6 Optimization

The **interior** of a set $S \subset \mathbb{R}^n$ is the set

$$\text{int } X = \{x \in S : \exists \text{ an open box } B \text{ such that } x \in B \subset S\}$$

Similarly, the **boundary** of S, denoted ∂S, is the set

$$\partial S := \{x \in \mathbb{R}^n : \text{every open box } B \text{ s.t. } x \in B \text{ contains points } y \in S \text{ and } z \notin S\}.$$

The following result is intuitive, and I encourage you to translate your intuition for why it must be true into a proof. In fact, you can do it in an exercise.

Theorem 29. *Suppose $S \subset \mathbb{R}$ and suppose that $f : S \to \mathbb{R}$ is differentiable function on the interior of S. If f achieves a maximum or minimum in the interior of S then $f'(x) = 0$ at the point x at which it achieves a maximum or minimum. If $x \in \text{int } S$ is a maximum then $f''(x) \leq 0$, and if it is a minimum then $f''(x) \geq 0$.*

Note that $f'(x) = 0$ and $f''(x) \leq 0$ are only necessary conditions for $x \in \text{int } S$ being a maximum. This exercise demonstrates how they are insufficient.

Exercise 55. Let $f : [-2, 5] \to \mathbb{R}$ be defined by $f(x) = x^3 - 4.5x + 6x + 1$. By the Weierstrass theorem, f achieves a maximum and minimum on its domain. We can find the values of x where $f'(x) = 0$. Which of these are maximum values of f and minimum values of f? Calculate $f(x)$ at points where $f'(x) = 0$. Then calculate $f(-1)$ and $f(4)$.

What are the issues with sufficiency? Part of the problem is that of "global" versus "local" minima and maxima. A **global** maximum of a real-valued function f defined on $S \in \mathbb{R}^n$ is a point $x \in X$ such that $f(x) \geq f(y)$ for all $y \in S$. A **local** maximum of the same function is a point $x \in S$ such that there exists an open box $B \subset S$ that contains x for which $f(x) \geq f(y)$ for all $y \in B$. Obviously, global maxima are local but not the other way around.

The problem of sufficiency does not arise in the one-dimensional case, $S \subset \mathbb{R}$, when S is a closed and convex set (i.e., closed interval) and f is either concave or convex.

Theorem 30. *If f is a concave (resp. convex) real-valued function on some closed interval $S \subset \mathbb{R}$ and $f'(x) = 0$ for some $x \in \text{int } S$, then x is a global maximum (resp. minimum) of f on S. If f is strictly concave (resp. strictly convex) then it is the unique global minimum (resp. maximum) of f on S. If f is differentiable on int S but there is no x such that $f'(x) = 0$, then the extrema (maxima and minima) of f lie on ∂S.*

Exercise 56. Prove (or convince yourself) that the two theorems above are true.

6.1 Unconstrained Optimization

Let's extend the ideas above to the case where f is a real-valued function defined on a set $S \subset \mathbb{R}^n$ that is closed and convex. As before, we say that f is **concave** if $\forall x \in S$ and $\forall y \in S$ such that $y \neq x$ we have,

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y) \quad \forall \alpha \in (0,1); \tag{70}$$

and f is **strictly concave** if in the inequality above, you replace $\geq$ with just $>$. It is **convex** if the function $-f$ is concave and **strictly convex** if $-f$ is strictly concave.

Exercise 57. Continue to let $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}^n$ is a closed and convex set. Prove that f is concave if and only if the function $g(t) = f(x+tz)$ is concave on the set $T = \{t \in \mathbb{R} \mid x + tz \in S\}$, where $x \in S$, $z \in S$ and $z \neq 0_n$.

Before proceeding any further, I will define an important concept. An $n \times n$ matrix A is said to be **negative semi-definite** if for all $x \in \mathbb{R}^n$, we have $x'Ax \leq 0$. If the inequality is strict for all $x \neq 0_n$ then A is **negative definite**. A is **positive definite** if $-A$ is negative definite and **positive semi-definite** if $-A$ is negative semi-definite.

