N}$, the notation $n!$ (read "n factorial") denotes $n \times (n-1) \times (n-2) \times ... \times 1$, and we define $0! := 1$. The notation $\binom{n}{k}$ (read "n choose k") is

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

and gives the number of ways in which k balls can be chosen from an urn of n balls without repetition and where the order in which they are chosen does not matter. A set S is **countably infinite** if there is a bijective function mapping the set of natural numbers $\mathbb{N}$ to S. A set is **countable** if it is finite or countably infinite.

Exercise 2. Prove **Pascal's rule**; i.e., for all $n \in \mathbb{N}$,

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}, \qquad \forall k = 1, ..., n$$

Theorem 1. (Binomial Theorem) *For all $n \in \mathbb{N}$,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Proof. The proof is by induction. The **basis case** is $(x+y)^1 = x+y$, which holds. Now for the **inductive step**: we assume that the result holds for n and must prove it holds for $n+1$. Note that $(x+y)^{n+1} = x(x+y)^n + y(x+y)^n$. This implies that the **coefficient** of $x^i y^j$ in the **polynomial** $(x+y)^{n+1}$ is given by the sum of coefficients of $x^{i-1}y^j$ and $x^i y^{j-1}$ in the polynomial $(x+y)^n$. Therefore, by the **inductive hypothesis** the coefficient of $x^{n+1-k}y^k$ in $(x+y)^{n+1}$ is $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$. (This follows from Pascal's rule above.) Coefficients of $x^i y^j$ in the polynomial $(x+y)^{n+1}$ such that $i+j \neq n+1$ are zero. Therefore, we have

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k,$$

which proves the inductive step. □

1.2 Sets, Relations and Functions

We've already use the word "set" above. A **set** is a "well-defined" collection of elements. "Well-defined" means that I can describe to you what kinds of things are in the set, and you will be able to know exactly whether something is in the set or not. Sometimes, we can describe a set by simply listing out its elements:

$$A = \{a_1, a_2, a_3, \ldots\},$$

Whenever we use curly brackets, that is { }, those are sets and inside the brackets is a list of elements in the set or a mathematical statement describing the common property satisfied by the elements belonging to the set. That is, we may write a set as $\{x \in X : P\}$ which you can read as "the set of $x \in X$ such that (some property) P holds."

The set A is a **subset** of the set B, written $A \subset B$, if every element of A is also an element of B. Two sets are equal if they are subsets of each other. If A is a set, then the set of all subsets of A is called the **power set** of A and is denoted $\mathcal{P}(A)$. If A is a set with a finite number of elements then $|A|$ (read "cardinality of A") denotes the number of elements in A. The set B where $|B| = 0$ is unique, it is called the **empty set**, and it is denoted $\emptyset$. You should realize that for every set A, we have $\emptyset \in \mathcal{P}(A)$ and $A \in \mathcal{P}(A)$. If A is a set and B is a subset of A, then the set $A \backslash B$ is the set of all elements that are in A but not in B. It is called the **complement** of B in A.

Exercise 3. Prove by induction that if a set has n elements then it has 2^n subsets.

The **cartesian product** of A and B, denoted $A \times B$, is the set of *all* **pairs** (a, b) such that $a \in A$ and $b \in B$. The product $A \times B \times C$ is the set of all triples (a, b, c), with $a \in A, b \in B, c \in C$, and so on. $A \times A \times ... \times A$ (n times) is often denoted A^n and is the set of all "**n-tuples**" $(a_1, a_2, ..., a_n)$ where each entry in the tuple is an element of A. Familiarize yourself with $\mathbb{R}^n$, which is the set of n-tuples (sometimes called "vectors") of real numbers.

A (binary) **relation** R, over $A \times B$, is a subset of $A \times B$. We often write aRb to mean the same thing as $(a, b) \in R$.

A **function** f, over $A \times B$ (often denoted $f : A \to B$), is a relation that has the following property: if $(a, b) \in f$ and $(a, b') \in f$ then $b = b'$. We say that f "maps" the set A to the set B, so we will sometimes refer to a function as a "mapping." A is called the **domain** while B is called the **range** of f. The statement $(a, b) \in f$ is often written $f(a) = b$, which you are probably more familiar with. For $A' \subseteq A$, the set $\{b \in B : \exists a \in A' \text{ with } f(a) = b\}$ is called the **image** of A' under f and is denoted $f(A')$. The image of the domain A under f is called the **codomain** of f.