Lemma 5. (**Multidimesnional Concavity Lemma**) *Consider the function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$ is convex and has a nonempty interior. Assume that f has a Hessian where all the entries are continuous. Then the following three statements are equivalent: (i) f is concave, (ii) $Hf(x)$ is negative semi-definite for all $x \in S$, and (iii) $f(x) \leq f(y) + \nabla f(y)(x-y)$ for all $x, y \in S$. In addition, if $Hf(x)$ is negative definite for all $x \in S$ then f is strictly concave.*

Proof. I demonstrate only that (i) and (ii) are equivalent, as well as the additional claim about strict concavity. I first show that (i) implies (ii), and hope that you are convinced that it is enough to establish the claim on the interior of S since continuity takes care of the boundary points. Now let g be the function and T be the set both defined for Exercise 57. Notice that

$$g'(t) = \nabla f(x+tx) \cdot z = \sum_{i=1}^{n} \frac{\partial f(x+tx)}{\partial x_i} z_i \tag{71}$$

because $g(t) = f(x+tz)$ implies that

$$\lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h \to 0} \frac{f(x+tz+hz) - f(x+tz)}{h} \tag{72}$$

and the right hand side is the definition of directional derivative of f at $x + tz$ in the direction of z. (Remember that that is $\nabla f(x) \cdot z$.) Now take the second derivative of g knowing that that is the directional derivative of each of the partial derivatives of f added:

$$g''(t) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f(x + tz)}{\partial x_j x_i} z_i z_j. \tag{73}$$

But this is just the equation $g''(t) = z' Hf(x+tz)z$. Now f is concave, which by Exercise 57 means that g is concave, and because z is arbitrary (as long as we are in the set T), that means that $z' Hf(x+tz)z \leq 0$, i.e. Hf is negative semi-definite since the argument holds for any x and any z such that $x + tz$ is in the domain of f. Proving (ii) implies (i) is simply writing the argument backwards, and the claim about strict concavity is merely replacing the $\leq$ sign by the $<$ sign in that proof. □

Exercise 58. Complete the proof of the multidimensional concavity lemma.

A **critical point** of a real-valued function f defined on a set $S \subset \mathbb{R}^n$ is a vector $x \in \text{int}\, S$ such that $\nabla f(x) = 0_n$.

Theorem 31. (**First Order Conditions Theorem**) *If the differentiable function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$, reaches a local interior extremum (i.e., maximum or minimum) at x^* then x^* is a critical point of f.*

Proof. Again write $f(x^* + tz)$ in terms of the familiar g of Exercise 52 (yes, this time x^* instead of x). Note that $g(0) = f(x^*)$. Because g coincides with some value of f for every t, then $g(t)$ must reach a local extremum at $t = 0$ if f reaches an extremum at x. This means that $g'(0) = 0$. Now recall from the proof of the multidimensional concavity lemma why it should be that (71) should hold with x replaced by x^*. Combined with $g'(0) = 0$, this implies that $\nabla f(x^*)z = 0_n$. Since this holds for any z, it must hold for all $e_j, j = 1, \ldots, n$. Therefore, each of the partials are 0, and we have $\nabla f(x^*) = 0_n$. □

Theorem 32. (**Second Order Conditions Theorem**) *If the C^2 function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$, reaches a local interior maximum (resp. minimum) at x^*, then $Hf(x^*)$ is negative (resp. positive) semidefinite.*

Proof. Recall that (73) must be true. Then if f is maximized at x it must be that g is maximized at $t = 0$, and thus $g''(0) \leq 0$. That means that the right hand side of (73) is nonpositive, or that the Hessian of f at x is negative semidefinite. (This is again because z is arbitrary.) A similar argument holds for the "minimized" case. □

Theorem 33. (**Global Maximum Theorem**) *If the C^2 function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$, has a negative semidefinite Hessian at every x in the interior of S, then if x^* is a critical point of f, the function f achieves a global maximum at x^*.*

Proof. Because the Hessian is everywhere negative semidefinite, f is concave by the multidimensional concavity lemma. The same lemma tells us that $f(x) \leq f(x^*) + \nabla f(x^*)(x - x^*)$ for all $x \in S$. Now if $\nabla f(x^*) = 0_n$ then $f(x) \leq f(x^*)$ for all $x \in S$. So f reaches a global maximum. □

A similar theorem holds for global minima. However, in most applications if what you are using the first and second order conditions to find interior minima of f, you can do this by finding interior maxima of $-f$.

Exercise 59. Re-read the proof of the global maximum theorem. Then prove that if the Hessian of f is negative definite at every x in its domain, and x^* maximizes f, then x^* is the *unique* global maximizer of f.

Now here is a useful result to check the definiteness of the Hessian. To state the result, I'll define a **submatrix** of a matrix A informally to be a matrix that is a portion of A. There is no need to be more formal; you'll understand what I mean.

Theorem 34. *Let $D_1(x)$ be the determinant of the top left 1×1 submatrix of the $n \times n$ matrix $Hf(x)$, $D_2(x)$ be the determinant of the top left 2×2 submatrix of $Hf(x)$,..., and $D_k(x)$ be the determinant of the top left $k \times k$ submatrix of $Hf(x)$. If $(-1)^i D_i(x) > 0$ for all $i = 1, \ldots, n$ then $Hf(x)$ is negative definite, and if this is true for all x in the domain of f then f is strictly concave. If $D_i(x) > 0$ for all $i = 1, \ldots, n$ then $Hf(x)$ is positive definite, and if this is true for all x in the domain of f then f is strictly convex.*

Exercise 60. Read and understand the theorem above.

Exercise 61. Refer back to the application called "The Algebra of Least Squares." Review it and prove that the $(k+1)$-dimensional vector of partial derivatives of $S(\beta)$ with respect to the components of the vector of variables $\beta = (m_1, ..., m_k, b)$, which we abbreviate as $\partial S/\partial\beta$, is given by

$$\frac{\partial S(\beta)}{\partial \beta} = -2y'X + 2X'X\beta.$$

Then show that the value of β that minimizes $S(\beta)$ is the value of β that solves

$$0_n = \frac{\partial S(\beta)}{\partial \beta}.$$

Finally, compute the value of β that minimizes $S(\beta)$ as a function of the data only, i.e. as a function of $(\bar{X}, y)$.

6.2 Equality-Constrained Optimization

Let $f, g_1, ...,$ and g_n be C^1 functions defined on some set $S \subset \mathbb{R}^n$. The problem we are interested in is the problem of maximizing a function f (or, equivalently, minimizing $-f$) subject to the constraints that $g_i(x) = 0$. That is, we are looking for a point $x \in S$ that maximizes f on the set

$$\mathcal{D} := \{x \in S : g_i(x) = 0, \forall i\}.$$

We write the problem as

$$\max_{x \in S} f(x) \qquad \text{subject to} \qquad g_i(x) = 0, \qquad \forall i \tag{P1}$$

Alternatively, we could have written

$$\max_{x \in \mathcal{D}} f(x) \tag{74}$$

The following theorem gives us some ideas about how we may search for a solution.

Theorem 35. (**Lagrange's Theorem**) *Let $S \subset \mathbb{R}^n$. Let $f : S \to \mathbb{R}$ and $g_i : S \to \mathbb{R}, i = 1, \dots, k$, be C^1 functions. Let x^* be a point in the interior of S and suppose that x^* is an optimum (local maximum or local minimum) of f subject to the constraints, $g_i(x) = 0, i = 1, \dots, k$. If the gradient vectors $\nabla g_i(x^*), i = 1, \dots, k$, are linearly independent then there exists a vector $\Lambda = (\lambda_1, \dots, \lambda_k)' \in \mathbb{R}^k$ such that*

$$\nabla f(x^*) + \sum_{i=1}^{k} \lambda_i \nabla g_i(x^*) = 0 \tag{75}$$

This theorem is kind of important, so let me provide some intuition and warnings before giving the proof. Recall that int S denotes the interior of S and write the problem as

$$\max_{x \in \text{int } S} f(x) \text{ subject to } g_i(x) = 0, i = 1, \dots, k.$$

Now define the so-called **Lagrangean** for this problem,

$$\mathcal{L}(x, \Lambda) := f(x) + \sum_{i=1}^{k} \lambda_i g_i(x),$$

and apply the first order conditions theorem to $\mathcal{L}$ while deleting the last equation:

$$\nabla_n \mathcal{L}(x, \Lambda) = \nabla f(x) + \sum_{i=1}^{k} \lambda_i \nabla g_i(x) = 0, \tag{76}$$

where $\nabla_n \mathcal{L}$ simply means the vector with only the first n entries of usual gradient of $\mathcal{L}$, since differentiating with respect to Λ gives us back the constraints. Thus the information that we discard was something we already knew.