We say f is **surjective** if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$. We say that f is **injective** if $f(a) = f(a')$ implies $a = a'$. A function that is both injective and surjective is called **bijective**. Bijective functions are **invertible**, that is, given $b \in B$, there is a unique $a \in A$ such that $f(a) = b$. Functions that are not bijective are not invertible. (Please convince yourself that this is true.) If $f : A \to B$ is invertible, then there is a function $f^{-1} : B \to A$ such that for all $a \in A$, $f^{-1}(f(a)) = a$; f^{-1} is called the **inverse** of f. Notice that $f(f^{-1}(b)) = b$ for all $b \in B$.

Exercise 4. Consider the functions $f : A \to B$ and $g : B \to C$. The set

$$\{(a, g(f(a))) : a \in A\}$$

is read "the set of pairs $(a, g(f(a)))$ such that $a \in A$". Verify that this set is also a function. (*Hint:* Is it a relation? Over what? Does it satisfy the property that relations must satisfy to be functions?) We call such a function the **composition** of f and g and we denote the function as $g \circ f$. Its domain is A and range is C.

For any two sets, A and B, define their **union** by $A \cup B = \{x : x \in A \text{ or } x \in B\}$ and their **intersection** by $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Let $y > x$. Then $[x, y]$ is the set of all numbers between x and y, including both x and y. Alternatively, we can write $(x, y]$ to exclude x or $[x, y)$ to exclude y or (x, y) to exclude both x and y. Remember that you should not confuse the interval (x, y) with the pair (x, y): when this notation is used the context will make it clear which of these we refer to. All of these sets, $[x, y]$, $(x, y]$, $[x, y)$ and (x, y), which happen to be subsets of $\mathbb{R}$, are called **intervals**. $[x, y]$ is a **closed interval**, while (x, y) is an **open interval**; $[x, y)$ and $(x, y]$ are **half open** intervals.

1.3 Application: Rational Choice Theory

A **preference relation** on a set X, denoted by $\succeq$, is a binary relation on $X \times X$. We write $x \succeq y$ to say that the pair (x, y) belongs to the set $\succeq$.

A preference relation satisfies **completeness** if for all $x, y \in X$ either $x \succeq y$ or $y \succeq x$. It satisfies **transitivity** if $x \succeq y$ and $y \succeq z$ then $x \succeq z$ for all $x, y, z \in X$.

A decision maker who is confronted with a set of possible choices X is said to be **rational** if and only if she possesses a preference relation on X satisfying completeness and transitivity and would choose x over y only if $x \succeq y$. The decision maker's preference relation, $\succeq$, is also said to be rational. For such a decision maker with preference relation $\succeq$, we read $x \succeq y$ as saying "the decision maker weakly prefers x to y."

If $x \succeq y$ and $y \succeq x$ then we may write $x \sim y$, and read this as saying "the decision maker is indifferent between x and y," and so x and y are tied in the decision-maker's ranking. If $x \succeq y$ but it is not the case that $y \succeq x$ then we may write $x \succ y$ and read this as saying "the decision maker strictly prefers x to y." Note that both $\sim$ and $\succ$ are subsets of $\succeq$ that each satisfy transitivity. We refer to $\sim$ as the **indifference relation** (or symmetric part of $\succeq$) and $\succ$ as the **strict** preference (or asymmetric part).

Given a preference relation $\succeq$ over X, we say that a real valued function U defined on X **represents** $\succeq$ when

$$x \succeq y \text{ if and only if } U(x) \geq U(y).$$

A function U that represents a preference relation is called a **utility function**.

Theorem 2. *Suppose X is finite and $\succeq$ is a preference relation over X. There is a real valued function U on X that represents $\succeq$ if and only if $\succeq$ is rational.*

Proof. Suppose that U represents $\succeq$. For all $x, y \in X$, either $U(x) \geq U(y)$ or $U(y) \geq U(x)$, which implies that either $x \succeq y$ or $y \succeq x$. This shows that $\succeq$ satisfies completeness.