Now we have to be convinced of two things. First, we must be convinced that the solution to the maximization problem, x is a vector of the first n entries of a critical point of $\mathcal{L}(x, \Lambda)$. To convince yourself, understand that the directional derivative $\nabla f(x) \cdot h = 0$ at the maximum for all small length vectors h that take x to points in which the constraints g_i continue to be satisfied. If the g_i continue to be satisfied after moving a small amount in the direction h, then no change occurs in any g_i, i.e. $\nabla g_i \cdot h = 0$ for all i. But that means that $\nabla_n \mathcal{L}(x, \Lambda) \cdot h = \nabla f(x) \cdot h$ for any movement h that keeps the constraints satisfied. This says that you cannot increase or decrease the objective function f by making small movements in any "permissible" direction. So we must be at a critical point of $\mathcal{L}$ (except that we don't know the λ_i). Suppose we were at a maximum or minimum at x. Then the constraints would be satisfied and so would (75).

The second thing that we must convince ourselves is that the vector Λ exists. That's a little harder, and you will have to wait until we see the proof of the theorem.

Caution with Lagrange Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = -y$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be the function $g(x, y) = y^3 - x^2$. Notice that the maximum of f subject to $g(x, y) = 0$ is at $(0, 0)$, since if y is negative x would have to be the square root of a negative number for the constraint to be met. If you set up the Lagrangean, $\mathcal{L}$ of this problem (defined in the previous lecture note) and take the first order conditions then you get

$$\begin{aligned} -2\lambda x &= 0 \\ -1 + 3\lambda y^2 &= 0 \\ -x^2 + y^3 &= 0 \end{aligned}$$

Now for the first equation to be true, either $\lambda = 0$ or $x = 0$. If $\lambda = 0$ then the second equation is a contradiction. If $x = 0$ then the third implies that $y = 0$. Plug this back into the second and obtain a contradiction again.

Exercise 62. What went wrong in the example above?

Now suppose that f is defined by

$$f(x, y) = \frac{1}{3}x^3 - \frac{3}{2}y^2 + 2x \tag{77}$$

instead, and g was the benign $g(x,y) = x - y$. The constraint qualification is met since $\nabla g(x,y) = (1,-1)$ for all x and y. Now set up the Lagrangean and take the first order conditions:

$$x^2 + 2 + \lambda = 0$$
$$-3y - \lambda = 0$$
$$x - y = 0$$

There are two solutions: $(x,y) = (2,2)$ or $(1,1)$, and seeing that $f(2,2) = 2/3$ while $f(1,1) = 5/6$ you could guess that $(2,2)$ is a minimum and $(1,1)$ is a maximum. But in fact, $f(0,0) = 0$ and $f(3,3) = 1.5$.

Exercise 63. What went wrong here?

I now give the proof of Lagrange's theorem.

Proof of Theorem 35. I prove the statement only for $k = 1$, noting that the general result follows a very similar argument. Let $g(x) = g_1(x)$ so that we can drop the subscript from now on. Let the local optimum be x^*. The rank condition on g tells us that $\nabla g(x^*) \neq 0_n$. Without loss of generality, assume that the first component of this vector is nonzero. Denote the first coordinate of a vector $x \in \mathcal{D}$ by w and the last $n-1$ of them by z. (Recall the definition of $\mathcal{D}$ from above.) Write $x = (w,z)$. Let $x^* = (w^*, z^*)$ denote the local optimum. Let $\nabla f_w(w,z)$ denote the derivative of f with respect to w alone and $\nabla f_z(w,z)$ the derivative with respect to z alone. The derivative of g is partitioned analogously into a number, $\nabla g_w(w,z)$, and a vector $\nabla g_z(w,z)$ of size $n-k$.

To prove the theorem we must show that there exists $\lambda \in \mathbb{R}$ such that

1. $\nabla f_w(w^*, z^*) + \lambda \nabla g_w(w^*, z^*) = 0$
2. $\nabla f_z(w^*, z^*) + \lambda \nabla g_z(w^*, z^*) = 0$

To show this, we have to use the implicit function theorem (Theorem 18). This says that there is an open box $B \in \mathbb{R}^{n-1}$ containing z^* and a C^1 function $h : B \to \mathbb{R}$ such that $h(z^*) = w^*$ and $g(h(z), z) = 0$ for all $z \in B$. Also,

$$\nabla h(z) = -\frac{\nabla g_z(h(z), z)}{\nabla g_w(h(z), z)} \tag{78}$$

which is none other than the formula in the implicit function theorem.

Define λ now as

$$\lambda = -\frac{\nabla f_w(w^*, z^*)}{\nabla g_w(w^*, z^*)}$$

which rearranges to

$$\nabla f_w(w^*, z^*) + \lambda \nabla g_w(w^*, z^*) = 0.$$

That's the first thing we had to show, which is a bit simpler than the second. Define the function $\phi : B \to \mathbb{R}$ by $\phi(z) = f(h(z), z)$. Since f has a local optimum at $(w^*, z^*) = (h(z^*); z^*)$, then ϕ has a local optimum at z^*. Since B is open, z^* is an unconstrained local optimum of ϕ and the first-order conditions for an unconstrained optimum imply $\nabla\phi(z^*) = 0_{n-1}$, i.e. by the chain rule:

$$\nabla f_w(w^*, z^*)\nabla h(z^*) + \nabla f_z(w^*, z^*) = 0. \tag{79}$$

Substitute (78) in this to get

$$\nabla f_z(w^*, z^*) + \lambda \nabla g_z(w^*, z^*) = 0.$$

and that's it. □

Exercise 64. We did not prove the chain rule appearing in (79) in the multidimensional case. In the case of two variables, let $x(t)$, $y(t)$ be two differentiable functions of t and let $f(x, y)$ be a differentiable function. For the purposes of this demonstration, define $\partial x = x(t + h) - x(t)$ and $\partial y = y(t + h) - y(t)$. Then

$$\begin{aligned}
f'(x(t), y(t)) &= \lim_{h\to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t))}{h} \\
&= \lim_{h\to 0} \frac{f(x + \partial x, y + \partial y) - f(x, y + \partial y) + f(x, y + \partial y) - f(x, y)}{h} \\
&= \lim_{h\to 0} \frac{f(x + \partial x, y + \partial y) - f(x, y + \partial y)}{h} + \lim_{h\to 0} \frac{f(x, y + \partial y) - f(x, y)}{h}
\end{aligned}$$

On the right is the definition of the partial of f with respect to t through y, which by the single variable chain rule is

$$\frac{\partial f}{\partial y}\frac{dy}{dt}$$

Apply the mean value theorem to the limit on the left by picking an $x' \in [x, x + \partial x]$ such that the limit is equal to

$$\lim_{h\to 0} \frac{\partial x}{h}\frac{\partial f(x')}{\partial x} = \frac{\partial f}{\partial x}\frac{dx}{dt}.$$

That gives us

$$f'(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Use an extended argument based on this demonstration to argue that (79) is true.

Second Order Conditions Let us make a few modifications to Lagrange's Theorem. Instead of assuming that f and the g_i are C^1, let us assume that they are all C^2 since we will want to take their second derivatives. Let us also assume that S is an open set so that it is equal to its interior. It will be useful now to state the problem as

$$\max_{x \in \mathcal{D}} f(x),$$

(or the corresponding minimization; recall how $\mathcal{D}$ is defined above.). The Lagrangean for the problem is

$$\mathcal{L}(x, \Lambda) = f(x) + \sum_{i=1}^{k} \lambda_i g_i(x).$$

Taking the second derivative we get

$$H\mathcal{L}(x, \Lambda) = Hf(x) + \sum_{i=1}^{k} \lambda_i H g_i(x).$$

Let (x^*, Λ^*) denote a critical point of $\mathcal{L}(x, \Lambda)$. For simplicity of notation let $A = H\mathcal{L}(x^*, \Lambda^*)$ and $B = [\nabla g_1 1(x^*), \ldots, \nabla g_k(x^*)]'$. In other words B is the $k \times n$ matrix created by putting all of the vectors $\nabla g_i(x^*)$ together into columns (in order), and then taking the transpose.

Theorem 36. (**Lagrange's Second Order Conditions Theorem**) *Let (x^*, Λ^*) be a critical point of $\mathcal{L}(x, \Lambda)$ and suppose that B has rank k (that's the same as saying the $\nabla g_i(x^*)$ are linearly independent). Let*

$$\mathcal{Z}(x^*) = \{z \in \mathbb{R}^n : Bz = 0_k\}.$$

Then

1. *If f has a local maximum at $x^* \in \mathcal{D}$, then $z'Az \leq 0$ for all $z \in \mathcal{Z}(x^*)$.*

2. *If f has a local minimum at $x^* \in \mathcal{D}$, then $z'Az \geq 0$ for all $z \in \mathcal{Z}(x^*)$.*

3. *If $z'Az < 0$ for all $z \in \mathcal{Z}(x^*) - \{0_n\}$, then $x^* \in \mathcal{D}$ is a local maximum of f.*

4. *If $z'Az > 0$ for all $z \in \mathcal{Z}(x^*) - \{0_n\}$, then $x^* \in \mathcal{D}$ is a local minimum of f.*

This is not much different from the second order conditions theorem for the unconstrained case, except now we can't talk of A being negative or positive semidefinite since the permissible zs are restricted to directions where the constraints are still satisfied. That is, we must have $\nabla g_i(x^*) \cdot z = 0$, which is to say that the value of the constraint doesn't change along z. (And if that starting value was 0, then it remains 0...)