Also, $x \succeq y \succeq z$ implies $U(x) \geq U(y) \geq U(z)$, so $U(x) \geq U(z)$. Therefore, $x \succeq z$. This shows that $\succeq$ satisfies transitivity. Therefore, $\succeq$ is rational.

Now suppose $\succeq$ is rational. Let $U(x) = |\{z \in X : x \succeq z\}|$. We show that U represents $\succeq$. Suppose $x \succeq y$. If $y \succeq z$ then $x \succeq z$ since $\succeq$ is complete and transitive. Therefore $U(x) \geq U(y)$. Suppose next that not $x \succeq y$. Then $y \notin \{z \in X : x \succeq z\}$. Since $\succeq$ is complete, then $y \succeq x$. By transitivity $x \succeq z$ implies $y \succeq z$, so $U(y) \geq |\{z \in X : x \succeq z\} \cup \{y\}| = U(x) + 1$ so not $x \succeq y$ implies not $U(x) \geq U(y)$. □

Note that the argument that if U represents $\succeq$ on X then $\succeq$ is rational does not require the assumption that X is finite. We use the assumption only to prove the reverse—that if $\succeq$ is rational then there is a function u that represents $\succeq$ on X. If X is not finite, then a version of the above result holds under conditions whose statements require mathematics that is is outside the scope of this class.

Exercise 5. Let U represent $\succeq$ on a set X. Show that V also represents $\succeq$ on X if and only if there is a strictly increasing function $f : U(X) \to V(X)$ such that $f \circ U = V$.

1.4 Application: Social Choice Theory

Rational choice theory is about individual preferences and decision-making, but there is a corresponding theory of choice about group decision making that builds upon analogous axiomatic structure. We describe a couple results from this literature—one about social choice functions, and another about social preference relations.

Social Choice Functions There is a society of individuals $1, \ldots, n$ and a set of alternatives, A. Let $\mathscr{P}$ be the set of all strict preference relations over A. Each individual i has a complete, transitive (i.e. rational) and asymmetric (i.e. strict) preference relation $\succ_i$ over A. A **social choice function** is a function $\alpha : \mathscr{P}^n \to A$, where $\mathscr{P}^n$ denotes the set $\mathscr{P} \times \cdots \times \mathscr{P}$ (n times) of which $(\succ_1, \ldots, \succ_n)$ is a typical element. The social choice function α chooses an alternative for each profile of strict preferences.

Say that the social choice function α respects **monotonicity** if $\alpha(\succ_1, \ldots, \succ_n) = a$ and for all i and all $b \in A$, if $a \succ_i b$ implies $a \succ'_i b$, then $\alpha(\succ'_1, \ldots, \succ'_n) = a$. This means that if α selects a when the preference profile is $(\succ_1, \ldots, \succ_n)$ and for all individuals i, everything that is defeated by a when their ranking of alternatives is given by $\succ_i$ is also defeated by a when their ranking is given by $\succ'_i$, then α selects a when the preferences profile is $(\succ_1, \ldots, \succ_n)$.

Say that α respects **unanimity** if for all individuals i, $a \succ_i b$ for all $b \neq a$, then $\alpha(\succ_1, ..., \succ_n) = a$; i.e., when a is at the top of everyone's ranking, then α selects a.

Say that α is **dictatorial** if there is an individual j such that $\alpha(\succ_1, \ldots, \succ_n) = a$ if and only if $a \succ_j b$ for all $b \neq a$.

Theorem 3. (Mueller & Satterthwaite 1977) *Suppose $|A| \geq 3$ and the social choice function α respects monotonicity and unanimity. Then α is dictatorial.*

The theorem states that there is an individual j that is a dictator. To prove the theorem we proceed in two steps. Step 1 is to use the **Geanakoplos algorithm** to find the individual that we would like to accuse of being the dictator. Step 2 then proves that this individual is in fact a dictator.