How do you test any of the conditions on the matrix A in any of the four proposition above? For $l \leq n$, let A_{ll} denote the top left $l \times l$ submatrix of A and B_{kl} the top left $k \times l$ submatrix of B. Now let π denote a permutation, i.e. a jumbling up of the order of numbers $1, \ldots, n$ and let A^π denote the matrix resulting from jumbling up the order of rows of A, and the order of columns in exactly the same way as the rows were jumbled. Now convince yourself that A^π is also a symmetric matrix like the Hessian, A. Let B^π denote the $k \times n$ matrix obtained by jumbling only the columns of B in the same way that the rows and columns of A were jumbled. As before let A^π_{ll} denote the top left $l \times l$ submatrix of A^π, and B^π_{kl} the top left $k \times l$ submatrix of B^π. Now define C_l to be the $k + l \times k + l$ matrix obtained:

$$C_i = \begin{bmatrix} 0_{nn} & B_{kl} \\ (B_{kl})' & A_{ll} \end{bmatrix}$$

where 0_{nn} denotes the $n \times n$ matrix of 0s in every entry. And, define C^π_l analogously where A_{ll} is replaced by A^π_{ll} and B_{kl} by B^π_{kl} (and its transpose...). Assume that $\det B_{kk} \neq 0$, which we can do without loss of generality since the rank of B is k.

Theorem 37. (**Lagrange Characterization Theorem**). *The following are true.*

1. *$x'Ax \geq 0$ for every x that $Bx = 0$ if and only if for all permutations π of the first n integers, and for all $r \in \{k+1, \ldots, n\}$, you got $(-1)^k \det C^\pi_r \geq 0$.*

2. *$x'Ax \leq 0$ for every x that $Bx = 0$ if and only if for all permutations π of the first n integers, and for all $r \in \{k+1, \ldots, n\}$, you got $(-1)^r \det C^\pi_r \geq 0$.*

3. *$x'Ax > 0$ for all $x \neq 0$ that $Bx = 0$ if and only if for all $r \in \{k+1, \ldots, n\}$, we have $(-1)^k \det C^\pi_r > 0$.*

4. *$x'Ax < 0$ for all $x \neq 0$ that $Bx = 0$ if and only if for all $r \in \{k+1, \ldots, n\}$, we have $(-1)^r \det C^\pi_r > 0$.*

6.3 Inequality-Constrained Optimization

In many applications, the constraints in a constrained optimization problem take the form of inequalities rather than equalities. For example, "maximize welfare subject to not spending more than a certain budget." The constraint is that spending must be smaller than or equal to the budget; the optimum may be at a point where the entire budget is spent, or it may not. Then, inheriting the notation of the previous section, we consider the problem

$$\max_{s \in S} f(x) \qquad \text{subject to} \qquad g_i(x) \geq 0, \qquad \forall i \tag{P2}$$

Theorem 38. (**The Karush-Kuhn-Tucker Theorem**) *Let $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}$ and $g_i : S \to \mathbb{R}$, $i = 1, \ldots, k$, be C^1 functions. Let x^* be a point in the interior of S and suppose that x^* is an optimum (local maximum or local minimum) of f subject to the constraints, $g_i(x) \geq 0$, $i = 1, \ldots, k$. If the gradient vectors $\nabla g_i(x^*)$, $i = 1, \ldots, k$, are linearly independent then there exists a vector $\Lambda = (\lambda_1, \ldots, \lambda_k)' \in \mathbb{R}^k$ such that*

$$\nabla f(x^*) + \sum_{i=1}^{k} \lambda_i \nabla g_i(x^*) = 0 \text{ and } \lambda_i g_i(x^*) = 0 \text{ for all } i = 1, \ldots, k.$$

In addition, λ_i, $i = 1, ..., k$, are all nonnegative if x^* is a maximum and nonpositive if it is a minimum.

This one I'm not going to prove, but I will give some intuition for it. Begin with the simple problem

$$\max_{x} f(x) \text{ subject to } x \geq 0$$

and notice that if x^* is the solution then it satisfies one of the following three cases:

1. $x^* = 0$ and $f'(x^*) < 0$,
2. $x^* = 0$ and $f'(x^*) = 0$,
3. $x^* > 0$ and $f'(x^*) = 0$.

These imply that

1. $f'(x^*) \leq 0$,
2. $x^*[f'(x^*)] = 0$,
3. $x^* \geq 0$.

In the world of $\mathbb{R}^n$, these correspond to

1. $\frac{\partial f(x^*)}{\partial x_i} \leq 0$,

2. $x_i^* \left[\frac{\partial f(x^*)}{\partial x_i}\right] = 0$

3. $x_i^* \geq 0$,

which must hold for all $i = 1, \ldots, n$ if x^* maximizes $f(x)$ subject to $x_i \geq 0$ for all i. Now convince yourself that the problem

$$\max_{x_1,x_2} f(x_1, x_2) \text{ subject to } g(x_1, x_2) \geq 0,$$

is equivalent to

$$\max_{x_1,x_2,z} f(x_1, x_2) \text{ subject to } g(x_1, x_2) - z = 0 \text{ and } z \geq 0.$$

Take the first order conditions of the Lagrangian, $\mathcal{L}$ of this latter beast while ignoring the inequality constraint, and arrive at

$$\begin{aligned} \frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} &= 0 \\ \frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} &= 0 \\ \frac{\partial f}{\partial \lambda} = g(x_1, x_2) - z &= 0 \end{aligned}$$

These give the critical points of $\mathcal{L}$. The only additional necessary conditions come from the inequality constraint $z \geq 0$. The insight from the previous discussion tells us the equivalent of properties (1) – (3) for z, i.e that:

$$\begin{aligned} -\lambda &\leq 0 \\ z(-\lambda) &= 0 \\ z &\geq 0 \end{aligned}$$

Summarizing, we have

$$\begin{aligned} \frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} &= 0 \\ \frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} &= 0 \\ \lambda g(x_1, x_2) &= 0 \\ \lambda &\geq 0 \\ g(x_1, x_2) &\geq 0 \end{aligned}$$

These are called **Kuhn-Tucker conditions**.

Exercise 65. Re-read the cautionary examples for Lagrange and consider the problem of finding (x, y) to maximize $f(x, y) = -(x^2 + y^2)$ subject to $h(x, y) = (x-1)^3 - y^2 \geq 0$. Find a solution to this problem just by looking at the functions. (You don't have to set up any Lagrangean or Karush-Kuhn-Tucker conditions). Now show that this problem cannot be analyzed using the Karush-Kuhn-Tucker theorem. Which one of the assumptions is violated?

The exercise above demonstrates that the same caution should be taken when applying the Karush-Kuhn-Tucker theorem as should be taken when applying Lagrange's theorem. However, the following useful theorem should ease some of your fears.

Theorem 39. (**Sufficient Conditions Kuhn-Tucker Theorem**) *Let $S \subset \mathbb{R}^n$ be an open set and $f : S \to \mathbb{R}$ be a concave C^1 function and $g_i : S \to \mathbb{R}$, $i = 1, \ldots, k$, be C^1 functions such that*

$$\mathcal{D} = \{x \in S : g_i(x) \geq 0, \ \forall i\}$$

is a convex set. If $x^* \in X$ and there are numbers $\lambda_1, \ldots, \lambda_k \geq 0$ such that

$$\nabla f(x^*) + \sum_{i=1}^{k} \lambda_i \nabla g_i(x^*) = 0 \text{ and } \lambda_i g_i(x) = 0 \text{ for all } i = 1, \ldots, k,$$

then x^* solves the program

$$\max_{x \in \mathcal{D}} f(x).$$

Proof. Suppose the theorem isn't true, i.e. there is an $x \in \mathcal{D}$ such that $f(x) > f(x^*)$. Let $v = x^* - x$ and begin by writing the definition of the directional derivative of f at x^* in the direction $-v$:

$$\begin{aligned} -\nabla f(x^*) \cdot v &= \lim_{t \to 0} \frac{f(x^* - tv) - f(x^*)}{t} \\ &= \lim_{t \to 0} \frac{f((1-t)x^* + tx) - f(x^*)}{t} \\ &\geq \frac{(1-t)f(x^*) + tf(x) - f(x^*)}{t} \\ &= f(x) - f(x^*) > 0 \end{aligned} \tag{80}$$

where the weak inequality in (80) is due to the concavity of f and the fact that small $t > 0$ probably means $t < 1$ at least. The last line is due to the assumption.