Proof. Step 1– Consider a profile of strict preferences where a is ranked highest and b lowest for all individuals i. Unanimity implies that α gives a at this profile. Raise b one spot at a time in individual 1's ranking until it rises above a. Monotonicity implies that either b is the social choice or a is. (Why can't it be another alternative, c?) If a is still the social choice then move on to individual 2 and do the same thing: raise b from the bottom until it rises above a. As you do this, monotonicity says that a is the social choice except possibly just as when b rises above it. Keep doing this and move across individuals until you hit individual $k \leq n$ for whom raising b above a makes b the social choice for that particular configuration of strict preference relations. (We know $k \leq n$ because by the time $k = n$, unanimity tells us that b would have to be chosen.)

Step 2– We show that k is a dictator. Consider the following two preference profiles generated by the Geanakoplos algorithm: $\mathbf{P}^1$, where b is at the top of the ranking for $i < k$, just below for a for $i = k$ and at the bottom for $i > k$; and $\mathbf{P}^2$ which is otherwise the same as $\mathbf{P}^1$ except that b is at the top for $i = k$ as well. These are supposed to depict the "just before" and "just after" situations where the social choice switches from a to b. To construct $\mathbf{P}^3$, take $\mathbf{P}^2$ and lower a to the very bottom for $i < k$ and to only just above b for $i > k$, leaving it unmoved for $i = k$. By monotonicity, b is still the social choice in $\mathbf{P}^3$. If we constructed $\mathbf{P}^4$ from $\mathbf{P}^1$ in just the same way, the social choice would still be either b or a since $\mathbf{P}^3$ and $\mathbf{P}^4$ differ only in how k ranks a and b, which in his ranking are adjacent to each other. (This means that the social choice cannot be some other alternative c.) But if the social choice in $\mathbf{P}^4$ is b, then the social choice in $\mathbf{P}^1$ would have to be b, by monotonicity. But the social choice in $\mathbf{P}^1$ is a. So, the social choice in $\mathbf{P}^4$ must be a.

Now consider $\mathbf{P}^5$ constructed from $\mathbf{P}^4$ first lowering b to just above a for all $i < k$, then taking a third alternative $c \neq a, b$ and lowering it just above b for $i < k$, placing it between a and b for $i = k$ and just above a for $i > k$. Since the social choice in

1	...	$k-1$	k	$k+1$	...	n
b	...	b	a	a	...	a
a	....	a	b	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	$\vdots$	b	...	b
$\boldsymbol{a}$						$\mathbf{P}^1$

$\downarrow$

1	...	$k-1$	k	$k+1$	...	n
b	...	b	b	a	...	a
a	....	a	a	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	$\vdots$	b	...	b
$\boldsymbol{b}$						$\mathbf{P}^2$

$\downarrow$

1	...	$k-1$	k	$k+1$	...	n
b	...	b	a	$\vdots$	...	$\vdots$
$\vdots$	....	$\vdots$	b	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	$\vdots$	a	...	a
a	...	a	$\vdots$	b	...	b
b	or	$\boldsymbol{a}$				$\mathbf{P}^4$

1	...	$k-1$	k	$k+1$	...	n
b	...	b	b	$\vdots$	...	$\vdots$
$\vdots$	....	$\vdots$	a	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	$\vdots$	a	...	a
a	...	a	$\vdots$	b	...	b
$\boldsymbol{b}$						$\mathbf{P}^3$

$\downarrow$

1	...	$k-1$	k	$k+1$	...	n
$\vdots$	...	$\vdots$	a	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	c	$\vdots$	...	$\vdots$
$\vdots$	....	$\vdots$	b	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
c	...	c	$\vdots$	c	...	c
b	...	b	$\vdots$	a	...	a
a	...	a	$\vdots$	b	...	b
$\boldsymbol{a}$						$\mathbf{P}^5$

$\rightarrow$

1	...	$k-1$	k	$k+1$	...	n
$\vdots$	...	$\vdots$	a	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	c	$\vdots$	...	$\vdots$
$\vdots$	....	$\vdots$	b	$\vdots$	...	$\vdots$
$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
c	...	c	$\vdots$	c	...	c
b	...	b	$\vdots$	b	...	b
a	...	a	$\vdots$	a	...	a
b	or	$\boldsymbol{a}$				$\mathbf{P}^6$

$\mathbf{P}^4$ was a and the relative ranking of a against any other alternative was not changed when we constructed $\mathbf{P}^5$, the social choice in $\mathbf{P}^5$ must, by monotonicity, also be a. Now construct $\mathbf{P}^6$ from $\mathbf{P}^5$ by interchanging the spots of a and b for all individuals $i > k$. By monotonicity, the social choice is either a or b. But the social choice cannot be b since c is higher than b in every individual's ranking and monotonicity would imply that b is still the choice if c is raised to the top of everyone's ranking. This would contradict UNA. Thus, the social choice in $\mathbf{P}^6$ must be a.

Now, any profile of rankings with a at the top of individual k's ranking can be constructed from $\mathbf{P}^6$ with monotonicity requiring that a is the social choice in each of these arbitrarily constructed profiles. Thus, a is the social choice whenever it is at the top of individual k's ranking. Since a was arbitrary we have shown that for any alternative, there is an individual that whenever that alternative is at the top of the said individual's ranking, that alternative is the unique social choice. But it would be a contradiction of α being a function if there was any such individual besides k for any other alternative. Thus k is a dictator. $\square$

Social Preference Relations Again there are n individuals. $\mathscr{P}_*$ is the set of rational preference relations $\succeq_i$ over a set of alternatives A. A function $\varphi : \mathscr{P}^n \to \mathscr{P}_*$ is a **social preference function**, mapping preference profiles to a preference order for society. Let $\succeq_{\varphi(\succ)} = \varphi(\succ)$ for $\succ = (\succ_1, ..., \succ_n)$ with asymmetric part denoted $\succ_{\varphi(\succ)}$.

A social preference function φ respects the **Pareto** principal if whenever $a \succ_i b$ for all individuals i, $a \succ_{\varphi(\succ)} b$. In general, we will say that two alternatives (sometimes also called "outcomes") $a, b \in A$ are **Pareto ranked** if there is some individual i for whom $a \succ_i b$, and for all other individuals j it is *not* the case that $b \succ_j a$. When this is the case we say that a "Pareto ranks better" than b. We say that an alternative $a \in A$ is **Pareto optimal** if there is no alternative that Pareto ranks better than it.

We say that the social preference function φ respects the **independence from irrelevant alternatives** (IIA) principal if, informally speaking, whenever the ranking of a and b is unchanged for all individuals i when the preference profile changes from $\succ = (\succ_1, ..., \succ_n)$ to $\succ' = (\succ'_1, ..., \succ'_n)$, the ranking of a and b by $\varphi(\succ')$ is the same as the ranking of a and b by $\varphi(\succ)$. This can be formalized. (I encourage you to do it.)

In this context we say that a social preference function φ is **dictatorial** if there exists an individual i such that $\varphi(\succ)$ always ranks any two alternatives, a and b, the same was that i ranks the two alternatives. This definition can also be formalized.

Theorem 4. (Arrow 1951) *Suppose $|A| \geq 3$, and the social preference function φ respects both the Pareto principal and the IIA principal. Then φ is dictatorial.*

Proof. Philip Reny ["Arrow's theorem and the Gibbard-Satterthwaite theorem: A unified approach," *Economics Letters*, Vol. 70, No. 1, 2001, pp. 99-105] shows that the proof of this result has the same structure as the proof of the Mueller-Satterthwaite theorem above. Please see his paper for the proof. □

Exercise 6. Suppose there are three policy-makers in government, 1, 2 and 3, who must decide which of the following projects to spend government money on: a new security measure s, health-care h, and education, e. The set of choices is $C = \{s, h, e\}$, and only one project can get chosen. 1 has a strict preference relation over C, denoted $\succ_1$. 2's strict preference is $\succ_2$, and 3's is $\succ_3$. Assume $e \succ_1 h$ and $h \succ_1 s$: 1 prefers to spend on education than to spend on health-care, and she also prefers spending on health-care to spending on security. Next, assume $h \succ_2 s$ and $s \succ_2 e$. Finally, let $s \succ_3 e$ and $e \succ_3 h$. Find a strict preference relation for the government, $\succ_G$, such that if the majority of policy-makers (at least two out of the three) strictly prefers alternative a to b then government prefers a to b (i.e. $a \succ_G b$). If you can't find one, what is the problem? And does the problem necessarily go away if you have more policy-makers?