Now, due to the convexity of $\mathcal{D}$, we have $x^* - tv = (1-t)x^* + tx \in \mathcal{D}$. So for each g_i such that $g_i(x^*) = 0$ we have

$$-\nabla g_i(x^*) \cdot v = \lim_{t \to 0} \frac{g_i(x^* - tv) - g_i(x^*)}{t} \geq 0. \tag{81}$$

For all the rest, we have $\lambda_i = 0$ as per one of the Karush-Kuhn-Tucker conditions. Gathering all observations, we have the following contradiction:

$$0 > \nabla f(x^*) \cdot v = -\left(\sum_{i=1}^{k} \lambda_i \nabla g_i(x^*)\right) \cdot v = -\left(\sum_{i=1}^{k} \lambda_i \nabla g_i(x^*) \cdot v\right) \geq 0. \tag{82}$$

We are done. □

6.4 Other Results

Consider the problem

$$\max_x f(x,a) \text{ subject to } g(x,a) = 0 \text{ and } x \geq 0, \tag{P3}$$

where x is the usual vector of variables (size n), and a is a vector of parameters (size m). Suppose that for each vector a, the solution to this problem is unique, and denote it $x(a)$. Now define the **maximum value function**,

$$M(a) = f(x(a), a);$$

in other words, M is a function of a subject to x having been chosen to solve Problem (P3). Now suppose we would like to analyze how M varies as a varies.

Theorem 40. (**Envelope Theorem**) *Consider Problem* (P3) *and assume that f and g are C^1 in a. For each a let $x(a)_j \geq 0$ for all $j = 1, \ldots, n$ and assume that the $x(a)_j$ are also C^1 in a. Let $\mathcal{L}(x, a, \lambda)$ be the Lagrangean for Problem* (P3) *and let $(x(a); \lambda(a))$ solve the Karush-Kuhn-Tucker conditions for the problem. Let $M(a)$ be the maximum value function for f. Then*

$$\frac{\partial M(a)}{\partial a_j} = \frac{\partial \mathcal{L}}{\partial a_j} \text{ evaluated at } (x(a), \lambda(a)), \text{ for all } j = 1, \ldots, m.$$

Proof. First write the Lagrangian

$$\mathcal{L} = f(x,a) + \lambda g(x,a).$$

If $x(a)$ solves Problem (P3), then Karush-Kuhn-Tucker says

$$\nabla_n f(x(a), a) + \lambda(a)\nabla_n g(x(a), a) = 0 \tag{83}$$
$$g(x(a), a) = 0 \tag{84}$$

which define solutions $(x(a), \lambda(a))$. Then note

$$\frac{\partial \mathcal{L}(x(a), \lambda(a))}{\partial a_j} = \frac{\partial f(x(a), a)}{\partial a_j} + \lambda(a)\frac{\partial g(x(a), a)}{\partial a_j}. \tag{85}$$

Also note,

$$\begin{aligned}
\frac{\partial M(a)}{\partial a_j} &= \sum_{i=1}^{n}\left[\frac{\partial f(x(a), a)}{\partial x_i}\right]\frac{\partial x_i(a)}{\partial a_j} + \frac{\partial f(x(a), a)}{\partial a_j} \\
&= \lambda(a)\sum_{i=1}^{n}\left[-\frac{\partial g(x(a), a)}{\partial x_i}\frac{\partial x_i(a)}{\partial a_j}\right] + \frac{\partial f(x(a), a)}{\partial a_j} \\
&= \lambda(a)\frac{\partial g(x(a), a)}{\partial a_j} + \frac{\partial f(x(a), a)}{\partial a_j} \\
&= \frac{\partial \mathcal{L}(x(a), \lambda(a))}{\partial a_j}
\end{aligned}$$

where the first equality comes from the chain rule you argued in Exercise 64; the second from substituting (83); the third from substituting the derivative of the left hand side of (84) and applying the chain rule; and the fourth from substituting (85). □

Exercise 66. Verify that the envelope theorem is true for the following problem.

$$\max_{x_1,x_2} x_1x_2 \text{ subject to } a - 2x_1 - 4x_2 = 0 \text{ and } x_i \geq 0 \text{ for } i = 1, 2$$

You can dispense with the nonnegativity constraints since the solution will satisfy the Karush-Kuhn-Tucker conditions.